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Structure Sheaves and Noncommutative Topologies

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In this note, we study abstract localization with respect to elements in the free semigroup generated by all Gabriel filters over a fixed base ring. The main purpose of this set-up, which behaves in many ways as the analogous one with respect to a single Gabriel filter, is to construct structure sheaves over noncommutative topologies associated to an arbitrary left noetherian ring. This construction is particularly useful, when the base ring is "very noncommutative" and does not allow for a sufficiently large spectrum of left or two-sided prime ideals. © 1997 Academic Press

INTRODUCTION

If R is a commutative ring, then the category of quasicoherent sheaves on the affine scheme $\mathrm{Spec}(R)$ is well known to be equivalent to the category R-mod. If $R = \bigoplus_{n \geq 0} R_n$ is a positively graded commutative ring,

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then, modulo some harmless restrictions on R, a similar result holds for the category of quasicoherent sheaves over the projective scheme Proj(R), which is thus equivalent to (R, σ_+)-gr, the quotient of the category R-gr of graded R-modules over the localizing subcategory, consisting of all graded R-modules, which are torsion with respect to $R_+ = \oplus_{n>0} R_n$. For lack of a suitable geometric object, this led Artin [1] to define the projective scheme associated to a positively graded noncommutative ring as a similarly defined category (R, σ_+)-gr, the category R-mod arising as an algebraic analogue of the (category of quasicoherent sheaves on the) affine scheme associated to R.

Although the above constructions appear to be very useful analogues of the commutative ones, the problem of exhibiting a genuine "geometric" realization of noncommutative rings remained unsolved. Of course, working over the prime spectrum Spec(R) of R, highly satisfactory constructions of this type exist if R is commutative [9] or if R satisfies Jategaonkar's second layer condition [3, 11], for example. In [16], an attempt was made to associate a geometric object to a reasonably general type of rings, working with "noncommutative" Grothendieck topologies and suitably adapted structure "sheaves." Unfortunately, the latter construction heavily depends upon the existence of "many" Ore sets and lacks geometric features. This is mainly due to the fact that the analogues of open covers it is based on are always induced by global ones and thus do not allow for non-trivial local-global results, as well as failing to generalize the classical construction of affine or projective schemes associated to a commutative ring.

The methods we develop in this note aim to remedy this and permit us to construct universal geometric objects representing arbitrary left noetherian rings, thus answering some questions raised by Manin [10] in the context of noncommutative geometry and quantum groups.

Let us start with some definitions. One calls two idempotent kernel functors [8] σ and τ over R *compatible*, cf. [3, 18], if $\sigma Q_\tau = Q_\tau \sigma$, where Q_τ is the localization functor associated to τ as in [3, 4, 7, 8, 14, 15]. For example, over a commutative noetherian ring any pair of idempotent kernel functors is compatible, cf. [18].

As expounded in [3, 17, 18], compatibility is, somewhat surprisingly, one of the main ingredients in the construction of structure sheaves over commutative and noncommutative rings. The main reason for this is a result, proved in [12], which states that σ and τ are compatible if and only if the following canonical sequence of functors is exact,

$$0 \to Q_{\sigma \wedge \tau} \to Q_\sigma \oplus Q_\tau \to Q_{\sigma \vee \tau}$$

(where $\sigma \wedge \tau$ resp. $\sigma \vee \tau$ denotes the meet resp. join of σ and τ, cf. [7], for example). Actually, in view of Deligne's formula [9], which states that for any ideal I of a noetherian commutative ring R and any R-module M, the sections of the quasicoherent sheaf $\mathcal{O}_M$ over the open set $D(I) \subseteq \operatorname{Spec}(R)$ are given by

$$\mathcal{O}_M(D(I)) = \Gamma(D(I), \mathcal{O}_M) = Q_I M,$$

the localization of M at I (see below), it is easily seen that the above exact sequence specializes to

$$0 \to \mathcal{O}_M(D(I) \cup D(J)) \to \mathcal{O}_M(D(I)) \oplus \mathcal{O}_M(D(J)) \to \mathcal{O}_M(D(I) \cap D(J))$$

which just describes the patching property for the sheaf $\mathcal{O}_M$ on $\operatorname{Spec}(R)$. It has been shown in [19] that in the general case, i.e., if σ and τ are not necessarily compatible, one still has an exact sequence (with obvious morphisms)

$$0 \to Q_{\sigma \wedge \tau} \to Q_\sigma \oplus Q_\tau \to Q_\sigma Q_\tau \oplus Q_\tau Q_\sigma .$$

In fact, it was verified that for any finite family $\{\sigma_\alpha;\ \alpha \in A\}$ of idempotent kernel functors over R, the localization functor $Q_{\wedge \sigma_\alpha}$ is the projective limit of the family

$$\left\{Q_{\sigma_\alpha} \to Q_{\sigma_\alpha} Q_{\sigma_\beta}, Q_{\sigma_\beta} \to Q_{\sigma_\alpha} Q_{\sigma_\beta}; \alpha, \beta \in A\right\}.$$

Some first indications were given in [19] about extensions of this last result to generalized localization functors, associated to words composed of idempotent kernel functors, i.e., of elements of the free semigroup generated by all idempotent kernel functors over R. In the present text, we expound this construction and its main properties in full detail and show how it may be applied to construct structure sheaves over generalized, noncommutative topologies over the base ring R, thus generalizing ‘‘classical’’ sheaf constructions both in the commutative and the noncommutative case.

In our set-up, we have preferred to use the language of Gabriel filters and not the (equivalent) one of idempotent kernel functors. The main reason for this being that we frequently will have to deal with a notion of composition of idempotent kernel functors (recalled below), which does not yield an idempotent kernel functor, in general, but which possesses a very natural interpretation, when expressed in terms of Gabriel or more general filters (it yields a so-called *uniform* filter).

This note is organized as follows. In the first section, we quickly recollect some of the main notions, constructions, and results, which will be needed throughout. No proofs have been included; references to the literature being given instead. In the second section, we study localization at elements in the free semigroup generated by all Gabriel filters over R and prove its main properties. In particular, we will show how the main results in [19] may be generalized to this set-up. In the last section, we introduce noncommutative topologies over an arbitrary ring R and show how the former results may be applied to construct structure sheaves over these. Some examples are included, showing our methods to generalize previous sheaf constructions in the literature, as pointed out above.

1. GENERALITIES

(1.1) Throughout this note, R denotes an associative ring with unit. A *filter* over R is a non-empty set of left ideals L such that for any pair of left ideals $I \subseteq J$ of R with $I \in \mathsf{L}$, we also have $J \in \mathsf{L}$. Let us say that a left R-module M is L-torsion if for all $m \in M$, we may find some $L \in \mathsf{L}$ with $Lm = 0$. If L and H are two filters over R, then we define the filter $\mathsf{L} \circ \mathsf{H}$ (the *composition* of L and H) as consisting of all left ideals I of R with the property that there exists some $H \supseteq I$ in H such that H/I is L-torsion.

A filter L is said to be a *cofilter* if it is closed under taking intersections. It is easy to see that if L and H are cofilters, then so is $\mathsf{L} \circ \mathsf{H}$. A *uniform* filter is a cofilter L with the property that $\mathsf{L} \subseteq \mathsf{L} \circ \{R\}$, equivalently, if $(I : r) \in \mathsf{L}$ for all $I \in \mathsf{L}$ and $r \in R$. For any uniform filter L, we may define a left exact subfunctor σ_{L} of the identity in R-mod, the category of left R-modules, by putting

$$\sigma_{\mathsf{L}} M = \left\{ x \in M; \operatorname{Ann}^l_R(x) \in \mathsf{L} \right\},$$

for every left R-module M. Clearly, $M \in R$-mod is L-*torsion* if $\sigma_{\mathsf{L}} M = M$. We say that M is L-*torsionfree* if $\sigma_{\mathsf{L}} M = 0$.

(1.2) A uniform filter L is said to be a *Gabriel filter*, if $\mathsf{L} \circ \mathsf{L} = \mathsf{L}$. This is easily seen to be equivalent to the usual definition, given in [4, 14], for example, and to $\mathsf{L} \circ \mathsf{L} \subseteq \mathsf{L}$, since the other inclusion follows from

$$\mathsf{L} \subseteq \mathsf{L} \circ \{R\} \subseteq \mathsf{L} \circ \mathsf{L},$$

as L is assumed to be uniform. It is fairly easy to see that if L is a Gabriel filter, then the associated functor σ_{L} is actually an idempotent kernel functor in R-mod in the sense of [8], i.e., we also have that $\sigma_{\mathsf{L}}(M/\sigma_{\mathsf{L}} M) = 0$, for any left R-module M. Conversely, any idempotent kernel functor σ in R-mod yields a Gabriel filter $\mathsf{L}(\sigma)$ consisting of all left ideals L of R with the property that R/L is σ-torsion.

(1.3) As an example, recall that any left Ore set S in R defines a Gabriel filter $\mathcal{L}(S)$, consisting of all left ideals L of R with $L \cap S = \emptyset$. We denote by σ_S the associated idempotent kernel functor. In [16], the authors work with the free monoid $\mathbf{W}(R)$ on all left Ore sets in R. If $W = S_1 \cdots S_n \in \mathbf{W}(R)$, then we let the filter $\mathcal{L}(W)$ consist of all left ideals L containing a product $s_1 \cdots s_n$ with $s_i \in S_i$. In general, $\mathcal{L}(W)$ is only a uniform filter, i.e., not necessarily a Gabriel filter.

In fact, we have:

(1.4) LEMMA. *Let $S_1, \dots, S_n$ be left Ore sets in R. Then*

$$\mathcal{L}(S_1 \cdots S_n) = \mathcal{L}(S_1) \circ \cdots \circ \mathcal{L}(S_n).$$

Proof. It clearly suffices to prove this for two left Ore sets S_1 and S_2.

First, pick $L \in \mathcal{L}(S_1 S_2)$, then there exist $s_1 \in S_1$ and $s_2 \in S_2$ with $s_1 s_2 \in L$. Clearly $L + Rs_2 \in \mathcal{L}(S_2)$. On the other hand, if $r \in R$ is chosen arbitrarily, then we may find $r' \in R$ and $s_1' \in S_1$, with $s_1' r = r' s_1$, hence $s_1'(rs_2) = r' s_1 s_2 \in L$. So, $L + Rs_2/L$ is S_1-torsion, which shows that $\mathcal{L}(S_1 S_2) \subseteq \mathcal{L}(S_1) \circ \mathcal{L}(S_2)$.

Conversely, if $L \in \mathcal{L}(S_1) \circ \mathcal{L}(S_2)$, then we may pick $H \in \mathcal{L}(S_2)$, say with $s_2 \in H \cap S_2$, with $L \subseteq H$ and H/L torsion at S_1. So, there exists $s_1 \in S_1$ with $s_1 s_2 \in L$, i.e., $L \in \mathcal{L}(S_1 S_2)$. ∎

(1.5) If $\mathcal{L}$ is a Gabriel filter over R, then we denote by $Q_{\mathcal{L}}$ the associated *localization functor*, cf. [3, 4, 7, 8, 14, 15]. It may be defined by

$$Q_{\mathcal{L}} M = \varinjlim_{I \in \mathcal{L}} \operatorname{Hom}_R(I, M/\sigma_{\mathcal{L}} M),$$

for any $M \in R$-mod. Alternatively, $Q_{\mathcal{L}} M$ may be viewed as consisting of all $q \in E(M/\sigma_{\mathcal{L}} M)$ (the injective hull of $M/\sigma_{\mathcal{L}} M$) with the property that $Iq \subseteq M/\sigma_{\mathcal{L}} M$, for some $I \in \mathcal{L}$. The canonical morphism $j_{\mathcal{L}, M}: M \to Q_{\mathcal{L}} M$ is called the *localization map* at $\mathcal{L}$. If $\mathcal{L}$ is associated to the idempotent kernel functor σ in R-mod, then we will sometimes also write $j_{\sigma, M}$ or just j_σ, when no ambiguity arises.

(1.6) It is easy to see that the composition of uniform filters is a uniform filter, the trivial filter $\{R\}$ acting as an identity element, and that it satisfies the associativity property.

Unfortunately, the composition of Gabriel filters does not yield a Gabriel filter, in general. As a typical example, define for any two-sided ideal I of a left noetherian ring R the Gabriel filter $\mathcal{L}_I$ as consisting of all left ideals L of R containing some positive power I^n of I. We then leave it as an easy exercise to the reader to verify that for any pair of two-sided ideals I and J of R, the uniform filter $\mathcal{L}_I \circ \mathcal{L}_J$ consists of all left ideals L

of R, which contain I^nJ^m for some positive integers n and m. However, this uniform filter is not necessarily a Gabriel filter (it is, if I and J commute, for example).

In fact, we will define two Gabriel filters $\mathcal{L}$ and $\mathcal{L}'$ to be *mutually compatible*, if their composition is again a Gabriel filter. One may prove that this is equivalent to asserting that $\mathcal{L} \circ \mathcal{L}' = \mathcal{L}' \circ \mathcal{L}$ (cf. [3]). In this case, $\mathcal{L} \circ \mathcal{L}'$ is a Gabriel filter and, actually, $\mathcal{L} \circ \mathcal{L}' = \mathcal{L} \vee \mathcal{L}'$, where $\mathcal{L} \vee \mathcal{L}'$ is the *join* of $\mathcal{L}$ and $\mathcal{L}'$, i.e., the smallest Gabriel filter, which contains both $\mathcal{L}$ and $\mathcal{L}'$. In the general case, $\mathcal{L} \vee \mathcal{L}'$ is the Gabriel filter generated by $\mathcal{L} \circ \mathcal{L}'$, as one easily verifies.

(1.7) EXAMPLE. Let us again consider a left Ore set S in R and denote by Q_S the localization at the Gabriel filter $\mathcal{L}(S)$ associated to S. Of course, for any left R-module M, we then have $Q_S M = S^{-1}M$. Using this, the previous remarks, and (1.3), it now easily follows that $\mathcal{L}(S)$ and $\mathcal{L}(T)$ are mutually compatible if and only if $\mathcal{L}(ST) = \mathcal{L}(TS)$, and that this is also equivalent to asserting $S^{-1}T^{-1}M = T^{-1}S^{-1}M$ for all $M \in R$-mod. In fact, in this case, these localizations then coincide with $(S \vee T)^{-1}M$, where $S \vee T$ is the left Ore set generated by S and T. We refer to [16] for some applications of these remarks.

(1.8) EXAMPLE. Recall from [4, 14], for example, that a Gabriel filter $\mathcal{L}$ is said to be *stable*, if the associated torsion class is closed under taking injective hulls. As a typical example, let I be a two-sided ideal of a left noetherian ring R and let us define the Gabriel filter $\mathcal{L}_I$ as consisting of all left ideals of R containing some positive power of I. It is then easy to see, cf. [3], that $\mathcal{L}_I$ is stable if and only if I satisfies the left Artin–Rees condition, i.e., if for any left ideal L of R we may find some positive integer n such that $I^n \cap L \subseteq IL$. Of course, due to the Artin–Rees lemma, this implies that over a commutative noetherian ring all Gabriel filters $\mathcal{L}_I$ are stable—note that with a little more work, one may prove that in this case actually *all* Gabriel filters are stable. Let us also point out that if R satisfies the second layer condition, cf. [11], then it was proved in [3] that I satisfies the Artin–Rees condition if and only if it satisfies the weak Artin–Rees condition, i.e., if for any two-sided ideal K of R there exists some positive integer n with $KI^n \subseteq IK$.

The main interest of stable Gabriel filters in the present context resides in the fact that any pair $\mathcal{L}$, $\mathcal{H}$ of these is mutually compatible, cf. [3, 18]. Moreover, one then even has that $Q_{\mathcal{L}}$ and $Q_{\mathcal{H}}$ commute, so, in particular, that $Q_{\mathcal{L}} Q_{\mathcal{H}} = Q_{\mathcal{L} \vee \mathcal{H}}$.

This follows from:

(1.9) LEMMA. *Let $\mathcal{L}$ and $\mathcal{H}$ be Gabriel filters over the left noetherian ring R. If $Q_{\mathcal{L}}$ and $Q_{\mathcal{H}}$ commute, then $Q_{\mathcal{L}} Q_{\mathcal{H}} = Q_{\mathcal{L} \vee \mathcal{H}}$.*

Proof. We know that $Q_L Q_H = Q_H Q_L$ implies that $L \circ H = H \circ L = L \vee H$. For any $M \in R$-mod, the kernel of the canonical map $M \to Q_L Q_H M$ is thus just $\sigma_{L \vee H} M$, and its cokernel is $L \vee H$-torsion. It thus remains to show that $Q_L Q_H M$ is $L \vee H$-closed. It is clear that $Q_L Q_H M = Q_H Q_L M$ is $L \vee H$-torsionfree, as it is both torsionfree at L and H. On the other hand, it is also $L \vee H$-injective, since any R-linear map $f \colon I \to Q_L Q_H M$ extends to the whole ring. Indeed, as $L \vee H = L \circ H$, there exists $H \supseteq I$ in H such that $(I : h) \in L$ for all $h \in H$. So, mapping $r \in R$ to $f(rh)$ defines a morphism $(I : h) \to Q_L Q_H M$ and as the latter module is L-closed, it extends uniquely to $g_h \colon R \to Q_L Q_H(M)$. Define $p \colon H \to Q_L Q_H M$ by putting $p(h) = g_h(1)$. By the uniqueness of g_h for all $h \in H$, the map p is well defined and R-linear. It is also clear that p is an extension of f, since $(I : h) = R$ for all $h \in I$. Finally, as $Q_L Q_H M = Q_H Q_L M$ is H-injective, p can be extended to the whole of R, hence $Q_L Q_H M$ is $L \vee H$-closed, which proves the assertion. ∎

Denoting for any two-sided ideal I of the left noetherian ring R by Q_I the localization functor at L_I, it thus follows, in particular, for any pair of R-ideals I and J satisfying the Artin−Rees condition (or the weak Artin−Rees condition, if R satisfies the left second layer condition), that $Q_I Q_J = Q_{IJ}$. This result is the basis for most of the sheaf constructions developed in [3].

2. FREE SEMIGROUPS OVER GABRIEL FILTERS

(2.1) Let $K = \{ L_\alpha; 1 \le \alpha \le n\}$ be a finite family of Gabriel filters over R and put $L = \bigcap_\alpha L_\alpha$ (which is also a Gabriel filter, as one easily checks). Let M be a left R-module and denote by $j_{M,\alpha} \colon M \to Q_\alpha M$ the localization map with respect to L_α, where Q_α is the localization functor at L_α. Then, denoting by $Q_K M$ the projective system

$$\left\{Q_\alpha j_{M,\beta} \colon Q_\alpha M \to Q_\alpha Q_\beta M, j_{Q_\beta M,\alpha} \colon Q_\beta M \to Q_\alpha Q_\beta M; 1 \le \alpha,\ \beta \le n\right\},$$

it was proved in [19] that

$$\varprojlim Q_K M = Q_L M.$$

In particular, if L and L' are mutually compatible Gabriel filters, then it is easy to see that $Q_L Q_{L'}$ is $L \vee L'$-torsionfree, so, in particular, the canonical map $Q_L Q_{L'} \to Q_{L \vee L'}$ is injective. Applying the previous remark to a family $\{ L_\alpha ; 1 \leq \alpha \leq n\}$ of mutually compatible Gabriel filters, it thus follows that there is an exact sequence

$$0 \to Q_L M \to \oplus_\alpha Q_\alpha M \rightrightarrows \oplus_{\beta,\gamma} Q_{\beta \vee \gamma} M,$$

where $Q_{\beta \vee \gamma}$ is the localization functor at $L_\beta \vee L_\gamma$, of course. For the implications of this result to the construction of structure sheaves on various "classical" spectra of R, we refer to [3, 18].

(2.2) Let $G(R)$ denote the set of all Gabriel filters over R and let $\langle G(R)\rangle$ be the free semigroup generated by it, i.e., elements in $\langle G(R)\rangle$ are words $\mathbf{L} = L_1 \cdots L_n$, where $L_1, \ldots, L_n$ are Gabriel filters. The uniform filter $L_1 \circ \cdots \circ L_n$ is denoted by $\varepsilon(\mathbf{L})$, and the associated left exact subfunctor $\sigma_{\varepsilon(\mathbf{L})}$ of the identity by $\sigma_{\mathbf{L}}$. When no ambiguity arises, we will refer to $\varepsilon(\mathbf{L})$-torsion resp. $\varepsilon(\mathbf{L})$-torsionfree R-modules, as being $\mathbf{L}$-torsion resp. $\mathbf{L}$-torsionfree. We denote by $Q_{\mathbf{L}}$ the composition $Q_{L_n} \cdots Q_{L_1}$, and if M is a left R-module, then $j_{\mathbf{L}, M}$ denotes the composition of the localization morphisms $j_{n,M} \circ \cdots \circ j_{1,M}$, where $j_{1,M} = j_{L_1, M}$ and where, for all $2 \leq i \leq n$,

$$j_{i,M} \colon Q_{L_{i-1}} \cdots Q_{L_1} M \to Q_{L_i}\big(Q_{L_{i-1}} \cdots Q_{L_1} M\big)$$

is the localization map at L_i for the left R-module $Q_{L_{i-1}} \cdots Q_{L_1} M$.

The functors $Q_{\mathbf{L}}$ behave in many respects like localization functors with respect to some Gabriel filter. To see this, let us start by mentioning the following basic result, which was also pointed out in [19]:

(2.3) PROPOSITION. *Let M be a left R-module and consider a word $\mathbf{L} = L_1 \cdots L_n$ in $\langle G(R)\rangle$. Then*

$$\operatorname{Ker}(j_{\mathbf{L},M}) = \sigma_{\mathbf{L}} M$$

and $\operatorname{Coker}(j_{\mathbf{L},M})$ *is an* $\mathbf{L}$*-torsion* R*-module.*

Proof. If $n = 1$, this is well known, cf. [3, 4, 7, 8, 14]. Otherwise, assume the statement holds true for words in $\langle G(R)\rangle$ of length up to $n-1$. Let $\mathbf{L}' = L_1 \cdots L_{n-1}$ and put $\sigma = \sigma_{L_n}$.

If $x \in \operatorname{Ker}(j_{\mathbf{L}})$, then $j_\sigma j_{\mathbf{L}'}(x) = 0$ (where $j_\sigma \colon Q_{\mathbf{L}'} M \to Q_\sigma Q_{\mathbf{L}'} M$ is the localization map), i.e., $j_{\mathbf{L}'}(x) \in \operatorname{Ker}(j_\sigma) = \sigma Q_{\mathbf{L}'} M$. So, there exists $I \in L(\sigma) = L_n$, such that $j_{\mathbf{L}'}(Ix) = Ij_{\mathbf{L}'}(x) = 0$, i.e., $Ix \subseteq \operatorname{Ker}(j_{\mathbf{L}'}) = \sigma_{\mathbf{L}'} M$. Hence, for all $i \in I$, there exists $J \in \varepsilon(\mathbf{L}') = L_1 \circ \cdots \circ L_{n-1}$, with $Jix = 0$. Let $L = \operatorname{Ann}^l_R(x)$, then it follows that $L \in \varepsilon(\mathbf{L}') \circ L_n = \varepsilon(\mathbf{L})$, so $x \in \sigma_{\mathbf{L}} M$, indeed. Since the converse is obvious, this proves the first statement.

To prove the second one, let $q \in Q_{\mathbf{L}} M = Q_\sigma Q_{\mathbf{L}'} M$ and let L resp. H consist of all $r \in R$ with the property that $rq \in \operatorname{Im}(j_{\mathbf{L}})$ resp. $rq \in \operatorname{Im}(j_\sigma)$. Clearly, $L \subseteq H$ and $H \in L(\sigma) = L_n$. On the other hand, let $h \in H$, then

$hq = j_\sigma(q')$ for some $q' \in Q_{\mathbf{L}'}M$. By assumption, there exists some $J \in \varepsilon(\mathbf{L}')$ with $Jq' \subseteq \mathrm{Im}(j_{\mathbf{L}'})$, so

$$Jhq = j_\sigma(Jq') \subseteq j_\sigma(\mathrm{Im}(j_{\mathbf{L}'})) = \mathrm{Im}(j_{\mathbf{L}}).$$

Hence, $Jh \subseteq L$, proving that H/L is $\mathbf{L}'$-torsion. It thus follows that $L \in \varepsilon(\mathbf{L}') \circ \mathsf{L}_n = \varepsilon(\mathbf{L})$, which proves the assertion. ∎

Let $\mathbf{L} = \mathsf{L}_1 \cdots \mathsf{L}_n$ be a word in $\langle \mathsf{G}(R) \rangle$ and let us write $f_{\mathbf{L}} = Q_{\mathbf{L}} f$, for any left R-linear map $f\colon M \to N$. Since Q_{L_i} is left exact for $1 \le i \le n$, so is the composition $Q_{\mathsf{L}_n} \cdots Q_{\mathsf{L}_1} = Q_{\mathbf{L}}$. In particular, it thus follows that $\mathrm{Im}(f_{\mathbf{L}})$ is contained in $Q_{\mathbf{L}}(\mathrm{Im}(f))$, viewed as a submodule of $Q_{\mathbf{L}}N$.

(2.4) LEMMA. *Let $\mathbf{L}$ and $\mathbf{H}$ be words in $\langle \mathsf{G}(R) \rangle$. Let M be a left R-module and assume that $q' \in Q_{\mathbf{H}}M$ has the property that $j_{\mathbf{L}, Q_{\mathbf{H}}M}(q') \in \mathrm{Im}(Q_{\mathbf{L}}(j_{\mathbf{H},M}))$. Then there exists $L \in \varepsilon(\mathbf{L})$, such that $Lq' \in \mathrm{Im}(j_{\mathbf{H},M})$.*

Proof. Let us write $u = Q_{\mathbf{L}}(j_{\mathbf{H},M})$ and $v = j_{\mathbf{L}, Q_{\mathbf{H}}M}$ for the canonical maps. Assume $\mathbf{L} = \mathsf{L}_1 \cdots \mathsf{L}_n$ and put $\mathbf{L}^i = \mathsf{L}_1 \cdots \mathsf{L}_{n-i}$ for any $1 \le i \le n$. Finally, let us denote by $v^i\colon Q_{\mathbf{H}}M \to Q_{\mathbf{HL}^i}M$ resp. $t^i\colon Q_{\mathbf{HL}^i}M \to Q_{\mathbf{HL}^{i-1}}M$ the obvious maps. In particular, we then have that v^n is the identity on $Q_{\mathbf{H}}M$ and that $v^0 = v$, while it is also clear that $t^i v^i = v^{i-1}$ for any $1 \le i \le n$. Choose $q \in Q_{\mathbf{L}}M$ such that $u(q) = v(q')$. Since

$$u(q) \in \mathrm{Im}(u) \subseteq Q_{\mathbf{L}}\mathrm{Im}(j_{\mathbf{H}}) = Q_{\mathsf{L}_n} Q_{\mathbf{L}^1}\mathrm{Im}(j_{\mathbf{H}}),$$

by the above remarks, there exists $J_n \in \mathsf{L}_n$ with $J_n v(q') = J_n u(q) \subseteq t^1 Q_{\mathbf{L}^1}\mathrm{Im}(j_{\mathbf{H}})$. So, for any $j \in J_n$, there exists some $y_1 \in Q_{\mathbf{L}^1}\mathrm{Im}(j_{\mathbf{H}})$ with

$$t^1(jv^1(q')) = jv(q') = t^1(y^1).$$

We may thus pick $J'_n \in \mathsf{L}_n$, with $J'_n(jv^1(q') - y_1) = 0$, i.e., $J'_n jv^1(q') \subseteq Q_{\mathbf{L}^1}\mathrm{Im}(j_{\mathbf{H}})$. But then, since L_n is a Gabriel filter, there exists some $I_n \in \mathsf{L}_n$, such that $I_n v^1(q') \in Q_{\mathbf{L}^1}\mathrm{Im}(j_{\mathbf{H}})$.

Next, pick $i \in I_n$ and choose $J_{n-1} \in \mathsf{L}_{n-1}$ with $J_{n-1} i v^1(q') \subseteq t^2 Q_{\mathbf{L}^2}\mathrm{Im}(j_{\mathbf{H}})$. For any $j \in J_{n-1}$, there thus exists some $y_2 \in Q_{\mathbf{L}^2}\mathrm{Im}(j_{\mathbf{H}})$ with

$$t^2(jiv^2(q')) = jiv^1(q') = t^2(y_2).$$

Choosing $J'_{n-1} \in \mathsf{L}_{n-1}$ with the property that $J'_{n-1}(jiv^2(q') - y_2) = 0$ thus shows that $J'_{n-1} jiv^2(q') \in Q_{\mathbf{L}^2}\mathrm{Im}(j_{\mathbf{H}})$. So, L_{n-1} being a Gabriel filter, there exists $I'_{n-1} \in \mathsf{L}_{n-1}$ such that $I'_{n-1} iv^2(q') \subseteq Q_{\mathbf{L}^2}\mathrm{Im}(j_{\mathbf{H}})$. As this holds for any choice of $i \in I_n$, we thus find some $I_{n-1} \in \mathsf{L}_{n-1} \circ \mathsf{L}_n$, with the property that $I_{n-1} v^2(q') \subseteq Q_{\mathbf{L}^2}\mathrm{Im}(j_{\mathbf{H}})$.

It is now clear that this process may be iterated. So, we finally obtain some $L = I_1 \in \mathsf{L}_1 \circ \cdots \circ \mathsf{L}_n = \varepsilon(\mathbf{L})$, with $I_1 v^n(q') \subseteq Q_{\mathbf{L}^n}\mathrm{Im}(j_{\mathbf{H}})$, i.e., $Lq' \subseteq \mathrm{Im}(j_{\mathbf{H}})$, as claimed. This proves our assertion. ∎

(2.5) Let $K = \{\mathbf{L}_\alpha;\ 1 \leq \alpha \leq n\}$ be a finite family of words in $\langle \mathsf{G}(R)\rangle$ and denote $Q_{\mathbf{L}_\alpha}$ by Q_α. Let us fix a left R-module M and consider the projective system

$$\mathsf{Q}_K M = \left\{ j^\alpha_{\alpha\beta}\colon Q_\alpha M \to Q_\alpha Q_\beta M,\ j^\beta_{\alpha\beta}\colon Q_\beta M \to Q_\alpha Q_\beta M;\ 1 \leq \alpha,\ \beta \leq n \right\},$$

where $j^\alpha_{\alpha\beta} = Q_\alpha(j_{\mathbf{L}_\beta, M})$ and $j^\beta_{\alpha\beta} = j_{\mathbf{L}_\alpha, Q_\beta(M)}$ are the obvious arrows.

If we denote by L the intersection $\bigcap_{\alpha=1}^n \varepsilon(\mathbf{L}_\alpha)$, then

$$\sigma_{\mathsf{L}} M \subseteq \sigma_{\mathbf{L}_\alpha} M = \mathrm{Ker}\big(j_\alpha = j_{\mathbf{L}_\alpha, M}\colon M \to Q_\alpha M\big),$$

for all $1 \leq \alpha \leq n$. So, j_α factorizes through $\overline{j_\alpha}\colon M/\sigma_{\mathsf{L}} M \to Q_\alpha M$. In a similar way, we have

$$\sigma_{\mathsf{L}} M \subseteq \sigma_{\mathbf{L}_\beta \mathbf{L}_\alpha} M = \mathrm{Ker}\big(j_{\alpha\beta} = j_{\mathbf{L}_\beta \mathbf{L}_\alpha, M}\colon M \to Q_\alpha Q_\beta M\big),$$

so $j_{\alpha\beta}$ factorizes through $\overline{j_{\alpha\beta}}\colon M/\sigma_{\mathsf{L}} M \to Q_\alpha Q_\beta M$, for every $1 \leq \alpha,\ \beta \leq n$. Since the diagrams

$$\begin{array}{ccc} M/\sigma_{\mathsf{L}} M & \xrightarrow{\ \overline{j_\beta}\ } & Q_\beta M \\ {\scriptstyle \overline{j_\alpha}}\downarrow & & \downarrow{\scriptstyle j^\beta_{\alpha\beta}} \\ Q_\alpha M & \xrightarrow[\ j^\alpha_{\alpha\beta}\ ]{} & Q_\alpha Q_\beta \end{array}$$

commute for all $1 \leq \alpha,\ \beta \leq n$ (as $j^\alpha_{\alpha\beta}\overline{j_\alpha} = j^\beta_{\alpha\beta}\overline{j_\beta} = \overline{j_{\alpha\beta}}$), there exists a unique morphism

$$j\colon M/\sigma_{\mathsf{L}} M \to \varprojlim \mathsf{Q}_K M,$$

fitting into the commutative diagrams

$$\begin{array}{ccccc} & & M/\sigma_{\mathsf{L}} M & & \\ & {\scriptstyle \overline{j_\alpha}}\swarrow & \downarrow{\scriptstyle j} & \searrow{\scriptstyle \overline{j_{\alpha\beta}}} & \\ Q_\alpha M & \longrightarrow & \varprojlim \mathsf{Q}_K M & \longrightarrow & Q_\alpha Q_\beta M \end{array}$$

(where the horizontal maps are the obvious ones).

We claim that j is a monomorphism. Indeed, since $\sigma_{\mathsf{L}} M \subseteq \sigma_{\mathbf{L}_\alpha} M$ and $\mathrm{Ker}(j_\alpha) = \sigma_{\mathbf{L}_\alpha} M$ by (2.3), $\overline{j_\alpha}$ factorizes as

$$\overline{j_\alpha}\colon M/\sigma_{\mathsf{L}} M \to M/\sigma_{\mathbf{L}_\alpha} M \hookrightarrow Q_\alpha M,$$

so $\operatorname{Ker}(\overline{j_\alpha}) = \sigma_{\mathbf{L}_\alpha} M/\sigma_{L} M$. It thus follows that

$$\operatorname{Ker}(j) = \bigcap_{\alpha=1}^{n} \operatorname{Ker}(\overline{j_\alpha}) = \bigcap_{\alpha=1}^{n} \sigma_{\mathbf{L}_\alpha} M/\sigma_{L} M = \sigma_{L} M/\sigma_{L} M = 0,$$

as claimed.

We may now prove:

(2.6) PROPOSITION. *With the previous notations, for any left R-module M and for any family* $K = \{\mathbf{L}_\alpha \in \langle G(R)\rangle;\ 1 \le \alpha \le n\}$, *the cokernel* $\operatorname{Coker}(j)$ *is* $L \circ L$*-torsion.*

Proof. Consider an element

$$q = (q_\alpha)_\alpha \in \varprojlim Q_K M \subseteq \prod_{\alpha=1}^{n} Q_\alpha M.$$

Since for every $\alpha, \beta \in \{1, \dots, n\}$ we have $j^{\alpha}_{\alpha\beta}(q_\alpha) = j^{\beta}_{\alpha\beta}(q_\beta)$, we may find some $L_{\alpha\beta} \in \varepsilon(\mathbf{L}_\alpha)$ with $L_{\alpha\beta} q_\beta \subseteq \operatorname{Im}(j_\beta) = \operatorname{Im}(\overline{j_\beta})$. Let us fix β for a moment and consider

$$L_\beta = \sum_{\alpha=1}^{n} L_{\alpha\beta} \in \bigcap_{\alpha=1}^{n} \varepsilon(\mathbf{L}_\alpha) = L.$$

Then $L_\beta q_\beta \subseteq \operatorname{Im}(\overline{j_\beta})$. Putting $L = \bigcap_{\beta=1}^{n} L_\beta \in L$, we obtain that $Lq_\beta \subseteq \operatorname{Im}(\overline{j_\beta})$ for all $1 \le \beta \le n$.

For all $l \in L$ and for all $1 \le \alpha \le n$, there thus exists $y_{l,\alpha} \in M/\sigma_L M$, with $\overline{j_\alpha}(y_{l,\alpha}) = jq_\alpha$. Fixing α, it follows from $j^{\alpha}_{\alpha\beta} \circ \overline{j_\alpha} = j^{\beta}_{\alpha\beta} \circ \overline{j_\beta}$ and $j^{\alpha}_{\alpha\beta}(q_\alpha) = j^{\beta}_{\alpha\beta}(q_\beta)$ for all $1 \le \beta \le n$ that $j^{\beta}_{\alpha\beta}(\overline{j_\beta}(y_{l,\alpha})) = j^{\beta}_{\alpha\beta}(\overline{j_\beta}(y_{l,\beta}))$, i.e., that

$$\overline{j_\beta}(y_{l,\alpha}) - \overline{j_\beta}(y_{l,\beta}) \in \operatorname{Ker}\left(j^{\beta}_{\alpha\beta}\right) = \sigma_{\mathbf{L}_\alpha} Q_\beta M.$$

Since we are working over a finite set of indices, there thus exists $K_\alpha \in \varepsilon(\mathbf{L}_\alpha)$ such that $K_\alpha(\overline{j_\beta}(y_{l,\alpha}) - \overline{j_\beta}(y_{l,\beta})) = 0$, for all β.

This yields for all $k \in K_\alpha$ that

$$klq = (klq_\beta)_\beta = \left(\overline{j_\beta}(ky_{l,\beta})\right)_\beta = \left(\overline{j_\beta}(ky_{l,\alpha})\right)_\beta = j(ky_{l,\alpha}),$$

i.e., $K_\alpha lq \subseteq \operatorname{Im}(j)$.

For any $l \in L$ we now put $K_l = \sum_{\alpha=1}^{n} K_\alpha \in \bigcap_{\alpha=1}^{n} \varepsilon(\mathbf{L}_\alpha) = L$, so clearly $K_l lq \subseteq \operatorname{Im}(j)$. With $K = \sum_{r \in J} K_l l \in L \circ L$, we thus finally obtain that $Kq \subseteq \operatorname{Im}(j)$, which proves the assertion. ∎

As a consequence, let us mention:

(2.7) COROLLARY. *With the previous notations, assume the finite family* $K = \{\mathbf{L}_\alpha \in \langle \mathsf{G}(R)\rangle;\ 1 \le \alpha \le n\}$ *to satisfy the following conditions*:

(1) $\mathsf{L} = \bigcap_{\alpha=1}^{n} \varepsilon(\mathbf{L}_\alpha)$ *is a Gabriel filter*;

(2) $Q_{\mathsf{L}} Q_\alpha = Q_\alpha$, *for all* $1 \le \alpha \le n$.

Then $\varprojlim \mathsf{Q}_K M = Q_{\mathsf{L}} M$.

Proof. By the previous result, Coker(j) is L-torsion. To prove the assertion, it thus clearly suffices to verify that $\varprojlim \mathsf{Q}_K M$ is L-closed.

Since $Q_{\mathsf{L}} Q_\alpha = Q_\alpha$, obviously $Q_\alpha N$ is L-closed for any left R-module N, so, in particular, $Q_\alpha M$ and $Q_\alpha Q_\beta M$ are L-closed for all $1 \le \alpha,\ \beta \le n$. Hence so is $\varprojlim \mathsf{Q}_{\mathbf{K}}(M)$, as a projective limit of L-closed left R-modules. ∎

(2.8) *Note*. Since $\mathsf{L} \subseteq \varepsilon(\mathbf{L}_\alpha)$ for all indices α, the second condition in the previous result is clearly satisfied, whenever Q_α is a localization functor and, in particular, when $\mathbf{L}_\alpha$ has length one, hence reduces to a single Gabriel filter. This applies, in particular, in the noetherian commutative case, using (1.9) and the fact that one then knows localization functors always to commute, cf. [18].

Let us point out that the second condition also holds, if we assume $\mathsf{L} \subseteq \mathsf{L}_i^\alpha$, for all $1 \le \alpha \le n$ and $1 \le i \le m_\alpha$, where the words $\mathbf{L}_\alpha$ are of the form $\mathbf{L}_\alpha = \mathsf{L}_1^\alpha \cdots \mathsf{L}_{m_\alpha}^\alpha$. In [16], one always works in the situation where L is trivial ($\mathsf{L} = \{R\}$ or $\mathsf{L} = \mathsf{L}_+$, if R is graded, cf. (3.1) and (3.2) below), so both conditions are then trivially satisfied.

Since it is well known (and easy to prove) that localization commutes with finite projective limits, the previous result may be strengthened to:

(2.9) THEOREM. *Consider a word* $\mathbf{H} \in \langle \mathsf{G}(R)\rangle$ *and a family*

$$K = \{\mathbf{L}_\alpha \in \langle \mathsf{G}(R)\rangle;\ 1 \le \alpha \le n\}.$$

Consider the projective system

$$Q_{\mathbf{H}} \mathsf{Q}_K = \{Q_{\mathbf{H}} Q_\alpha M \to Q_{\mathbf{H}} Q_\alpha Q_\beta M,\ Q_{\mathbf{H}} Q_\beta M \to Q_{\mathbf{H}} Q_\alpha Q_\beta M;\ 1 \le \alpha,\ \beta \le n\}$$

and assume that $\mathsf{L} = \bigcap_{\alpha=1}^{n} \varepsilon(\mathbf{L}_\alpha)$ *is a Gabriel filter. Suppose that one of the following conditions holds*:

(1) $Q_{\mathsf{L}} Q_\alpha = Q_\alpha$ *for all* $1 \le \alpha \le n$;

(2) $Q_{\mathbf{H}} Q_{\mathsf{L}} = Q_{\mathbf{H}}$.

Then $\varprojlim Q_{\mathbf{H}} Q_K M = Q_{\mathbf{H}} Q_{\mathsf{L}} M$.

Proof. Suppose $\mathbf{H} = H_1 \cdots H_p$. Since localization commutes with finite projective limits, as we just pointed out, we have under the first assumption

$$\varprojlim Q_{\mathbf{H}K} M = Q_{H_p}\big(\varprojlim Q_{H_1 \cdots H_{p-1}\mathbf{K}} M\big) = \cdots = Q_{\mathbf{H}}(\varprojlim Q_K M),$$

and by (2.7), this is $Q_{\mathbf{H}} Q_L M$.

In the second case, note that (2.6) implies the existence of an exact sequence

$$0 \to M/\sigma_L M \to \varprojlim Q_K M \to T \to 0,$$

where T is L-torsion. This yields an exact sequence

$$0 \to Q_H(M/\sigma_L M) \to \varprojlim Q_{\mathbf{H}} Q_K M \to Q_{\mathbf{H}} T.$$

Here $Q_{\mathbf{H}} T = Q_{\mathbf{H}} Q_L T = 0$, whereas

$$Q_{\mathbf{H}}(M/\sigma_L M) = Q_{\mathbf{H}} Q_L (M/\sigma_L M) = Q_{\mathbf{H}} Q_L M = Q_{\mathbf{H}} M.$$

This proves the assertion. ∎

(2.10) *Note.* The first condition in the previous result has already been discussed above. As for the second one, let us point out that if the components H_i of $\mathbf{H}$ yield mutually commuting localization functors, it just means that $H_1 \vee \cdots \vee H_p \subseteq L$.

3. STRUCTURE SHEAVES

(3.1) In order to allow us to restrict to *finite* covers, throughout this section R is assumed to be a left noetherian ring.

As pointed out in the Introduction, an attempt at associating geometric objects to some interesting classes of noncommutative rings may be found in [16]. Although this construction will be modified and strengthened below, let us briefly describe it, as it easily follows from the results in the previous section.

As in (1.4), let us consider the free monoid $\mathbf{W}(R)$ on all left Ore sets in R and for any $W = S_1 \cdots S_n \in \mathbf{W}(R)$, let us denote by Q_W the composition $Q_{S_n} \cdots Q_{S_1}$.

Consider a positively graded (left noetherian) ring $R = \oplus_{n \geq 0} R_n$, such that R_0 is a field and that R is generated by its elements of degree one. As in [16], let us call a finite set Σ of homogeneous left Ore sets of R such that $S \cap R_1 \neq \emptyset$ for every $S \in \Sigma$ an *affine cover* if $\bigcap_{S \in \Sigma} L(S) = L_+$. Here, L_+ denotes the Gabriel filter associated to the two-sided ideal

$R_+ = \oplus_{n>0} R_n$, i.e., consisting of all left R-ideals containing some positive power of R_+.

From the results in the previous section, it then easily follows:

(3.2) PROPOSITION. *Let Σ be an affine cover and let $W \in \mathbf{W}(R)$. Then, for any left R-module M, we have*

$$\varprojlim Q_{\Sigma W} M = Q_W M,$$

where $Q_{\Sigma W} M$ is the projective system

$$\{Q_{SW} M \to Q_{STW} M,\ Q_{TW} M \to Q_{STW} M;\ S, T \in \Sigma\}.$$

This result may be reformulated by asserting the sequence

$$0 \to Q_W M \to \oplus_{S \in \Sigma} Q_{SW} M \rightrightarrows \oplus_{S,T \in \Sigma} Q_{STW} M$$

(with obvious morphisms) to be exact. In the framework of [16], where a "noncommutative" Grothendieck topology is constructed over R based on the elements in $\mathbf{W}(R)$, this states exactly that assigning $Q_W M$ to $W \in \mathbf{W}(R)$ defines a "structure sheaf" on $\mathbf{W}(R)$. The main results in [16] may thus easily be recovered by applying the calculations in the previous section.

(3.3) Let $\langle G \rangle \subseteq \langle G(R) \rangle$ be the free monoid generated by some fixed class $G \subseteq G(R)$ of Gabriel filters over the ring R, the trivial filter $\{R\}$ acting as unit.

We say that two words $\mathbf{L}, \mathbf{L}' \in \langle G \rangle$ are equivalent if the associated functors $Q_{\mathbf{L}}$ and $Q_{\mathbf{L}'}$ coincide (in particular, we then have $j_{\mathbf{L},M} = j_{\mathbf{L}',M}$). This obviously defines an equivalence relation in $\langle G \rangle$, denoted by $\sim$ and compatible with the multiplication. The class of $\mathbf{L}$ in $\mathbb{T}(G) =: \langle G \rangle / \sim$ will be written as $[\mathbf{L}]$, and may be identified with the associated functor $Q_{\mathbf{L}}$. We will denote by $\varepsilon[\mathbf{L}]$ the uniform filter $\varepsilon(\mathbf{L})$. This is well defined, since $\mathbf{L} \sim \mathbf{L}'$ implies that $\varepsilon(\mathbf{L}) = \varepsilon(\mathbf{L}')$.

(3.4) EXAMPLE. Let R be a commutative ring, denote by $G_{Zar}(R)$ the set of all Gabriel filters over R of the form L_I, where I is some ideal of R, and put $\mathbb{T}_{Zar}(R) = \langle G_{Zar}(R) \rangle / \sim$. Since R is commutative and noetherian, it follows from [18] that $Q_L Q_H = Q_H Q_L = Q_{H \vee L}$ for all Gabriel filters $L, H \in G(R)$. On the other hand, we know that $L_I \circ L_J = L_I \vee L_J = L_{IJ}$ for any pair of ideals I, J of R. It thus follows for any family of ideals $I_1, \dots, I_n$ of R that $L_{I_1} \cdots L_{I_n} \sim L_{I_1 \cdots I_n}$, so every element in $\mathbb{T}_{Zar}(R)$ is of the form $[L_I]$ for some ideal I of R.

Denote by $D(I)$ the Zariski open subset of Spec(R) determined by the ideal I of R. As $Q_I = Q_J$ exactly when $L_I = L_J$ or, equivalently, when $D(I) = D(J)$, it follows that the elements of $\mathbb{T}_{Zar}(R)$ are thus in bijective correspondence with the Zariski open subsets of Spec(R).

(3.5) EXAMPLE. More generally, assume R to be a left noetherian, not necessarily commutative ring. Denote by $\mathcal{G}_{st}(R)$ the set of all Gabriel filters $\mathcal{L}_I$ with the property that $\mathcal{L}_I$ is stable or, equivalently, that I satisfies the (left) Artin–Rees condition and put $\mathbb{T}_{st}(R) = \langle \mathcal{G}_{st}(R) \rangle / \sim$. From (1.9), it then follows for each family of ideals $I_1, \dots, I_n$ satisfying the Artin–Rees condition that $\mathcal{L}_{I_1} \cdots \mathcal{L}_{I_n} \sim \mathcal{L}_{I_1 \cdots I_n}$. Just as in the commutative case, we then have that $[\mathcal{L}_I] = [\mathcal{L}_J]$ in $\mathbb{T}_{st}(R)$ if and only if $D(I) = D(J)$. It thus follows that the elements of $\mathbb{T}_{st}(R)$ are in bijective correspondence with the open sets of the stable topology on Spec(R), cf. [3]. (See also [5], for other examples.)

(3.6) Let $\mathcal{G}$ be an arbitrary family of Gabriel filters over R, then we make $\mathbb{T}(\mathcal{G}) = \langle \mathcal{G} \rangle / \sim$ into a category, by letting $\mathrm{Hom}_{\mathbb{T}(\mathcal{G})}([\mathbf{K}], [\mathbf{L}])$ consist for any $\mathbf{K}, \mathbf{L} \in \langle \mathcal{G} \rangle$ of the natural transformations $\phi: Q_{\mathbf{L}} \to Q_{\mathbf{K}}$ defined over the identity, i.e., with the property that for all $M \in R$-mod, the local morphism ϕ_M fits into a commutative diagram

$$\begin{array}{ccc} Q_{\mathbf{L}} M & \xrightarrow{\phi_M} & Q_{\mathbf{K}} M \\ {\scriptstyle j_{\mathbf{L},M}} \nwarrow & & \nearrow {\scriptstyle j_{\mathbf{K},M}} \\ & M & \end{array}$$

For example, any factorization $[\mathbf{K}] = [\mathbf{L}'][\mathbf{L}][\mathbf{L}'']$ in $\mathbb{T}(\mathcal{G})$ yields a morphism $\phi: [\mathbf{K}] \to [\mathbf{L}]$, given by the natural transformation

$$Q_{\mathbf{L}} \xrightarrow{Q_{\mathbf{L}} j_{\mathbf{L}'}} Q_{\mathbf{L}} Q_{\mathbf{L}'} \xrightarrow{j_{\mathbf{L}''}} Q_{\mathbf{L}''} Q_{\mathbf{L}} Q_{\mathbf{L}'} = Q_{\mathbf{L}'\mathbf{L}\mathbf{L}''} = Q_{\mathbf{K}}.$$

Note that this is actually the only type of morphism we will have to use below. The morphisms used in [16] are essentially of the same kind as well. Note also that if there exists a morphism $\phi: [\mathbf{K}] \to [\mathbf{L}]$, then $\sigma_{\mathbf{L}} \le \sigma_{\mathbf{K}}$. Indeed, this follows immediately from the fact that for any $\mathbf{L} \in \langle \mathcal{G} \rangle$ we have $\sigma_{\mathbf{L}} = \mathrm{Ker}(\mathrm{id}_{R\text{-mod}} \to Q_{\mathbf{L}})$.

Somewhat stronger:

(3.7) LEMMA. *For any pair of idempotent kernel functors σ and τ, the following assertions are equivalent*:

(1) *there exists a natural transformation $\phi: Q_\sigma \to Q_\tau$*;

(2) $\sigma \le \tau$.

Moreover, the natural transformation ϕ is then unique as such.

Proof. The previous remarks show that the existence of some natural transformation $\phi: Q_\sigma \to Q_\tau$ implies that $\sigma \le \tau$. Conversely, if $\sigma \le \tau$, this induces a canonical transformation $\phi: Q_\sigma \to Q_\tau Q_\sigma = Q_\tau$. That ϕ is unique as such follows from the universal properties of the functor Q_τ. ∎

(3.8) A *presheaf* (of abelian groups, for example) on a category $\mathbb{C}$ is, as always, just a contravariant functor $\mathcal{P}$ on $\mathbb{C}$ with values in the category of abelian groups. So, $\mathcal{P}$ associates to any object $C \in \mathbb{C}$ an abelian group $\mathcal{P}(C)$ and to any morphism $f\colon C \to C'$ a homomorphism of abelian groups $\mathcal{P}(f)\colon \mathcal{P}(C') \to \mathcal{P}(C)$. We usually refer to the $\mathcal{P}(f)$ as the *restriction maps*. As before, let $\mathcal{G}$ denote an arbitrary family of Gabriel filters over R and denote by $\mathbb{T}(\mathcal{G})$ the associated category. For any left R-module M, we may canonically define a presheaf of left R-modules $\mathcal{O}_M$ on $\mathbb{T}(\mathcal{G})$ as follows. For any $[\mathbf{L}] \in \mathbb{T}(\mathcal{G})$, put $\mathcal{O}_M[\mathbf{L}] = Q_{\mathbf{L}} M$. On the other hand, if the morphism $f\colon [\mathbf{K}] \to [\mathbf{L}]$ is given by the natural transformation $\phi\colon Q_{\mathbf{L}} \to Q_{\mathbf{K}}$, then we define the restriction map $\mathcal{O}_M f\colon \mathcal{O}_M[\mathbf{L}] \to \mathcal{O}_M[\mathbf{K}]$ to be the local morphism $\phi_M\colon Q_{\mathbf{L}} M \to Q_{\mathbf{K}} M$.

For arbitrary idempotent kernel functors σ and τ in R-mod, we do not necessarily obtain that σ is an idempotent kernel functor in $Q_\tau R$-mod. In particular, it then follows that for $[\mathbf{L}] = [\,\mathcal{L}_1 \cdots \mathcal{L}_n]$, with $n \geq 2$ (just as in [16]) the left R-module $\mathcal{O}_R[\mathbf{L}]$ does not have to be a ring. One easily sees, however, that if the Gabriel filters $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are pairwise compatible, then $\sigma_{\mathcal{L}_i}$ is an idempotent kernel functor in $(Q_{\mathcal{L}_{i-1}} \cdots Q_{\mathcal{L}_1} R)-$mod, so $\mathcal{O}_R[\mathbf{L}]$ is a ring and $\mathcal{O}_M[\mathbf{L}]$ is a left $\mathcal{O}_R[\mathbf{L}]$-module for every $M \in R-$mod.

Let us recall that stable idempotent kernel functors are always pairwise compatible, so if $\mathcal{G}$ contains "many" (or only!) stable Gabriel filters, then on "many" open sets $\mathcal{O}_R$ will yield a ring.

(3.9) EXAMPLE. Let us reconsider the examples given above. First, assume R to be commutative and consider a morphism $\phi\colon [\mathbf{K}] \to [\mathbf{L}]$ in $\mathbb{T}_{Zar}(R)$. We have already pointed out that $[\mathbf{K}] = [\,\mathcal{L}_K]$ resp. $[\mathbf{L}] = [\,\mathcal{L}_L]$ for some ideals K, L of R, so ϕ corresponds to a natural transformation $\phi\colon Q_L \to Q_K$. From (3.7), it follows that ϕ is induced by $\sigma_L \leq \sigma_K$, which is equivalent to $D(K) \subseteq D(L)$. It follows that morphisms in $\mathbb{T}_{Zar}(R)$ thus reduce to the canonical inclusions between open sets in Spec(R) endowed with the Zariski topology.

Of course, the same remarks remain valid for $\mathbf{T}_{st}(R)$ when R is no longer assumed to be commutative.

It thus clearly follows that the presheaves $\mathcal{O}_M$ constructed above essentially reduce to the canonical structure sheaves on Spec(R) endowed with the corresponding topology, cf. [3, 9] for example.

(3.10) From the foregoing, it appears that the monoids $\mathbb{T}(\mathcal{G})$ are rather close to being a (Grothendieck) topology, the morphisms $[\mathbf{K}] \to [\mathbf{L}]$ playing the role of inclusions and the product $[\mathbf{L}][\mathbf{L}'] = [\mathbf{L}\mathbf{L}']$ that of the intersection (or fibre product). Although many properties of ordinary topologies have an analogue in the present context, e.g., the fact that any morphism $f\colon [\mathbf{K}] \to [\mathbf{L}]$ yields for any $[\mathbf{H}] \in \mathbb{T}(\mathcal{G})$ an induced $f'\colon [\mathbf{K}][\mathbf{H}] \to [\mathbf{L}][\mathbf{H}]$, our set-up should be referred to as a *noncommutative* topology, as, in general, $[\mathbf{L}][\mathbf{L}'] \neq [\mathbf{L}'][\mathbf{L}]$ for arbitrary $[\mathbf{L}], [\mathbf{L}'] \in \mathbb{T}(G)$.

As our main purpose is to construct structure sheaves for arbitrary left noetherian rings and left modules over them, let us enhance the previous construction with a suitable notion of "covering."

Let $[\mathbf{H}] \in \mathbb{T}(G)$. A *cover* for $[\mathbf{H}]$ is a finite family

$$\mathfrak{U} = \{[\mathbf{L}_\alpha];\ 1 \leq \alpha \leq n\},$$

satisfying the following conditions:

(C1) $L =: \bigcap_{\alpha=1}^{n} \varepsilon[\mathbf{L}_\alpha]$ is a Gabriel filter;

(C2) $[L][\mathbf{H}] = [\mathbf{H}]$.

We denote by $\mathrm{Cov}_G[\mathbf{H}]$ the set of all covers of $[\mathbf{H}] \in \mathbb{T}(G)$. As it will appear that these covers are tightly connected to the usual notion of an open cover, we will usually refer to the covers $\mathrm{Cov}_G[\mathbf{H}]$ as defining a *noncommutative Grothendieck topology* Cov_G on $\mathbb{T}(G)$. We will call the couple $(\mathbb{T}(G), \mathrm{Cov}_G)$ the *noncommutative site* associated to G.

Covers may be composed:

(3.11) PROPOSITION. *With notations as before, let* $\{[\mathbf{L}_\alpha]\}_\alpha \in \mathrm{Cov}_G[\mathbf{H}]$ *and, for each index* α, *let* $\{[\mathbf{L}_\beta^\alpha]\}_\beta \in \mathrm{Cov}_G[\mathbf{L}_\alpha]$. *Then*

$$\{[\mathbf{L}_\beta^\alpha][\mathbf{L}_\alpha]\}_{\alpha,\beta} \in \mathrm{Cov}_G[\mathbf{H}].$$

Proof. By definition, $L = \bigcap_\alpha \varepsilon[\mathbf{L}_\alpha]$ is a Gabriel filter, and so is $L_\alpha = \bigcap_\beta \varepsilon[\mathbf{L}_\beta^\alpha]$, for every α. The assertion thus follows immediately from the observation that

$$\begin{aligned}\bigcap_{\alpha,\beta} \varepsilon[\mathbf{L}_\beta^\alpha \mathbf{L}_\alpha] &= \bigcap_\alpha \bigcap_\beta \varepsilon[\mathbf{L}_\beta^\alpha] \circ \varepsilon[\mathbf{L}_\alpha] = \bigcap_\alpha L_\alpha \circ \varepsilon[\mathbf{L}_\alpha] \\ &= \bigcap_\alpha \varepsilon[L_\alpha \mathbf{L}_\alpha] = \bigcap_\alpha \varepsilon[\mathbf{L}_\alpha] = L. \quad \blacksquare\end{aligned}$$

(3.12) EXAMPLE. Let R be commutative. A cover for $[L_I] \in \mathbb{T}_{Zar}(R)$ is a finite family $\{[L_{I_\alpha}]\}_\alpha$ satisfying (C1)–(C2). Of course, the only non-trivial property to be satisfied is (C2), stating that $\bigcap_\alpha L_{I_\alpha} \vee L_I = L_I$, i.e., $\bigcap_\alpha L_{I_\alpha} \subseteq L_I$, which is equivalent to $D(I) \subseteq \cup_\alpha D(I_\alpha)$. Any cover of $[L_I]$, which represents an arbitrary element in $\mathbb{T}_{Zar}(R)$ as we saw before, thus yields an ordinary open cover of $D(I) \subseteq \mathrm{Spec}(R)$ for the Zariski topology. Conversely, any (finite) Zariski open cover $D(I) \subseteq \cup_\alpha D(I_\alpha)$ canonically defines a corresponding cover $\{[L_{I_\alpha}]\}_\alpha \in \mathrm{Cov}_{G_{Zar(R)}}[L_I]$.

It thus follows that covers in the above sense correspond exactly to finite ordinary open covers in the Zariski topology on Spec(R). Of course, since R is assumed to be noetherian, we may always restrict to finite covers, since open subsets of Spec(R) are then quasicompact.

Note also that a similar result is valid in the noncommutative case, if we work over the stable topology on Spec(R).

(3.13) Although it should be clear at this point that applying (2.9) allows us, as we will see below, to reach our aims, i.e., to construct for any left R-module M an associated structure sheaf $\mathcal{O}_M$ with respect to the above notion of cover, it appears that some applications (in particular, the functorial behaviour of our constructions [6] and the cohomological treatment, in the vein of [2] of quasicoherent sheaves over these "noncommutative sites") require a slightly modified notion of cover.

Let us call a cover $\mathfrak{U} = \{[\mathbf{L}_\alpha]\}_\alpha \in \mathrm{Cov}_{\mathcal{G}}[\mathbf{H}]$ *stable* if $\mathcal{L} = \bigcap_\alpha \varepsilon[\mathbf{L}_\alpha]$ is a stable Gabriel filter. Let us already point out here that the above examples are all of this type.

Note also:

(3.14) LEMMA. *The finite family* $\mathfrak{U} = \{[\mathbf{L}_\alpha]\}_\alpha$ *is a stable cover of* $[\mathbf{H}] \in \mathbb{T}(\mathcal{G})$ *if and only if the following assertions are valid*:

(1) $\mathcal{L} = \bigcap_\alpha \varepsilon[\mathbf{L}_\alpha]$ *is a stable Gabriel filter*;

(2) *if* T *is* $\mathcal{L}$*-torsion, then* $Q_{\mathbf{H}}T = 0$.

Proof. If $\mathfrak{U}$ is a stable cover of $[\mathbf{H}]$, then (1) is obvious, whereas (2) follows from the fact that for every $\mathcal{L}$-torsion left R-module T we have

$$Q_{\mathbf{H}}T = Q_{\mathbf{H}}Q_{\mathcal{L}}T = 0.$$

Conversely, assuming (1) and (2), to show that $\mathfrak{U}$ is a stable cover of $\mathbf{H}$, we have to check that $Q_{\mathbf{H}}Q_{\mathcal{L}} = Q_{\mathbf{H}}$. Let M be an arbitrary left R-module, then it follows from the exact sequence

$$0 \to M/\sigma_{\mathcal{L}} M \to Q_{\mathcal{L}} M \to T \to 0$$

that $Q_{\mathbf{H}}(M/\sigma_{\mathcal{L}} M) = Q_{\mathbf{H}}Q_{\mathcal{L}} M$, as $Q_{\mathbf{H}}T = 0$, since T is $\mathcal{L}$-torsion.

On the other hand, from the exact sequence

$$0 \to \sigma_{\mathcal{L}} M \to M \to M/\sigma_{\mathcal{L}} M \to 0$$

we deduce the exact sequence

$$0 \to Q_{\mathbf{H}}\sigma_{\mathcal{L}} M \to Q_{\mathbf{H}}M \to Q_{\mathbf{H}}(M/\sigma_{\mathcal{L}} M) \to \left(R^1Q_{\mathbf{H}}\right)\sigma_{\mathcal{L}} M \to \cdots.$$

Since our assumptions imply that $Q_{\mathbf{H}}\sigma_{\mathcal{L}} M = 0$, it suffices to show that $(R^1Q_{\mathbf{H}})Q_{\mathcal{L}} M = 0$. However, every minimal injective resolution of $\sigma_{\mathcal{L}} M$ is easily seen to consist of L-torsion left R-modules, since $\sigma_{\mathsf{L}} M$ is L-torsion and L is stable. ∎

The next result shows that "intersections" of stable covers again yield stable covers:

(3.15) PROPOSITION. *Assume that the stable covers* $\mathfrak{U} = \{[\mathbf{L}_\alpha]\}_\alpha$ *and* $\mathfrak{V} = \{[\mathbf{K}_\beta]\}_\beta$ *belong to* $\mathrm{Cov}_\beta[\mathbf{H}]$. *Then so does*

$$\mathfrak{U} \wedge \mathfrak{V} =: \left\{\left[\mathbf{L}_\alpha \mathbf{K}_\beta\right]\right\}_{\alpha, \beta}.$$

Proof. By assumption, both $\mathsf{L} = \bigcap_\alpha \varepsilon[\mathbf{L}_\alpha]$ and $\mathsf{K} = \bigcap_\beta \varepsilon[\mathbf{K}_\beta]$ are stable Gabriel filters. Let us first note that

$$\bigcap_{\alpha, \beta} \varepsilon\left[\mathbf{L}_\alpha \mathbf{K}_\beta\right] = \bigcap_{\alpha, \beta} \varepsilon\left[\mathbf{L}_\alpha\right] \circ \varepsilon\left[\mathbf{K}_\beta\right] = \mathsf{L} \circ \mathsf{K}.$$

Since the Gabriel filters L and K are stable, they are compatible, so $\mathsf{L} \circ \mathsf{K} = \mathsf{K} \circ \mathsf{L} = \mathsf{L} \vee \mathsf{K}$ is a (stable) Gabriel filter as well.

Next, if $T \in R$-mod is $\mathsf{L} \circ \mathsf{K}$-torsion, then $Q_{\mathbf{H}} T = 0$. Indeed, from the exact sequence

$$0 \to \sigma_{\mathsf{L}} T \to T \to T/\sigma_{\mathsf{L}} T \to 0$$

we derive an exact sequence

$$0 \to Q_{\mathbf{H}} \sigma_{\mathsf{L}} T \to Q_{\mathbf{H}} T \to Q_{\mathbf{H}}(T/\sigma_{\mathsf{L}} T).$$

Our assumptions imply that $Q_{\mathbf{H}} \sigma_{\mathsf{L}} T = 0$. To prove our claim, let us verify that $T/\sigma_{\mathsf{L}} T$ is K-torsion, as this will imply that $Q_{\mathbf{H}}(T/\sigma_{\mathsf{L}} T) = 0$. If $t \in T$, then there exists $L \in \mathsf{L} \circ \mathsf{K}$ with $Lt = 0$. Pick $K \supseteq L$ in K such that K/L is L-torsion. For any $k \in K$, there exists $L_k \in \mathsf{L}$ with $L_k k \subseteq L$, so $L_k kt = 0$, i.e., $kt \in \sigma_{\mathsf{L}} T$. Hence $Kt \subseteq \sigma_{\mathsf{L}} T$, i.e., $K\bar{t} = \bar{0} \in T/\sigma_{\mathsf{L}} T$, so $\bar{t} \in \sigma_{\mathsf{K}}(T/\sigma_{\mathsf{L}} T)$. This finishes the proof. ∎

(3.16) Again in the general case, i.e., not necessarily assuming covers to be stable, define the auto-intersection of a cover $\mathfrak{U} = \{[\mathbf{L}_\alpha];\ 1 \leq \alpha \leq n\}$ of $[\mathbf{H}] \in \mathbb{T}(\mathsf{G})$ as

$$\mathfrak{U} \wedge \mathfrak{U} = \left\{\left[\mathbf{L}_\alpha\right]\left[\mathbf{L}_\beta\right];\ 1 \leq \alpha,\ \beta \leq n\right\}.$$

Of course, even without supposing $\mathfrak{U}$ to be stable, clearly $\mathfrak{U} \cap \mathfrak{U} \in \mathrm{Cov}_{\mathsf{G}}[\mathbf{H}]$, since with $\mathsf{L} = \bigcap_\alpha \varepsilon[\mathbf{L}_\alpha]$, we have

$$\bigcap_{\alpha, \beta} \varepsilon\left[\mathbf{L}_\alpha \mathbf{L}_\beta\right] = \bigcap_{\alpha} \varepsilon\left[\mathbf{L}_\alpha\right] \circ \bigcap_{\beta} \varepsilon\left[\mathbf{L}_\beta\right] = \mathsf{L} \circ \mathsf{L} = \mathsf{L}.$$

The obvious morphisms $[\mathbf{L}_\alpha \mathbf{L}_\beta] \to [\mathbf{L}_\alpha]$ resp. $[\mathbf{L}_\alpha \mathbf{L}_\beta] \to [\mathbf{L}_\beta]$ yield maps $Q_{\mathbf{L}_\alpha} \to Q_{\mathbf{L}_\alpha \mathbf{L}_\beta}$ resp. $Q_{\mathbf{L}_\beta} \to Q_{\mathbf{L}_\alpha \mathbf{L}_\beta}$, which combine into morphisms

$$\oplus_{[\mathbf{L}_\alpha] \in \mathfrak{U}} Q_{\mathbf{L}_\alpha} \rightrightarrows \oplus_{[\mathbf{L}_\beta \mathbf{L}_\gamma] \in \mathfrak{U} \wedge \mathfrak{U}} Q_{\mathbf{L}_\beta \mathbf{L}_\gamma}.$$

Applying (2.9), it follows that we obtain an equalizer diagram

$$0 \to Q_{\mathbf{H}} \to \oplus_{[\mathbf{L}_\alpha] \in \mathfrak{U}} Q_{\mathbf{L}_\alpha \mathbf{H}} \rightrightarrows \oplus_{[\mathbf{L}_\beta \mathbf{L}_\gamma] \in \mathfrak{U} \wedge \mathfrak{U}} Q_{\mathbf{L}_\beta \mathbf{L}_\gamma \mathbf{H}}.$$

(3.17) A presheaf (of abelian groups) P is said to be a *sheaf* on the noncommutative site $(\mathbb{T}(G), \mathrm{Cov}_G)$ if the following conditions are satisfied for every $[\mathbf{H}] \in \mathbb{T}(G)$ and any $\mathfrak{U} = \{[\mathbf{L}_\alpha];\ 1 \leq \alpha \leq n\} \in \mathrm{Cov}_G[\mathbf{H}]$:

(1) if $m_1, m_2 \in P[\mathbf{H}]$ with the property that $m_1 \| [\mathbf{L}_\alpha \mathbf{H}] = m_2 \| [\mathbf{L}_\alpha \mathbf{H}]$ in $P[\mathbf{L}_\alpha \mathbf{H}]$ for all $1 \leq \alpha \leq n$ (with obvious notations), then $m_1 = m_2$;

(2) if $\{m_\alpha \in P[\mathbf{L}_\alpha \mathbf{H}];\ 1 \leq \alpha \leq n\}$ has the property that $m_\alpha \| [\mathbf{L}_\beta \mathbf{L}_\gamma \mathbf{H}] = m_\beta \| [\mathbf{L}_\beta \mathbf{L}_\gamma \mathbf{H}]$ for all $1 \leq \beta, \gamma \leq n$, then there exists a (unique!) $m \in P[\mathbf{H}]$ with $m \| [\mathbf{L}_\alpha \mathbf{H}] = m_\alpha$ for all α.

(Of course, it is clear what should be meant by a sheaf of left R-modules on $\mathbb{T}(G)$.)

Note that this definition fully takes into account the noncommutativity of the "Grothendieck topology" on $\mathbb{T}(G)$, as it asserts that restrictions of sections should coincide on *all* "intersections." Note also that the first of these properties obviously defines what should rightfully be called a *separated* presheaf on $\mathbb{T}(G)$.

The previous results thus prove:

(3.18) THEOREM. *Consider an arbitrary, non-empty set G of Gabriel filters on R and a left R-module M. Then, associating to any $[\mathbf{H}] \in \mathbb{T}(G)$ the left R-module $Q_{\mathbf{H}} M$ defines a sheaf of left R-modules O_M on $(\mathbb{T}(G), \mathrm{Cov}_G)$, such that $O_M[\mathbf{L}]$ is a left $Q_{L_n}R$-module for $[\mathbf{L}] = [L_1 \cdots L_n] \in \mathbb{T}(G)$.*

(3.19) Using the identifications of $\mathbb{T}_{Zar}(R)$ and $\mathbb{T}_{st}(R)$ with $\mathrm{Spec}(R)$ endowed with the Zariski topology resp. the stable topology, it thus follows that the present construction of a structure sheaf associated to arbitrary left R-modules M generalizes both the usual construction in the commutative case and its noncommutative analogue, described and studied in [3].

Let us also point out that, in particular, the space $\mathrm{Spec}(R)$ itself corresponds to the class of the trivial filter $\{R\}$ in $\mathbb{T}(G)$, and as the associated localization functor reduces to the identity, it follows that the global sections of the structure sheaf O_M may be identified with M, for any left R-module M. In this way, the above constructions provide a genuine geometric realization of arbitrary (left noetherian) noncommutative rings.

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