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Noncommutative smoothness and coadjoint orbits

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Dedicated to Claudio Procesi on the occasion of his 60th birthday

Abstract

In [math.AG/0010030; Math. Z., to appear] R. Bocklandt and the author proved that certain quotient varieties of representations of deformed preprojective algebras are coadjoint orbits for the necklace Lie algebra $\mathfrak{N}_Q$ of the corresponding quiver $\vec{Q}$. A conjectural ring-theoretical explanation of these results was given in terms of noncommutative smoothness in the sense of C. Procesi [J. Algebra 107 (1987) 63–74]. In this paper we prove these conjectures. The main tool in the proof is the étale local description due to W. Crawley-Boevey [math.AG/0105247]. Along the way we determine the smooth locus of the Marsden–Weinstein reductions for quiver representations.
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1. Introduction

In [2] Yu. Berest and G. Wilson asked whether the Calogero–Moser phase space is a coadjoint orbit for a central extension of the automorphism group of the Weyl algebra. This is indeed the case as was first proved by V. Ginzburg [8] and subsequently generalized independently by V. Ginzburg [9] and R. Bocklandt and the author [3] to certain quiver-varieties. Both proofs use

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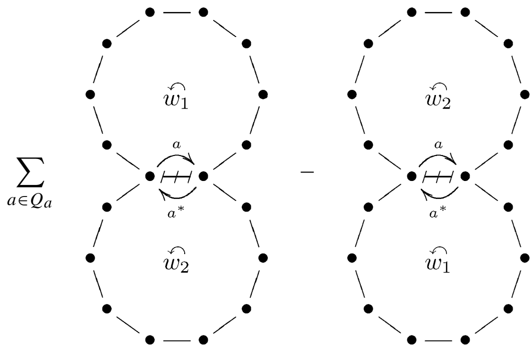


Fig. 1. Lie bracket $[w_1, w_2]$ in $\mathfrak{N}_Q$.

noncommutative symplectic geometry as outlined by M. Kontsevich [10] in an essential way.

Recall that a quiver $\vec{Q}$ is a finite directed graph on a set of vertices $Q_v = \{v_1, \ldots, v_k\}$ and having a finite set of arrows $Q_a = \{a_1, \ldots, a_l\}$ where we allow both multiple arrows between vertices and loops in vertices. The double quiver $\overline{Q}$ of the quiver $\vec{Q}$ is the quiver obtained by adjoining to every arrow $a \in Q_a$ an arrow a^* in the opposite direction. Two oriented cycles in $\overline{Q}$ are equivalent if they are equal up to a cyclic permutation of the arrow components. A *necklace word* w for $\overline{Q}$ is an equivalence class of oriented cycles in $\overline{Q}$. The *necklace Lie algebra* $\mathfrak{N}_Q$ of the quiver $\vec{Q}$ has as basis the set of all necklace words w for $\overline{Q}$ and with Lie bracket $[w_1, w_2]$ determined by Fig. 1. That is, for every arrow $a \in Q_a$ we look for an occurrence of a in w_1 and of a^* in w_2. We then open up the necklaces by removing these factors and regluing the open ends together to form a new necklace word. We repeat this operation for *all* occurrences of a (in w_1) and a^* (in w_2). We then replace the roles of a^* and a and redo this operation with a minus sign. Finally, we add up all these obtained necklace words for all arrows $a \in Q_a$.

The path algebra $\mathbb{C}\overline{Q}$ has as $\mathbb{C}$-basis the set of all oriented paths $p = a_{i_u} \ldots a_{i_1}$ of length $u \geqslant 1$ together with the vertex-idempotents e_i considered as paths of length zero. Multiplication in $\mathbb{C}\overline{Q}$ is induced by concatenation (on the left) of paths. Let $V = \mathbb{C} \times \cdots \times \mathbb{C}$ be the k-dimensional semisimple subalgebra generated by the vertex-idempotents. In [3] the noncommutative *relative* differential forms $\Omega_V^i \mathbb{C}\overline{Q}$ (introduced and studied by J. Cuntz and D. Quillen in [6]) were used to describe the noncommutative relative deRham (or Karoubi) complex

$$\mathsf{dR}_V^0\,\mathbb{C}Q \xrightarrow{d} \mathsf{dR}_V^1\,\mathbb{C}Q \xrightarrow{d} \mathsf{dR}_V^2\,\mathbb{C}Q \xrightarrow{d} \cdots$$

where we define the vector-space quotients dividing out the super-commutators

$$\mathsf{dR}_V^n\,\mathbb{C}Q = \frac{\Omega_V^n \mathbb{C}Q}{\sum_{i=0}^n [\Omega_V^i \mathbb{C}Q, \Omega_V^{n-i} \mathbb{C}Q]}.$$

In particular, the noncommutative functions $\mathsf{dR}_V^0\, \mathbb{C}\overline{Q}$ coincide with $\mathfrak{N}_Q$. A noncommutative symplectic structure is defined on $\mathbb{C}\overline{Q}$ by the element $\omega = \sum_{a \in Q_a} \mathrm{d}a^* \,\mathrm{d}a \in \mathsf{dR}_V^2\, \mathbb{C}\overline{Q}$ and we have a noncommutative version of the classical result in symplectic geometry relating the Lie algebra of functions to Hamiltonian vector-fields: there is a central extension of Lie algebras

$$0 \to V \to \mathfrak{N}_Q \to \mathrm{Der}_\omega \mathbb{C}\overline{Q} \to 0$$

where $\mathrm{Der}_\omega \mathbb{C}\overline{Q}$ is the Lie algebra of *symplectic* derivations, that is, $\theta \in \mathrm{Der}_V \mathbb{C}\overline{Q}$ such that $L_\theta \omega = 0$ where L_θ is the degree preserving derivation on the relative differential forms determined by $L_\theta(a) = \theta(a)$ and $L_\theta(\mathrm{d}a) = \mathrm{d}\theta(a)$, see [3, Theorem 4.2]. As $\mathrm{Der}_\omega \mathbb{C}\overline{Q}$ corresponds to the group of V-automorphisms of $\mathbb{C}\overline{Q}$ preserving the element $m = \sum_{a \in Q_a}[a, a^*]$ it is natural to consider for $\lambda = \sum_{i=1}^k \lambda_i e_i$ with $\lambda_i \in \mathbb{Q}$ the *deformed preprojective algebra*

$$\Pi_\lambda(\overline{Q}) = \frac{\mathbb{C}\overline{Q}}{(m - \lambda)}.$$

For a given dimension vector $\alpha = (a_1, \dots, a_k) \in \mathbb{N}^k$ one defines the affine scheme $\mathtt{rep}_\alpha\, \Pi_\lambda$ of α-dimensional representations of Π_λ. There is a natural action of the base-change group $GL(\alpha) = \prod_{i=1}^k GL_{a_i}$ on this scheme and the corresponding quotient variety $\mathtt{iss}_\alpha\, \Pi_\lambda$ represents the isomorphism classes of semisimple α-dimensional representations of Π_λ. The main coadjoint orbit result of [9] and [3, Theorem 5.5] is the following theorem.

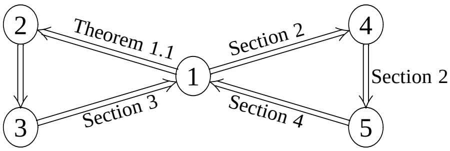
Theorem 1.1. *If α is a minimal element of Σ_λ, the set of dimension vectors of simple representations of Π_λ, then $\mathtt{iss}_\alpha\, \Pi_\lambda$ is a coadjoint orbit for the necklace Lie algebra $\mathfrak{N}_Q$.*

The first description of Σ_λ is due to W. Crawley-Boevey [4]. In [14] the author gave an alternative characterization. In [3] we gave a conjectural ring-theoretical explanation for these coadjoint orbit results in terms of noncommutative notions of smoothness which we will recall in Section 2. The main result of this paper is the following affirmative solution to this conjecture.

Theorem 1.2. *The following are equivalent*:

(1) *α is a minimal element of Σ_λ.*
(2) *$\mathtt{iss}_\alpha\, \Pi_\lambda$ is a coadjoint orbit for $\mathfrak{N}_Q$.*
(3) *$\mathtt{iss}_\alpha\, \Pi_\lambda$ is a smooth variety.*
(4) *$\int_\alpha \Pi_\lambda$ is an Azumaya algebra over the smooth variety $\mathtt{iss}_\alpha\, \Pi_\lambda$.*
(5) *Π_λ is α-smooth in the sense of Procesi* [18].

The outline of this paper, as well as the proof of this result is summarized in the following picture:



2. Noncommutative smoothness

Path algebras of quivers are examples of formally smooth algebras as defined and studied in [6], that is they have the lifting property for algebra morphisms modulo nilpotent ideals. As a consequence they have a good theory of differential forms, see for example [3]. If A is a formally smooth V-algebra, then for each dimension vector α the scheme of α-dimensional representations $\mathtt{rep}_\alpha A$ is smooth and noncommutative (relative) differential forms of A induce ordinary $GL(\alpha)$-invariant differential forms on these manifolds and hence on the corresponding quotient varieties $\mathtt{iss}_\alpha A$.

On the other hand we will see in the next sections that the deformed preprojective algebra Π_λ is *not* formally smooth as many of the representation schemes $\mathtt{rep}_\alpha \Pi_\lambda$ are singular. The quotient $\mathbb{C}\overline{Q} \twoheadrightarrow \Pi_\lambda$ indicates that Π_λ corresponds to a singular noncommutative subscheme of the noncommutative manifold corresponding to the formally smooth algebra $\mathbb{C}\overline{Q}$. As a consequence, the differential forms of $\mathbb{C}\overline{Q}$, when restricted to the singular subvariety Π_λ, may have rather unpredictable behavior.

Still, it may be that the induced $GL(\alpha)$-invariant differential forms on some of the representation schemes $\mathtt{rep}_\alpha \Pi_\lambda$ have desirable properties, in particular if $\mathtt{rep}_\alpha \Pi_\lambda$ is a smooth variety. For this reason we need a notion of noncommutative smoothness relative to a specific dimension vector α. This notion was introduced by C. Procesi in [18] and investigated further in [13]. We briefly recall the definition and main results from [18].

With $\mathtt{alg@}\alpha$ we denote the category of V-algebras C equipped with a trace map $C \xrightarrow{t} C$ satisfying $t(ab) = t(ba)$, $t(a)b = bt(a)$, $t(t(a)b) = t(a)t(b)$ for all $a, b \in C$ and such that $t(e_i) = a_i$ if $\alpha = (a_1, \ldots, a_k)$ and C satisfies the formal Cayley–Hamilton identity of degree n where $n = \sum_i a_i$. To explain the last definition, consider the characteristic polynomial $\chi_M(t)$ of a general $n \times n$ matrix M which is a polynomial in a central variable t with coefficients which can be expressed as polynomials with rational coefficients in $\mathrm{Tr}(M), \mathrm{Tr}(M^2), \ldots, \mathrm{Tr}(M^n)$. Replacing M by a and $\mathrm{Tr}(M^i)$ by $t(a^i)$ we have a formal characteristic polynomial $\chi_a(t) \in C[t]$ and we require that $\chi_a(a) = 0$ for all $a \in C$. Morphisms in $\mathtt{alg@}\alpha$ are V-algebra morphisms which are trace preserving.

An algebra C in $\mathtt{alg}@\alpha$ is said to be α-smooth if it satisfies the lifting property for morphisms modulo nilpotent ideals in $\mathtt{alg}@\alpha$. That is, every diagram

$$\begin{array}{ccc} B & \xrightarrow{\pi} & \frac{B}{I} \\ & \nwarrow^{\exists\tilde{\phi}} & \uparrow{\scriptstyle \phi} \\ & & C \end{array}$$

with $B, B/I$ in $\mathtt{alg}@\alpha$, I a nilpotent ideal and π and ϕ trace preserving maps, can be completed with a trace preserving algebra map $\tilde{\phi}$.

The forgetful functor $\mathtt{alg}@\alpha \to V\text{-}\mathtt{alg}$ has a left inverse which we will denote by $\int_\alpha$. It is defined in the following way. Let A be a V-algebra and consider the algebra

$$\int A = A \otimes_{\mathbb{C}} \mathbb{C}\left[\frac{A}{[A,A]_{\mathrm{vec}}}\right],$$

where the second component is the symmetric algebra of the quotient space where $[A,A]_{\mathrm{vec}}$ is the subspace spanned by all commutators. $\int A$ has a trace map defined by

$$\int A \xrightarrow{t} \int A, \quad t(a \otimes c) = 1 \otimes c.\bar{a},$$

where $\bar{a}$ is the image of a in the quotient space $A/[A,A]_{\mathrm{vec}}$. Using this trace map one can define the formal nth Cayley–Hamilton polynomials where $n = \sum_i a_i$ as defined before. The algebra $\int_\alpha A$ is now the quotient in algebras with trace of $\int A$ modulo the ideal generated by the elements $\chi_a(a)$ for all $a \in \int A$ and the elements $t(e_i) = a_i$. For more details we refer to [16]. From [18] we recall geometric reconstruction results for $\int_\alpha A$ and its central subalgebra $\oint_\alpha A = t \int_\alpha A$ as well as the characterization of α-smoothness.

Theorem 2.1 (C. Procesi). *With notations as above we have*:

(1) *The algebra $\int_\alpha A$ is the ring of $GL(\alpha)$-equivariant maps from $\mathtt{rep}_\alpha A$ to $M_n(\mathbb{C})$ where $GL(\alpha)$ acts on the latter by conjugation via the diagonal embedding $GL(\alpha) \hookrightarrow GL_n$; that is,*

$$\int_\alpha A = M_n\big(\mathbb{C}[\mathtt{rep}_\alpha A]\big)^{GL(\alpha)}.$$

(2) *The image of the trace map on $\int_\alpha A$ is the ring of $GL(\alpha)$-invariant polynomial functions on $\mathtt{rep}_\alpha A$; that is,*

$$\oint_\alpha A = \mathbb{C}[\mathtt{iss}_\alpha A].$$

(3) *A is α-smooth in the sense of Procesi*; *that is, $\int_\alpha A$ is α-smooth in $\mathtt{alg}@\alpha$ if and only if $\mathtt{rep}_\alpha A$ is a smooth variety.*

Recall that an algebra C in $\mathtt{alg}@\alpha$ is said to be an Azumaya algebra if and only if every trace preserving morphism $C \to M_n(\mathbb{C})$ is an epimorphism. If we start with a V-algebra A, then a trace preserving algebra map $\int_\alpha A \to M_n(\mathbb{C})$ corresponds one-to-one to a V-algebra map $A \to M_n(\mathbb{C})$ hence to a geometric point of $\mathtt{rep}_\alpha A$. The Azumaya property for $\int_\alpha A$ is therefore equivalent to saying that the quotient map

$$\mathtt{rep}_\alpha A \xrightarrow{\pi} \mathtt{iss}_\alpha A$$

is a principal $PGL(\alpha)$-fibration in the étale topology. For, in general a geometric point in $\mathtt{iss}_\alpha A$ determines an isomorphism class of a semi-simple α-dimensional representation of A and the map π sends an α-dimensional representation to the direct sum of its Jordan–Hölder factors.

Proposition 2.2. *The following implications of Theorem* 1.1 *hold*:

(1) ⇒ (4): *If α is a minimal element of Σ_λ, then $\int_\alpha \Pi_\lambda$ is an Azumaya algebra over the smooth variety $\mathtt{iss}_\alpha \Pi_\lambda$.*

(4) ⇒ (5): *If $\int_\alpha \Pi_\lambda$ is an Azumaya algebra over the smooth variety $\mathtt{iss}_\alpha \Pi_\lambda$, then Π_λ is α-smooth in the sense of Procesi.*

Proof. (1) ⇒ (4). Consider the *complex moment map*

$$\mathtt{rep}_\alpha \overline{Q} \xrightarrow{\mu_{\mathbb{C}}} M^0_\alpha(\mathbb{C}), \quad V \mapsto \sum_{a \in Q_a} [V_a, V_{a^*}],$$

where $M^0_\alpha(\mathbb{C})$ is the subspace of k-tuples $(m_1, \dots, m_k) \in M_{a_1}(\mathbb{C}) \oplus \cdots \oplus M_{a_k}(\mathbb{C})$ such that $\sum_i \operatorname{tr}(m_i) = 0$. For $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Q}^k$ such that $\sum_i a_i\lambda_i = 0$ we consider the element $\underline{\lambda} = (\lambda_1 \mathbb{1}_{n_1}, \dots, \lambda_k \mathbb{1}_{n_k})$ in $M^0_\alpha(\mathbb{C})$. The inverse image $\mu_{\mathbb{C}}^{-1}(\underline{\lambda}) = \mathtt{rep}_\alpha \Pi_\lambda$. By a result of M. Artin [1] one knows that the geometric points of the quotient scheme $\mathtt{iss}_\alpha \Pi_\lambda$ are the isomorphism classes of α-dimensional semisimple representations of Π_λ. Because α is a minimal element of Σ_λ all α-dimensional representations of Π_λ must be simple (consider the dimension vectors of Jordan–Hölder components) so each fiber of the quotient map π is isomorphic to $PGL(\alpha)$. The fact that $\mu_{\mathbb{C}}^{-1}(\underline{\lambda})$ is smooth if α is a minimal non-zero element of Σ_λ follows from computing the differential of the complex moment map, see also [4, Lemma 5.5]. Because $\mathtt{rep}_\alpha \Pi_\lambda \xrightarrow{\pi} \mathtt{iss}_\alpha \Pi_\lambda$ is a principal $PGL(\alpha)$-fibration, $\int_\alpha A$ is an Azumaya algebra and as the total space $\mathtt{rep}_\alpha \Pi_\lambda$ is smooth it follows that also the base space $\mathtt{iss}_\alpha \Pi_\lambda$ is smooth.

(4) ⇒ (5). If $\int_\alpha \Pi_\lambda$ is an Azumaya algebra, it follows that

$$\mathtt{rep}_\alpha \Pi_\lambda \xrightarrow{\pi} \mathtt{iss}_\alpha \Pi_\lambda$$

is a principal $PGL(\alpha)$-fibration. If in addition the base-space is smooth, so is the top space $\mathtt{rep}_\alpha\, \Pi_\lambda$. The assertion follows from Procesi's characterization of α-smoothness, Theorem 2.1. □

3. Central singularities

Clearly, if $\mathtt{iss}_\alpha\, \Pi_\lambda$ is a coadjoint orbit of $\mathfrak{N}_Q$ it is a smooth variety. In this section we will show that unless α is a minimal element of Σ_λ the quotient variety $\mathtt{iss}_\alpha\, \Pi_\lambda$ always has singularities. The crucial ingredient in the proof is the étale local description of $\mathtt{iss}_\alpha\, \Pi_\lambda$ due to W. Crawley-Boevey [5].

Let χ_Q be the Euler-form of the quiver $\vec{Q}$, that is, the bilinear form

$$\chi_Q : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

is determined by the matrix $(\chi_{ij})_{i,j} \in M_k(\mathbb{Z})$ where $\chi_{ij} = \delta_{ij} - \#\{a \in Q_a$ starting at v_i and ending in $v_j\}$. We denote the symmetrization of χ_Q by T_Q (the Tits form) and $p(\alpha) = 1 - \chi_Q(\alpha, \alpha)$ for every dimension vector α.

Throughout we assume that $\alpha \in \Sigma_\lambda$ and we consider a geometric point $\xi \in \mathtt{iss}_\alpha\, \Pi_\lambda$ which determines an isomorphism class of a semisimple α-dimensional representation of Π_λ, say

$$M_\xi = S_1^{\oplus e_1} \oplus \cdots \oplus S_u^{\oplus e_u},$$

where the S_i are distinct simple representations of Π_λ with dimension vectors β_i and occurring in M_ξ with multiplicity e_i, that is, $\alpha = \sum_{i=1}^u e_i \beta_i$. We say that ξ is of representation type $\tau = (e_1, \beta_1; \ldots; e_u, \beta_u)$.

Construct a new (symmetric) quiver Γ_τ on u vertices $\{v'_1, \ldots, v'_u\}$ (corresponding to the distinct simple components) such that there are

- $2p(\beta_i)$ loops in vertex v'_i, and
- $-T_Q(\beta_i, \beta_j)$ directed arrows from v'_i to v'_j.

We also consider the dimension vector $\alpha_\tau = (e_1, \ldots, e_u)$ for Γ_τ.

Theorem 3.1 (W. Crawley-Boevey). *With notations as above there is an étale isomorphism between*

(1) *a neighborhood of ξ in $\mathtt{iss}_\alpha\, \Pi_\lambda(\overline{Q})$, and*
(2) *a neighborhood of the trivial representation $\bar{0}$ in $\mathtt{iss}_{\alpha_\tau}\, \Pi_0(\Gamma_\tau)$,*

where $\Pi_0(\Gamma_\tau)$ is the preprojective algebra corresponding to the double quiver Γ_τ.

In particular, it follows that $\mathtt{iss}_\alpha\, \Pi_\lambda$ is smooth in all points ξ of representation type $\tau = (1, \alpha)$ (the so called *Azumaya locus*) and that the dimension of $\mathtt{iss}_\alpha\, \Pi_\lambda$

is equal to $2p(\alpha)$. In [5, Proposition 8.6] it was proved that $\mathtt{iss}_\alpha\,\Pi_0(\overline{Q})$ has singularities in case $\vec{Q}$ is a quiver without loops and α is an imaginary indivisible root of Σ_0. Recall that a dimension vector is said to be indivisible if the greatest common divisor of its components is one.

Theorem 3.2. *For $\alpha \in \Sigma_\lambda$, the smooth locus of $\mathtt{iss}_\alpha\,\Pi_\lambda$ coincides with the Azumaya locus. In particular, if α is not a minimal element of Σ_λ, then $\mathtt{iss}_\alpha\,\Pi_\lambda$ is singular; that is, implication* (3) $\Rightarrow$ (1) *of Theorem* 1.2 *holds.*

Proof. With $\mathtt{iss}_\alpha(\tau)$ we will denote the locally closed subvariety of $\mathtt{iss}_\alpha\,\Pi_\lambda$ consisting of all geometric points ξ of representation type τ. Observe that there is a natural ordering on the set of representation types

$$\tau \leqslant \tau' \quad \Longleftrightarrow \quad \mathtt{iss}_\alpha(\tau) \subset \overline{\mathtt{iss}_\alpha(\tau')}$$

where the closure is with respect to the Zariski topology. Clearly, if we can prove that all points of $\mathtt{iss}_\alpha(\tau')$ are singular then so are those of $\mathtt{iss}_\alpha(\tau)$.

Let ξ be a point outside of the Azumaya locus of representation type $\tau = (e_1, \beta_1; \ldots; e_u, \beta_u)$ then by Theorem 3.1 is suffices to prove that $\mathtt{iss}_{\alpha_\tau}\,\Pi_0(\Gamma_\tau)$ is singular in $\bar{0}$.

Assume that Γ_τ has $2p(\beta_i) > 0$ loops in the vertex v_i' where $e_i > 1$. This means that there are infinitely many nonisomorphic simple Π_λ-representations of dimension vector β_i, but then $\tau < \tau'$ where

$$\tau' = \bigl(e_1, \beta_1; \ldots; e_{i-1}, \beta_{i-1}; \underbrace{1, \beta_i; \ldots; 1, \beta_i}_{e_i}; e_{i+1}, \beta_{i+1}; \ldots; e_u, \beta_u\bigr)$$

and by the above remark it suffices to prove singularity for τ'. That is, we may assume that the quiver setting $(\Gamma_\tau, \alpha_\tau)$ is such that the symmetric quiver Γ_τ has loops only at vertices v_i' where the dimension $e_i = 1$.

Assume moreover that α_τ is indivisible (which by the above can be arranged once we start from a type τ such that Γ_τ has loops). Recall from [15] that invariants of quivers are generated by traces along oriented cycles in the quiver. As a consequence we have algebra generators of the coordinate ring $\mathbb{C}[\mathtt{iss}_{\alpha_\tau}\,\Pi_0(\Gamma_\tau)]$ which is a graded algebra by homogeneity of the defining relations of the preprojective algebra $\Pi_0(\Gamma_\tau)$. To prove singularity it therefore suffices to prove that the coordinate ring is not a polynomial ring. Let Γ_τ' be the quiver obtained from Γ_τ by removing all loops (which by the above reduction exist only at vertices where the dimension is one). Because the relations of the preprojective algebra are irrelevant for loops in such vertices we have that $\mathbb{C}[\mathtt{iss}_{\alpha_\tau}\,\Pi_0(\Gamma_\tau)]$ is a polynomial ring (in the variables corresponding to the loops) over $\mathbb{C}[\mathtt{iss}_{\alpha_\tau}\,\Pi_0(\Gamma_\tau')]$. By [5, Proposition 8.6] we know that $\mathtt{iss}_{\alpha_\tau}\,\Pi_0(\Gamma_\tau')$ is singular, finishing the proof in this case.

The remaining case is when Γ_τ contains no loops (that is, all β_i are real roots for $\vec{Q}$) and when α_τ is divisible. Because α_τ is the dimension vector of a simple representation of $\Pi_0(\Gamma_\tau)$ we know from [14] that the quiver setting $(\Gamma_\tau, \alpha_\tau)$ is such that Γ_τ contains a subquiver say on the vertices $T = \{v'_{i_1}, \ldots, v'_{i_z}\}$ which is the double of a tame quiver such that $(\alpha_\tau \mid T) \geqslant \delta$ where $\delta = (d_{i_1}, \ldots, d_{i_z})$ is the imaginary root of this tame subquiver. Consider the representation type of α_τ for $\Pi_0(\Gamma_\tau)$,

$$\gamma = (1, \delta; e_1, \varepsilon_1; \ldots; e_{i_1} - d_{i_1}, \varepsilon_{i_1}, \ldots; e_{i_x} - d_{i_x}, \varepsilon_{i_x}; \ldots; e_u, \varepsilon_u).$$

If we can show that a points is the γ-stratum of $\mathtt{iss}_{\alpha_\tau}\, \Pi_0(\Gamma_\tau)$ is singular, then the quotient scheme is singular in the trivial representation and we are done. Consider the quiver Γ_γ; then it has loops in the vertex corresponding to $(1, \delta)$. Moreover, α_γ is indivisible so we can repeat the argument above. The fact that α_τ was assumed to be divisible asserts that γ is not the Azumaya type $(1, \alpha_\tau)$, finishing the proof. □

4. Noncommutative singularities

We can refine the notion of α-smoothness to allow for noncommutative singularities with respect to the dimension vector α. Let A be a V-algebra and consider the quotient map

$$\mathtt{rep}_\alpha A \xrightarrow{\pi} \mathtt{iss}_\alpha A$$

and consider the open subvariety $\mathtt{sm}_\alpha A$ of $\mathtt{iss}_\alpha A$ consisting of those geometric points ξ such that $\mathtt{rep}_\alpha A$ is smooth along the fiber $\pi^{-1}(\xi)$ and call $\mathtt{sm}_\alpha A$ the α-smooth locus of A. In particular, A is α-smooth in the sense of Procesi if and only if $\mathtt{sm}_\alpha A = \mathtt{iss}_\alpha A$.

Returning to Π_λ it is clear from the foregoing that the Azumaya locus is contained in the α-smooth locus for $\alpha \in \Sigma_\lambda$. We will show in this section that this these loci are actually identical showing that deformed preprojective algebras are as singular as possible. This result should be compared to a similar result on quantum groups at roots of unity [12].

Using the notations needed in Theorem 3.1 we observe that the method of proof actually proves a stronger result which is a symplectic version of Luna slices [17], see also [7, Section 41].

Theorem 4.1. *There is a $GL(\alpha)$-equivariant étale isomorphism between*

(1) *a neighborhood of the orbit of M_ξ in $\mathtt{rep}_\alpha \Pi_\lambda$, and*
(2) *a neighborhood of the orbit of $\overline{(\mathbb{1}_\alpha, 0)}$ in the principal fiber bundle*

$$GL(\alpha) \times^{GL(\alpha_\tau)} \mathtt{rep}_{\alpha_\tau} \Pi_0(\Gamma_\tau).$$

We are now in a position to prove the final implication of Theorem 1.2.

Theorem 4.2. *If $\alpha \in \Sigma_\lambda$, then the α-smooth locus of Π_λ coincides with the Azumaya locus. In particular, if Π_λ is α-smooth in the sense of Procesi, then there is only one representation type $(1, \alpha)$, that is, α is a minimal element of Σ_λ. Hence,* (5) $\Rightarrow$ (1) *of Theorem* 1.2 *holds.*

Proof. Assume that $\xi \in \mathtt{sm}_\alpha\, \Pi_\lambda$ and of representation type $\tau = (e_1, \beta_1; \ldots; e_u, \beta_u)$. Then, $\mathtt{rep}_\alpha\, \Pi_\lambda$ is smooth in a neighborhood of the *closed* $GL(\alpha)$-orbit of the semisimple representation M_ξ (closedness follows from [1]). By a result of Voigt [11] we know that the normal-space N_ξ to the orbit in $\mathtt{rep}_\alpha\, \Pi_\lambda$ is equal to the space of self-extensions

$$\mathrm{Ext}^1_{\Pi_\lambda}(M_\xi, M_\xi) = \bigoplus_{i,j=1}^{u} \mathrm{Ext}^1_{\Pi_\lambda}(S_i, S_j)^{\oplus e_i e_j}.$$

As a consequence we can identify this space with the representation space $\mathtt{rep}_{\alpha_\tau}\, \Delta_\tau$ where Δ_τ is the quiver on u vertices $\{w'_1, \ldots, w'_u\}$ having

- $\dim \mathrm{Ext}^1_{\Pi_\lambda}(S_i, S_i)$ loops in vertex w'_i, and
- $\dim \mathrm{Ext}^1_{\Pi_\lambda}(S_i, S_j)$ directed arrows from w'_i to w'_j.

Moreover, the action of the stabilizer subgroup of M_ξ (which is $GL(\alpha_\tau)$) on the normal; space is the base-change action of this group on $\mathtt{rep}_{\alpha_\tau}\, \Delta_\tau$. By the Luna slice theorem [17] we have a $GL(\alpha)$-equivariant étale isomorphism between

(1) a neighborhood of the orbit of M_ξ in $\mathtt{rep}_\alpha\, \Pi_\lambda$, and
(2) a neighborhood of the orbit of $\overline{(\mathbb{1}_\alpha, 0)}$ in the principal fiber bundle

$$GL(\alpha) \times^{GL(\alpha_\tau)} \mathtt{rep}_{\alpha_\tau}\, \Delta_\tau.$$

Combining this étale description with the one from Theorem 4.1 we deduce an étale $GL(\alpha_\tau)$-isomorphism between the representation scheme $\mathtt{rep}_{\alpha_\tau}\, \Pi_0(\Gamma_\tau)$ (in a neighborhood of the trivial representation) and the representation space $\mathtt{rep}_{\alpha_\tau}\, \Delta_\tau$ (in a neighborhood of the trivial representation).

But then $\overline{0} \in \mathtt{sm}_{\alpha_\tau}\, \Pi_0(\Gamma_\tau)$ and in [3, Theorem 6.3] it was shown that for a preprojective algebra the smooth locus coincides with the Azumaya algebra. The only way the trivial representation can be a simple representation of $\Pi_0(\Gamma_\tau)$ (or indeed, even of $\mathbb{C}\Gamma_\tau$) is when Γ_τ has only one vertex and the dimension vector is $\alpha_\tau = 1$. But then the representation type of ξ is $(1, \alpha)$ finishing the proof. □

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