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Symmetric quiver settings with a regular ring of invariants

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Abstract

In this paper we classify all the symmetric quivers and corresponding dimension vectors having a smooth space of semisimple representation classes. The result we obtain is that such quivers can be decomposed as a connected sum of a few number of basic quivers.

Keywords: Quiver representations; Invariant theory

1. Introduction

An interesting problem in invariant theory is the following. Consider a complex vector space V and a reductive algebraic group G with a linear action on V. The ring of polynomial functions over V will be $\mathbb{C}[X_1, \dots, X_n]$ where n is the dimension of V. This ring will have a corresponding action of G on it. One can now look at the subring of invariant polynomial functions,

$$\mathbb{C}[X_1, \dots, X_n]^{\mathsf{G}} := \left\{ f \in \mathbb{C}[X_1, \dots, X_n] \mid f^g = f \right\},$$

and ask whether this subring is also a polynomial ring or a regular ring. In general, this is not an easy problem. In this paper we will look at the special case of symmetric quiver representations.

A *quiver* $Q = (V, A, s, t)$ consists of a set of vertices V, a set of arrows A between those vertices and maps $s, t : A \to V$ which assign to each arrow its starting and terminating vertex. We also denote this as

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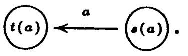


A quiver $Q = (V, A, s, t)$ is *symmetric* if and only if the number of arrows between two vertices is the same in both directions, that is,

$$\forall v, w \in V : \#\left\{a \in A \,\middle|\, v \xrightarrow{a} w\right\} = \#\left\{a \in A \,\middle|\, v \xleftarrow{a} w\right\}.$$

A *dimension vector* of a quiver is a map $\alpha : V \to \mathbb{N}$, the size of a dimension vector is defined as $|\alpha| := \sum_{v \in V} \alpha_v$. A couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in V$, α_v is referred to as the dimension of v. If we draw pictures of quiver settings, we will write down the dimension of a vertex inside that vertex.

An *α-dimensional complex representation* W of Q assigns to each vertex v, a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a, a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C}).$$

The space of all α-dimensional representations is denoted by $\mathsf{Rep}_\alpha Q$.

$$\mathsf{Rep}_\alpha Q := \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{t(a)} \times \alpha_{s(a)}}(\mathbb{C}).$$

To the dimension vector α, we can also assign a reductive group

$$\mathsf{GL}_\alpha := \bigoplus_{v \in V} \mathsf{GL}_{\alpha_v}(\mathbb{C}).$$

An element of this group, g, has a natural action on $\mathsf{Rep}_\alpha Q$:

$$W := (W_a)_{a \in A}, \quad W^g := (g_{t(a)} W_a g_{s(a)}^{-1})_{a \in A}.$$

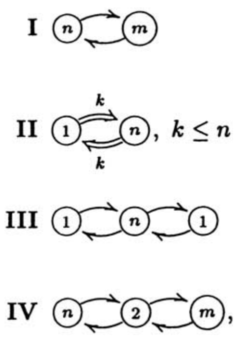
With these definitions, the special vector space we will look at is $\mathsf{Rep}_\alpha Q$ and the reductive group is GL_α. We also suppose that α does not contain any vertex with zero dimension because this problem can be reduced to the problem of a new quiver obtained by deleting all vertices that have zero dimension. Quiver settings having this property are called *genuine*.

The main theorem we prove here is a classification of all symmetric quiver settings for which the corresponding ring of invariants is a regular ring. Such quiver settings are called *coregular*.

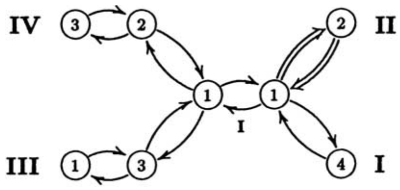
Theorem. *A symmetric quiver setting without loops (Q, α) is coregular if and only if the following conditions are satisfied*:

- *Q is treelike, by which we mean that the underlying graph, having the same vertices as Q and* 1 *edge between two vertices whenever there is at least one arrow between them in Q, is a tree*.
- *The branching vertices* (*i.e. vertices that have arrows connecting it to more than two other vertices*) *have dimension* 1.

- *The quiver setting is constructed by sticking together subquiver settings of the types shown below identifying only vertices with dimension* 1.



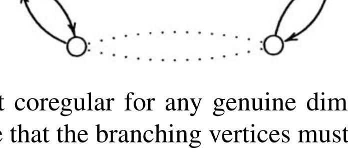
As an example illustrating this theorem we show a coregular quiver setting made by sticking together two settings of type I, and one of type II, III, and IV.



The proof of the theorem uses mainly two observations:

(1) If a quiver setting is coregular then all its subquiver settings and all its possible local quiver settings (see Section 3) are coregular.
(2) If one sticks together two quiver settings by identifying a vertex with dimension 1, the ring of invariants of this new setting will be the tensor product of the rings of invariants of the two original quiver settings.

First one proves that a coregular quiver setting must be treelike because of observation 1 and the fact that a quiver of the form



is not coregular for any genuine dimension vector. A similar argument is used to prove that the branching vertices must have dimension 1.

Observation 2 allows us now to cut the tree into pieces, (or in the algebraic way decomposing the ring of invariants into a tensor product) and to look at the pieces separately. Finally one concludes the proof by a classification of all coregular pieces that cannot be cut into smaller ones.

2. The quotient space and the ring of invariants

As we stated in the Introduction we want to study the ring of invariants $\mathbb{C}[\mathsf{Rep}_\alpha]^{\mathsf{GL}_\alpha}$. If we look at the problem in a geometric way this ring of invariants corresponds to a new affine variety that classifies the closed orbits of the GL_α-action on $\mathsf{Rep}_\alpha Q$.

Definition 2.1. If we divide out the action of GL_α on $\mathsf{Rep}_\alpha Q$, by taking the affine quotient we obtain a new space $\mathrm{iss}_\alpha Q$. The points of the space $\mathrm{iss}_\alpha Q$ are the closed GL_α-orbits in $\mathsf{Rep}_\alpha Q$. The coordinate ring of this variety is the ring of GL_α-invariant polynomial functions on $\mathsf{Rep}_\alpha Q$.

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \mathbb{C}[\mathsf{Rep}_\alpha]^{\mathsf{GL}_\alpha}.$$

For more details of this construction see [2].

The question whether the ring of invariants is regular or polynomial is the same as asking whether $\mathsf{iss}_\alpha Q$ is a smooth variety or an affine space.

Another way of looking at this problem comes from the representation's theoretic point of view. Two representations in $\mathsf{Rep}_\alpha Q$ are called equivalent, if they belong to the same orbit under the action of GL_α.

A representation W is called *simple* if the only collections of subspaces $(\mathrm{V}_v)_{v\epsilon V}$, $\mathrm{V}_v \subseteq \mathbb{C}^{\alpha_v}$ having the property

$$\forall a \in A : W_a V_{s(a)} \subset V_{t(a)}$$

are the trivial ones (i.e. the collection of zero-dimensional subspaces and $(\mathbb{C}^{\alpha_v})_{v\in V}$).

The direct sum $W \oplus W'$ of two representations W, W' has as dimension vector, the sum of the two dimension vectors and as matrices $(W \oplus W')_a := W_a \oplus W'_a$. A representation equivalent to a direct sum of simple representations is called *semisimple*.

In [1] it is proven that an orbit of a representation is closed if and only if this representation is semisimple. So one can also consider $\mathsf{iss}_\alpha Q$ as the space classifying all semisimple α-dimensional representation classes.

In order to study $\mathsf{iss}_\alpha Q$ more closely, we recall some of the results of the paper by Le Bruyn and Procesi [3], which studies the local structure of the invariant ring $\mathbb{C}[\mathsf{iss}_\alpha Q]$.

A sequence of arrows $a_1 \cdots a_p$ in a quiver Q is called a *path of length p* if $s(a_i) = t(a_{i+1})$, this path is called a cycle if $s(a_p) = t(a_1)$. To a cycle, we can associate a polynomial function

$$f_c : \mathsf{Rep}_\alpha Q \to \mathbb{C} : W \mapsto \mathrm{Tr}(W_{a1} \cdots W_{ap})$$

which is definitely GL_α-invariant. Two cycles that are a cyclic permutation of each other give the same polynomial invariant, because of the basic properties of the trace map. Two such cycles are called equivalent.

A cycle $a_1 \cdots a_p$ is called *primitive* if every arrow has a different starting vertex. This means that the cycle runs through each vertex at most 1 time. It is easy to see that every cycle has a decomposition in primitive cycles. It is however not true that the corresponding polynomial invariant decomposes to a product of the polynomial functions of the primitive cycles.

We will call a cycle *quasi-primitive* for a dimension vector α if the vertices that are ran through more than once, have dimension bigger than 1. By cyclicly permuting a cycle and splitting the trace of a product of two 1×1 matrices into a product of traces, we can always decompose an f_c into a product of traces of quasi-primitive cycles. We now have the following result.

Theorem 2.1 (Le Bruyn–Procesi). *$\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by all f_c where c is a quasi-primitive cycle of degree smaller than $|\alpha|^2 + 1$. We can turn $\mathbb{C}[\mathsf{iss}_\alpha Q]$ into a graded ring by giving f_c the length of its cycle as degree.*

Because $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is a graded ring, the only smooth varieties that can occur are affine spaces.

Theorem 2.2. *Let $R = \bigoplus_{i \in \mathbb{N}}$, R_i is a finitely generated positively graded $\mathbb{C}$-algebra, of which $R_0 = \mathbb{C}$. Define Ω to be the maximal graded ideal $\bigoplus_{i>0} R_i$. If Specm R is smooth in Ω then R is isomorphic to a polynomial ring $\mathbb{C}[X_1, \ldots, X_n]$ where the X_i correspond to homogeneous elements in R.*

Proof. The proof is based on Chapter III, Proposition 2.4, p. 136 of [7]. We prove this by induction on n = Krull Dim R. If $n = 0$ then R is $\mathbb{C}$ and the statement is trivial. Suppose now that the statement is true for dimensions smaller than n. Take $x \in R$ to be a homogeneous element of positive degree in Ω/Ω^2.

Now $R' := R/(x)$ is again graded with $R'_0 = \mathbb{C}$ and $\mathsf{Specm}\, R$ is smooth in $\Omega' := \Omega/(x)$ because

$$\text{Krull Dim } R/(x) = \text{Krull Dim } R - 1 = \text{Dim}_{\mathbb{C}} \Omega/\Omega^2 - 1 = \text{Dim}_{\mathbb{C}} \Omega'/\Omega'^2.$$

By the induction hypothesis, $R' \cong \mathbb{C}[X_1, \ldots, X_{n-1}]$. Let $Y_1, \ldots, Y_{n-1}$ be homogeneous representatives in R of the elements in R' that correspond to the X_i. We now have a graded epimorphism

$$\psi : \mathbb{C}[X_1, \ldots, X_n] \to R : X_i \mapsto \begin{cases} Y_i & i < n \\ x & i = n. \end{cases}$$

Because the Krull dimension of $\mathbb{C}[X_1, \ldots, X_n]$ and R are the same ψ is an isomorphism. □

Corollary 2.3. *If $\mathsf{iss}_\alpha Q]$ is smooth in the zero representation class then $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is a polynomial ring and hence $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is an affine space.*

Definition 2.2. Define a partial ordering $\leqslant$ on the set of quivers in the following way. A quiver $Q' = (V', A', s', t')$ is smaller than $Q = (V, A, s, t)$ if (up to isomorphism)

$$V' \subseteq V, \quad A' \subseteq A, \quad s' = s|'_A \quad \text{and} \quad t' = t|'_A,$$

where Q' is called a *subquiver* of Q.

Lemma 2.4. *If* $\mathsf{iss}_\alpha Q$ *is smooth and* $Q' \leqslant Q$ *then* $\mathsf{iss}_{\alpha'} Q'$ *is also smooth, where* $\alpha' := \alpha|_{V'}$.

Proof. We have an embedding

$$\mathsf{Rep}_{\alpha'} Q' \hookrightarrow \mathsf{Rep}_\alpha Q$$

by assigning to the additional arrows in Q zero matrices. So

$$\mathbb{C}[\mathsf{Rep}_\alpha Q] \twoheadrightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q'] \Rightarrow \mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha} \twoheadrightarrow \mathbb{C}[\mathsf{Rep}_{\alpha'} Q']^{\mathsf{GL}_\alpha}.$$

Because the action of GL_α on $\mathsf{Rep}_{\alpha'} Q'$ reduces to that of GL'_α, $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is a quotient ring of $\mathbb{C}[\mathsf{iss}_\alpha Q] = \mathbb{C}[X_1, \ldots, X_n]$. The only relations that we have to divide out are the X_i that correspond to a cycle containing one of the additional arrows we put zero, so $\mathbb{C}[\mathsf{iss}_{\alpha'} Q']$ is just a polynomial ring with less variables. □

Two vertices v and w are said to be *strongly connected* if there is a path from v to w and vice versa. It is easy to check that this relation is an equivalence so we can divide the set of vertices into equivalence classes V_i. The subquiver Q_i having V_i as set of vertices, and as arrows all arrows between vertices of V_i is called a *strongly connected component* of Q.

Lemma 2.5

(1) *If* (Q, α) *is a quiver setting then*

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \bigotimes_i \mathbb{C}[\mathsf{iss}_{\alpha_i} Q_i]$$

where $Q_i = (V_i, A_i, s_i, t_i)$ *are the strongly connected components of* Q *and* $\alpha_i := \alpha|_{V_i}$.

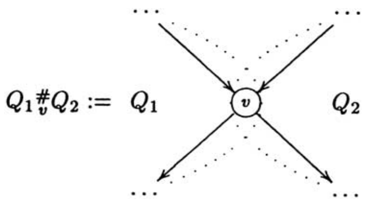
(2) $\mathsf{iss}_\alpha Q$ *is smooth if and only if the* $\mathsf{iss}_\alpha Q_i$ *of all its strongly connected components are smooth.*

Proof

(1) By Theorem 2.1 $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by the traces of cycles. Every cycle belongs to a certain connected component of Q. Between f_c's coming from cycles of different components there cannot be any relations, so we can consider the ring of invariants as a tensor product of the rings of invariants of the different strongly connected components.

(2) If all the strongly connected components are coregular the ring of invariants of the total quiver setting will be the tensor product of polynomial rings and hence a polynomial ring. The inverse implication follows directly from Lemma 2.4. □

Definition 2.3. A quiver $Q = (V, A, s, t)$ is said to be the *connected sum* of two subquivers $Q_1 = (V_1, A_1, S_1, t_1)$ and $Q_1 = (V_2, A_2, s_2, t_2)$ at the vertex v, if the two subquivers make up the whole quiver and only intersect in the vertex v. So in symbols $V = V_1 \cup V_2,\ A = A_1 \cup A_2,\ V_1 \cap V_2 = \{v\}$ and $A_1 \cap A_2 = \emptyset$.



If we connect three or more components we write $Q_1{}^{\#}_{v} Q_2{}^{\#}_{w} Q_3$ instead of $(Q_1{}^{\#}_{v} Q_2)^{\#}_{w} Q_3$ for sake of simplicity.

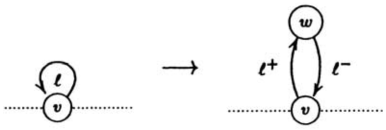
Lemma 2.6. *Suppose* $Q_1{}^{\#}_{v} Q_2$ *and* $\alpha_v = 1$. *Then*

$$\mathbb{C}[\mathsf{iss}_\alpha Q] := \mathbb{C}[\mathsf{iss}_{\alpha 1} Q_1] \otimes \mathbb{C}[\mathsf{iss}_{\alpha 2} Q_2],$$

where $\alpha_i := \alpha|_{Qi}$.

Proof. By Theorem 2.1 $\mathbb{C}[\mathsf{iss}_\alpha Q]$ is generated by the traces of quasi-primitive cycles. Because the dimension of v is one every quasi-primitive cycle is either in subquiver Q_1 or Q_2 and there cannot be any relations between invariants coming from cycles in different subquivers. This implies that the ring of invariants of (Q, α) is the tensor product of the rings of invariants of the two subquiver settings. □

Finally we can restrict to quivers without loops. If we have a quiver setting with loops than we can construct a new quiver setting $(Q^\times, \alpha^\times)$ such that $\mathsf{iss}_{\alpha^\times} Q^\times$ is isomorphic to the original $\mathsf{iss}_\alpha Q$. We alter every loop in the original quiver into a vertex and two arrows as in the picture.



The dimension at w is bigger or equal than on the vertex $v(\alpha^\times_\omega \geqslant \alpha_v)$. To every cycle c in Q corresponds exactly 1 cycle $c^\times$ in $Q^\times$ (replace ℓ by $\ell^+\ell^-$) and vice versa. We can embed $\mathsf{Rep}_{\alpha^\times} Q^\times$:

$$\iota : W \mapsto W^\times : W^\times_{\ell^+} = (1_{\alpha_v} \quad 0), \quad W^\times_{\ell^-} = \begin{pmatrix} W_\ell \\ 0 \end{pmatrix},$$

where the zeroes make up for the additional dimension in ω. We can map $\mathsf{Rep}_{\alpha^\times} Q^\times$ onto $\mathsf{Rep}_\alpha Q$

$$\pi : W^\times \mapsto W : W_\ell = W_{\ell^+} W_{\ell^-}.$$

Because the identity

$$f_c \circ \pi = f_{c^\times} \circ \iota$$

holds the maps $\pi^* : \mathbb{C}[\mathsf{iss}_\alpha Q] \to \mathbb{C}[\mathsf{iss}_{\alpha^\times} Q^\times] : f \mapsto f \circ \pi$ and its analogon ι^* are inverses of each other.

Lemma 2.7

$$\mathbb{C}[\mathsf{iss}_\alpha Q] \cong \mathbb{C}[\mathsf{iss}_{\alpha^\times} Q^\times]$$

Lemmas 2.5 and 2.7 allow us to consider only strongly connected quivers without any loops.

3. The Luna–Slice machinery

In this section we review briefly the Luna–Slice theorem and indicate in what way we will use it to obtain our classification. Most of the results in this section are taken from [4] or [6].

If we want to prove that a certain $\mathsf{iss}_\alpha Q$ is a smooth space, we have to check that it is smooth in every point. Take a point $p \in \mathsf{iss}_\alpha Q$, this point will correspond to the isomorphism class of a semisimple representation $V \in \mathsf{Rep}_\alpha Q$ which can be decomposed as a direct sum of simple representations.

$$V = S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}.$$

The Luna–Slice theorem connects the structure of $\mathsf{Rep}_\alpha Q$ around the closed orbit of V under the action of GL_α to the structure of $\mathsf{iss}_\alpha Q$ around p.

The orbit of V, $\mathcal{O}_V$, has a tangent space in $V : \mathsf{T}_V \mathcal{O}_V$. This tangent space forms a subspace of the complete tangent space $\mathsf{T}_V \mathsf{Rep}_\alpha Q$ and has a quotient space denoted by $\mathsf{N}_V := \mathsf{T}_V \mathsf{Rep}_\alpha Q / \mathsf{T}_V \mathcal{O}_V$. Every $g \in \mathsf{GL}_\alpha$ defines a natural linear map g^* on the tangent spaces.

$$g^* : \mathsf{T}_V \mathsf{Rep}_\alpha Q \to \mathsf{T}_{V^g} \mathsf{Rep}_\alpha Q.$$

Hence the stabilizer of V in GL_α, Stab_V, acts on $\mathsf{T}_V \mathsf{Rep}_\alpha Q$. Moreover is $\mathsf{T}_V \mathcal{O}_V$ mapped onto itself and therefore one can factor out $T_V \mathcal{O}_V$ to obtain an Stab_V-action on N_V. In [4] the following result is obtained:

Theorem 3.1 (Luna [4]). *There exists an étale isomorphism φ between an open neighborhood of the point $0 \in \mathsf{N}_V/\mathsf{Stab}_V$ and an open neighborhood of $p \in \mathsf{iss}_\alpha Q$ mapping 0 to V. So locally we have the following diagram*:

$$\begin{array}{ccc} \mathsf{Rep}_\alpha Q & \xrightarrow{/\mathsf{GL}_\alpha} & \mathsf{iss}_\alpha Q \\ & & \uparrow \varphi \\ \mathsf{N}_V & \xrightarrow{/\mathsf{Stab}_V} & \mathsf{N}_V/\mathsf{Stab}_V \end{array}$$

Near the point p $\mathsf{iss}_\alpha Q$ *is analytically isomorphic to the quotient* $\mathsf{N}_V/\mathsf{Stab}_V$.

Because we are only interested in smoothness which is an analytic property we can simplify the problem of studying $\mathsf{iss}_\alpha Q$ near p to the study of the simpler quotient $\mathsf{N}_V/\mathsf{Stab}_V$ around the zero. At this point the technique of local quivers comes in action [3, Section 6].

The stabilizer of a simple representation is isomorphic to the group of scalar matrices. The stabilizer of the direct sum of k copies of a simple representation is isomorphic to $\mathsf{GL}_k(\mathbb{C})$. Keeping this in mind and looking at the decomposition of V into simple representations, we obtain that the stabilizer of

$$V := S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k},$$

must be equal to the group

$$\mathsf{Stab}_V \cong \mathsf{GL}_{a_1}(\mathbb{C}) \times \cdots \times \mathsf{GL}_{a_k}(\mathbb{C}),$$

The tangent space in V will be identified with $\mathsf{Rep}_\alpha Q$. Due to the action of GL_α we can map the Lie-algebra

$$\mathfrak{gl}_\alpha : \mathrm{T}_e\mathsf{GL}_\alpha = \bigoplus_{v\in V} \mathfrak{gl}_{\alpha_v}(\mathbb{C})$$

subjectively onto the tangent space $\mathrm{T}_V\mathcal{O}_V$. A little calculation shows us that we can identify $\mathrm{T}_V\mathcal{O}_V$ with the following subset of $\mathsf{Rep}_\alpha Q$;

$$\{[m, V] \mid m \in \mathfrak{gl}_\alpha\}$$

Calculating the action of Stab_V on the space above and, on $\mathsf{Rep}_\alpha Q$ leads to the following theorem,

Theorem 3.2 (Le Bruyn–Procesi). *For a point* $p \in \mathsf{iss}_\alpha Q$ *corresponding to a semi-simple representation* $V := S_1^{\oplus a_1} \oplus \cdots \oplus S_k^{\oplus a_k}$, *we can identify* N_V *canonically with* $\mathsf{Rep}_{\alpha_p} Q_p$ *where* Q_p *is the local quiver of* $p \cdot Q_p$ *has k vertices corresponding to the set* $\{S_i\}$ *of simple factors of V and between* S_i *and* S_j *the number of arrows equals*

$$\delta_{ij} - \chi_Q(\alpha_i, \alpha_j),$$

where α_i *is the dimension vector of the simple component* S_i *and* χ_Q *is the Euler form of the quiver Q. The Euler form of Q is the bilinear form* $\chi_Q : \mathbb{Z}^{\#V} \times \mathbb{Z}^{\#V} \to \mathbb{Z}$ *defined by the matrix*

$$m_{ij} = \delta_{ij} - \#\left\{a \,\middle|\, (i) \xleftarrow{a} (j)\right\},$$

where δ *is the Kronecker delta.*

The dimension vector α_p *is defined to be* $(a_1, \ldots, a_k)$, *where the* a_i *are the multiplicities of the simple components in* V.

The action of Stab_V *on* N_V *corresponds to the normal action of* GL_{α_p} *on* $\mathsf{Rep}_{\alpha_p} Q_p$.

Putting all these results together we get (see [3, Theorem 5]):

Theorem 3.3 (Le Bruyn–Procesi). *For every point* $p \in \mathsf{iss}_\alpha Q$ *we have an étale isomorphism between an open neighborhood of the zero representation in* $\mathsf{iss}_{\alpha_p} Q_p$ *and an open neighborhood of* p.

How are we going to apply this theorem? If we want to compute whether a certain space $\mathsf{iss}_\alpha Q$ is smooth, then we can choose a certain point p and look at this locally. Because of the étale isomorphism, the corresponding local quiver Q_p must have a quotient space $\mathsf{iss}_{\alpha_p} Q_p$ that is smooth in the zero representation. Therefore by Corollary 2.3, $\mathbb{C}[\mathsf{iss}_{\alpha_p} Q_p]$ must be a polynomial ring and hence (Q_p, α_p) is coregular. To find out whether (Q, α) is coregular we have to check all possible points p.

Theorem 3.4. (Q, α) *is coregular if and only if for every possible semisimple* α-*dimensional representation* V, *the corresponding local quiver setting is coregular. One of the local quivers is equal to the original quiver, namely the one corresponding to the representation*

$$\bigoplus_{v \in V} S_v^{\bigoplus \alpha_v},$$

where S_v *corresponds to the simple representation with dimension vector*

$$\epsilon_v : V \to \mathbb{N} : w \mapsto \delta_{vw}$$

assigning to every arrow the zero matrix. This implies that we can only use this result to rule out quiver settings that are not coregular.

The structure of the local quiver setting only depends on the dimension vectors of the simple components. Therefore one can restrict to looking at decompositions of a into dimension vectors f.i.

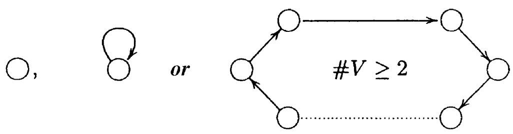
$$\alpha = a_1\beta_1 + \cdots + a_k\beta_k \quad \text{(the } \beta_i \text{ need not to be different).}$$

One can now ask whether there is a semisimple representation corresponding to such a decomposition. The answer to this question will be positive whenever for all the β_i there exist simple representations of that dimension vector and if there are two or more β_i equal, there are at least as much different simple representation classes with dimension vector β_i (otherwise you cannot make a direct sum with different simple representations having the same dimension vector).

To check the above conditions we must also have a characterization of the dimension vectors for which a quiver has simple representations. We recall a result from Le Bruyn and Procesi [3, Theorem 4].

Theorem 3.5. *Let (Q, α) be a genuine quiver setting. There exist simple representations of dimension vector α if and only if*

- *If Q is of the form*



  and $\alpha = 1$ (this is the constant map from the vertices to 1).
- *Q is not of the form above, but strongly connected and*

  $$\forall v \in V : \chi_Q(\alpha, \epsilon_v) \leqslant 0 \quad \textit{and } \chi Q(\epsilon_v s, \alpha) \leqslant 0.$$

In both cases the dimension of $\mathsf{iss}_\alpha Q$ is given by $1 - \chi_Q(\alpha, \alpha)$. In all cases except for the one vertex without loops this dimension is bigger than 0, so then there are infinite classes of simples with that dimension vector. In the case of the one vertex v without loops, there is one unique simple representation S_v.

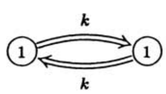
If (Q, α) is not genuine, the simple representations classes are in bijective correspondence to the simple representations classes of the genuine quiver setting obtained by deleting all vertices with dimension 0.

4. Necessary conditions

In this section we determine some necessary conditions for a quiver setting to be coregular. In the next section we will use these conditions to generate all coregular quiver settings.

We first look at a simple case.

Lemma 4.1. *The following quiver setting is not coregular if $k > 1$:*



(*)

Proof. By Definition 2.1, the ring of invariants is spanned by all the cycles

$$X_{ij} = f_{a_i b_j},$$

where a_i stands for one of the arrows to the right and b_j one to the left. All these cycles are necessary to generate the algebra, because for the representations

$$W^{mnt} : W^{mnt}_{a_i} = \delta_{im}, \quad W^{mnt}_{b_i} = t\delta_{in}, \qquad t \in \mathbb{C}, \quad m, n \leqslant k$$

all these invariants are zero except X_{mn} which is equal to t, so X_{mn} cannot be written as a polynomial in the other X_{ij}.

The relations between the cycles are of the form

$$X_{ij} X_{mn} = X_{in} X_{mj}.$$

These relations prevent $\mathsf{iss}_1 Q$ from being an affine space. The only way to make $\mathsf{iss}_1 Q$ into an affine space is that there is only one such cycle. □

We will use this lemma in the following way. Suppose we have a quiver setting (Q, α) and that W_1 and W_2 are different representations of Q with dimension vectors β_1 and β_2 such that for every vertex v, $\beta_{1v} + \beta_{2v} \leqslant \alpha_v$ and $\beta_1, \beta_2 \neq \epsilon_v$.

We can now construct a representation of the form

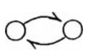
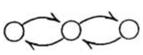
$$W_1 \oplus W_2 \oplus \underbrace{\bigoplus S_v^{\alpha v} - \beta_{1v} - \beta_{2v}}_{\text{rest}}$$

The local quiver setting of this representation now has at least two vertices corresponding to W_1 and W_2. By Theorem 3.2, the number of arrows between these two vertices is $k := -\chi_Q(\beta_1, \beta_2)$. So the local quiver setting contains a subsetting like *. So by the previous lemma, Lemma 2.4 and Theorem 3.4 neither the local quiver nor (Q, α) are coregular whenever $\chi_Q(\beta_1, \beta_2) < -1$.

If one of the two representations W_1 or W_2 is equal to an S_v for a certain $v \in V$ one can do the same trick provided one takes care that $\alpha_v = \beta_{1v} + \beta_{2v}$ such that S_v cannot occur anymore in the rest term.

We now use this technique (and some variations with other local quivers) for the following lemma's.

Lemma 4.2. *Suppose that (Q, α) is a coregular symmetric strongly connected quiver without loops and $\forall v \in V : \alpha(v) > 1$ then Q is either*

 *or* 

Proof. We can make an α-dimensional representation which is the direct sum of two different simple representations with dimension vector 1 and a rest term.

The number of arrows in the local quiver between the vertices corresponding to the first two components is

$$k = -\chi_Q(1, 1) = -\sum_{i,j \in V} \delta_{ij} - \#\left\{a \,\middle|\, (i) \xleftarrow{a} (j)\right\} = \#A - \#V.$$

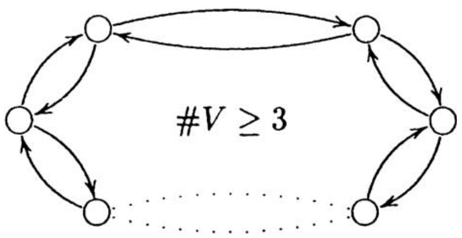
In order to be coregular this number must be at most 1. Because the quiver is symmetric and strongly connected without loops the inequality $\#A \geqslant 2(\#V - 1)$ holds.

If $\#V = 1$ this inequality is trivially true. Suppose the inequality holds for $\#V = n$ and that Q has $n + 1$ vertices. If there are two arrows terminating in every vertex the inequality holds for Q. If this is not the case there is a vertex where exactly one arrow starts and one terminates. Delete this vertex and the remaining quiver is still strongly connected without loops, so

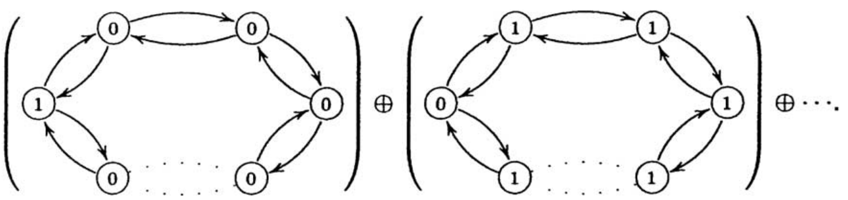
$$\#A - 2 \geqslant 2((\#V - 1) - 1) \Rightarrow \#A \geqslant 2(\#V - 1).$$

Using the inequality one gets that $1 \geqslant k \geqslant \#V - 2$. The only quivers satisfying $\#V \leqslant 3$ and $\#A \leqslant \#V + 1$ are the one listed above. □

Lemma 4.3. *The following quiver is not coregular for any genuine dimension vector*



Proof. By the previous lemma we can suppose that the dimension of the left vertex is 1. Construct a representation of the form

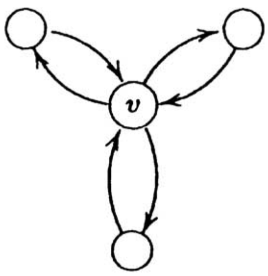


For the two quiver settings above there exist indeed simple representations by Theorem 3.5. The number of arrows in the local quiver between the first two components is 2 so (Q, α) is not coregular.

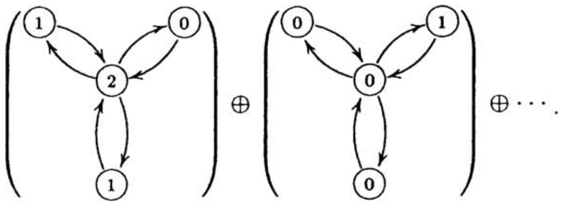
The lemma above, in combination with Lemma 2.4 shows us that if we look at the underlying graph of a symmetric strongly connected coregular quiver setting without loops (Q, α) (having the same vertices as Q, and 1 edge between two vertices if there is an arrow between them), this graph must have the form of a tree. Such quivers are called treelike.

There are also restrictions on the possible dimension vectors. We will determine at which vertices the dimension vector has to be 1. □

Lemma 4.4. *The following genuine quiver setting is not coregular if the dimension in the center v is bigger than* 1.



Proof. If the dimension vector of the center is bigger than 1 then by Lemma 4.2 we can suppose that at least 1 of the dimensions of the other vertices is 1 (take this to be the upper right one). Using Theorem 3.5 we can find a representation of the form

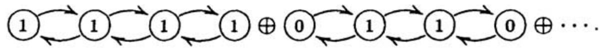


The number of arrows between the vertices in the local quiver corresponding to the first and the second simple representation is equals

$$-\begin{pmatrix}2 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & -1 & -1 & -1\\ -1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = 2. \qquad \square$$

Lemma 4.5. *The following genuine quiver setting is not coregular if* $v_2, v_3 \geqslant 2$.

Proof. Using Theorem 3.5 we can find a representation of the form



The number of arrows between the vertices in the local quiver corresponding to the first and the second simple representation is

$$-\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 0 & 0\\ -1 & 1 & -1 & 0\\ 0 & -1 & 1 & -1\\ 0 & 0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = 2. \qquad \square$$

Theorem 4.6. *If a symmetric strongly connected quiver setting without loops* (Q, α) *is coregular then* Q *is a connected sum*

$$Q := Q_{1_{v_1}}^{\#} Q_{2_{v_2}}^{\#} \cdots {}_{v_{k-1}}^{\#} Q_k,$$

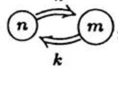
where the Q_i *have at most three vertices, and* $\alpha_{v_j} = 1$, $j = l, \ldots, k-1$.

Proof. For a quiver with at most three vertices this is obviously true. If a symmetric connected quiver setting is coregular and it has more than three vertices then it is treelike by Lemma 4.3. Cutting at the vertices with dimension 1, we can consider Q as a connected sum of smaller components. By Lemma 4.4 branching vertices (i.e. vertices that have arrows connecting it with more than three other vertices) have dimension 1 and by Lemma 4.5 there are no three consecutive vertices with dimension bigger than 1 unless they are at the end of a branch. This implies that the components of this connected sum have at most three vertices. $\square$

5. The classification result

In this section we determine all coregular quiver settings with two or three vertices. After that we combine them to bigger quivers in order to construct all symmetric quiver settings that are coregular.

Lemma 5.1. *The quiver setting*

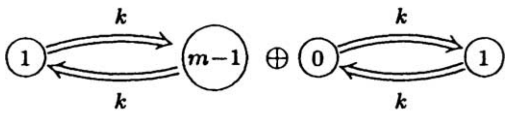
$(Q, \alpha) :=$ 

$, \; n \le m$

is coregular if and only if $k = l$ *or* $l = n \leqslant k \leqslant m$.

Proof. If $k = 1$ then Lemma 2.7 shows that the space is equal to that of the quiver with one vertex and one loop, such that the dimension vector is n. This problem is the same as the conjugacy problem of matrices, which is known to have a smooth quotient space (see classical invariant theory in Kraft [2]).

If $k > 1$ then by Lemma 4.2 at least n must be 1. If $k > m$ then we can make the following decomposition in simples:



Computing the number of arrows in the local quiver gives us

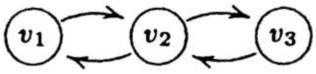
$$-\begin{pmatrix}1 & m-1\end{pmatrix}\begin{pmatrix}1 & -k\\ k & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = k - m + 1 > 1.$$

If $k = m$ then there are exactly k^2 quasi-primitive cycles. Moreover by Theorem 3.5 the dimension of $\mathsf{iss}_\alpha Q$ is $1 - \chi_Q(\alpha, \alpha) = k^2$. Hence there can be no relations between the generators of $\mathbb{C}[\mathsf{iss}_\alpha Q]$ otherwise the Krull dimension of $\mathbb{C}[\mathsf{iss}_\alpha Q]$ would be smaller then k_2.

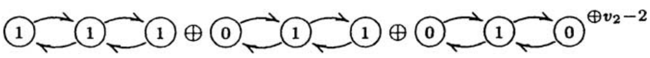
By Lemma 2.4 the case $k < m$ is also coregular. $\square$

For quivers with three vertices we only have to look at the settings where the dimension of the middle vertex is bigger than 1 because otherwise we can consider it as a connected sum of two quiver settings with two vertices.

Lemma 5.2. *The following genuine quiver setting is not coregular if* $v_2 \geqslant 3$, $v_3 \geqslant 2$



Proof. Using Theorem 3.5 we construct a representation of the form



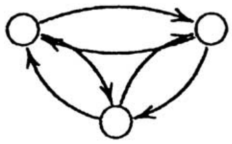
Computing the arrows between the three vertices corresponding to the three simple components

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ 1 & 1 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = -1$$

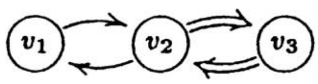
shows that the associated local quiver has a subquiver of the form



(∗∗)

which is not coregular according to Lemma 4.3. □

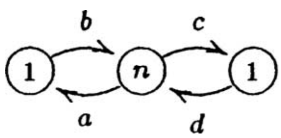
Lemma 5.3. *The following quiver setting is not coregular if* $v_2 \leqslant 2$.



Proof. Using Theorem 3.5 we construct a representation of the form

Computing the arrows as in the previous lemma shows that the associated local quiver has also a subquiver of the form (∗∗). □

Lemma 5.4. *The quiver setting*



is coregular.

Proof. There are only three quasi-primitive cycles, the left cycle and the right cycle and the combination of the 2. Those three are independent because the representation

$$V_a := \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix},$$

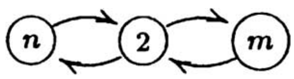
$$V_b := \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$V_c := \begin{pmatrix} 0 & 1 & \cdots & 0 \end{pmatrix},$$

$$V_d := \begin{pmatrix} d_1 \\ d_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

gives us as invariants $b_1, b_2, b_2 d_1$. □

Lemma 5.5. *The quiver setting*



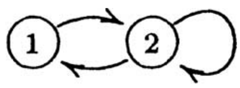
is coregular.

Proof. If both n and m are bigger then 1, this problem is the same as a vertex with two loops, or two simultaneously conjugated 2×2-matrices. As we know from [5], this problem is coregular and its ring of invariants is generated by

$$\mathsf{Tr}A,\ \mathsf{Tr}B,\ \mathsf{Tr}A^2,\ \mathsf{Tr}B^2 \text{ and } \mathsf{Tr}AB,$$

where A and B are the matrices corresponding to the two loops.

If $n = 1$ and $m > 1$ we can simplify it to the situation



This can be modeled as two matrices A and B under simultaneous conjugation and with the restriction that A has rank one. The invariants are the same as above except that $\mathsf{Tr}A^2 - (\mathsf{Tr}A^2)$. This means that the ring of invariants is indeed a polynomial ring generated by

$$\mathsf{Tr}A,\ \mathsf{Tr}B,\ \mathsf{Tr}B^2 \text{ and } \mathsf{Tr}AB,$$

If both n and m are 1 we are in the situation of the previous lemma. □

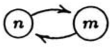
Keeping Lemmas 5.2 and 5.3 in mind, the last two lemmas give us all coregular quiver settings with three vertices that are not the connected sum of smaller ones.

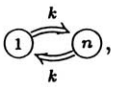
Combining all the results we get a characterization:

Theorem 5.6. *Let (Q, α) be a symmetric strongly connected quiver setting without loops. (Q, α) is coregular if and only if Q is a connected sum*

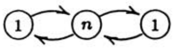
$$Q := Q_1 \underset{v1}{\#} Q_2 \underset{v2}{\#} \cdots \underset{vk-1}{\#} Q_k,$$

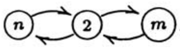
where the (Q_i, α_i) *are of the form*

I 

II 

, $k \leq n$

III 

IV 

,

and $\alpha_{v_j} = 1$, $j = 1, \ldots, k-1$ *dvips.*

Proof. The proof follows from Theorem 4.6 and the fact that the above list characterizes all coregular quiver settings with three or less vertices that cannot be written as a connected sum of smaller ones. □

References

[1] M. Artin, On Azumaya algebras and finite-dimensional representations of rings, J. Algebra 11 (1969) 523–563.

[2] H. Kraft, Geometrische Methoden in der Invariantentheorie, Aspekte der Mathematik, vol. Dl, Vieweg-Verlag, Braunschweig, 1984.

[3] L. Le Bruyn, C. Procesi, Semisimple representations of quivers, Trans. Am. Math. Soc. 317 (1990) 585–598.

[4] D. Luna, Slices etales, Bull. Soc. Math. France Suppl. Mém. 33 (1973) 81–105.

[5] C. Procesi, The invariant theory of $n \times n$ matrices, Adv. Math. 19 (1976) 306–381.

[6] P. Slodowy, Der Scheibensatz fur Algebraische Transformationsgruppen, in: H. Kraft, P. Siodowy, T.A. Springer (Eds.), Algebraische Transformationsgruppen und Invariantentheorie, Algebraic Transformation Groups and Invariant Theory, DMV Seminar, vol. 13, Birkhäuser, Basel, Boston, Berlin, 1989, pp. 89–114.

[7] F. Van Oystaeyen, C. Nastasescu, Graded and Filtered Rings and Modules, Springer-Verlag, Berlin, Heidelberg, New York, 1979.