Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 294 (2005) 307–320
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Generalized twisted group rings

E. Nauwelaerts [a], F. Van Oystaeyen [b,*]

[a] *Limburgs Universitair Centrum, 3590 Diepenbeek, Belgium*
[b] *University of Antwerp UA, 2020 Antwerpen, Belgium*

Received 5 August 2003

Available online 6 October 2005

Communicated by Corrado de Concini

Abstract

Let R be a domain and G a group. Let $\alpha : G \times G \to R \setminus \{0\}$ be a generalized 2-cocycle, i.e., not necessarily taking its values in the units of R, and consider the generalized twisted group ring $R *_\alpha G$. First we investigate the graded structure of $R *_\alpha G$, in particular we give conditions under which $R *_\alpha G$ is gr-hereditary, respectively a gr-maximal order. Next, we derive criteria for $R *_\alpha G$ to be a tame order, respectively a maximal order over some central subring. We also derive conditions under which $R *_\alpha G$ is an Azumaya algebra.

Introduction

Let R be a domain and G a group. Let $\alpha : G \times G \to R \setminus \{0\}$ be a generalized 2-cocycle, i.e., not necessarily taking its values in the units of R, and consider the generalized twisted group ring $R *_\alpha G$. In case R is a Dedekind domain, we may associate with α cocycles $m_p : G \times G \to \mathbb{Z}$ by taking exponents in the factorization of $R\alpha(x, y)$ into prime ideals p of R. In each cohomology class of finite order of $H^2(G, \mathbb{Z})$, there is a cocycle m such that $m(x, y) \in \{0, 1\}$ for all $x, y \in G$, see Proposition 1.3.

* Corresponding author.
E-mail address: francine.schoeters@ua.ac.be (F. Van Oystaeyen).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.11.017

In Section 2, we investigate the graded structure of $R *_\alpha G$. In particular, if $m_p(x, y) \in \{0, 1\}$ (and some additional condition is satisfied), then $R *_\alpha G$ is a gr-maximal order and it is gr-hereditary, see Theorem 2.5 and Proposition 2.12. In Section 3, we consider a finitely generated group G containing a central subgroup of finite index. In this case, we give conditions under which $R *_\alpha G$ is a tame order, respectively a maximal order over some central subring (Theorems 3.1 and 3.2). To conclude, we derive criteria for $R *_\alpha G$ to be separable over its center, see Section 4.

For finite groups similar results are known, see [6] and [7].

1. Generalized 2-cocycles

Let R be a domain with quotient field K and let G be a group. Then a *generalized 2-cocycle* is a map $\alpha : G \times G \to R \setminus \{0\}$ such that $\alpha(e, e) = 1$ and $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$ (e being the neutral element). To α we associate the R-algebra $R *_\alpha G$; as R-module $R *_\alpha G$ is freely generated by symbols $\{u_x \mid x \in G\}$ and multiplication is defined by $(au_x)(bu_y) = \alpha(x, y)abu_{xy}$ for $a, b \in R$, $x, y \in G$. Observe that u_e is the unit element of $R *_\alpha G$.

We set $H = \{x \in G \mid \alpha(x, x^{-1})$ is invertible in $R\}$. Observe that $\alpha(x, x^{-1}) = \alpha(x^{-1}, x)$ and that $x \in H$ if and only if u_x is invertible in $R *_\alpha G$.

Lemma 1.1. *For any $x \in G$, $h \in H$, $\alpha(x, h)$ and $\alpha(h, x)$ are invertible in R. Furthermore, H is a subgroup of G. Moreover, $R\alpha(hx, yh') = R\alpha(x, y)$ for all $x, y \in G$ and $h, h' \in H$.*

Proof. We have $\alpha(x, h)\alpha(xh, h^{-1}) = \alpha(h, h^{-1})$, hence $\alpha(x, h)$ is invertible in R for any $h \in H$, and similarly for $\alpha(h, x)$.

Further, $\alpha(x, y)\alpha(xy, y^{-1}x^{-1}) = \alpha(x, x^{-1})\alpha(y, y^{-1}x^{-1})$. So if $x, y \in H$, then $\alpha(xy, y^{-1}x^{-1})$ is invertible in R.

To prove the last assertion, consider

$$\begin{aligned} \alpha(h, x)\alpha\big(hx, yh'\big)\alpha\big(y, h'\big) &= \alpha\big(h, xyh'\big)\alpha\big(x, yh'\big)\alpha\big(y, h'\big) \\ &= \alpha\big(h, xyh'\big)\alpha(x, y)\alpha\big(xy, h'\big). \end{aligned}$$

Using the first assertion, we deduce that $R\alpha(hx, yh') = R\alpha(x, y)$ for $h, h' \in H$ and $x, y \in G$. □

Note 1.2. (1) Let R be a Dedekind domain and let G and α be as above. In case $R\alpha(x, y) \neq R$, $R\alpha(x, y)$ is uniquely expressible as a product of powers of distinct prime ideals p of R, say $R\alpha(x, y) = \prod p^{k_p(x,y)}$ with $k_p(x, y) \in \mathbb{N}$. If a nonzero prime ideal p does not occur in this factorization, then we set $k_p(x, y) = 0$. Also, if $R\alpha(x, y) = R$, set $k_p(x, y) = 0$ for all prime ideals p.

Clearly, for each nonzero prime ideal p, $[k_p] \in H^2(G, \mathbb{Z})$ with G acting trivially on $\mathbb{Z}$ and $k_p(e, e) = 0$.

Let H be as above and suppose that $[G : H] < \infty$. Let $x_1, \ldots, x_r$ respectively $y_1, \ldots, y_r$ be a set of right respectively left coset representatives of H in G. Then for $x, y \in G$ there are $i, j \in \{1, \dots, r\}$ such that $R\alpha(x, y) = R\alpha(x_i, y_j)$, by the last assertion in Lemma 1.1. In this case, there are only finitely many prime ideals of R appearing in the different $R\alpha(x, y)$.

(2) In part (1), let R be a principal ideal domain and let $p = (\pi_p)$. Then $\alpha(x, y) = \gamma(x, y)\prod \pi_p^{k_p(x,y)}$ with $k_p(x, y)$ as above and $\gamma(x, y) \in U(R)$, the group of units of R. Clearly, $\gamma : G \times G \to U(R)$ is a 2-cocycle and $\gamma(e, e) = 1$.

Conversely, given a cocycle $\gamma : G \times G \to U(R)$, a finite number of prime ideals of R and a finite number of cocycles $G \times G \to \mathbb{Z}$ (with values in $\mathbb{N}$), we may construct generalized 2-cocycles.

(3) Let $k : G \times G \to \mathbb{Z}$ be a 2-cocycle having values in $\mathbb{N}$ and set $H_0 = \{x \in G \mid k(x, x^{-1}) = 0\}$. Then $k(x, h) = 0$ and $k(h, x) = 0$ for $x \in G$, $h \in H_0$, and H_0 is a subgroup of G. Moreover, $k(hx, yh') = k(x, y)$ for all $x, y \in G$ and $h, h' \in H_0$. The proof is analogous to the proof of Lemma 1.1.

The following result will be useful later on.

Proposition 1.3. *Let G be a group, let $k : G \times G \to \mathbb{Z}$ be a 2-cocycle with $k(e, e) = 0$ and suppose that $[k] \in H^2(G, \mathbb{Z})$ has finite order.*

Then there is a 2-cocycle $m \in [k]$ such that $m(x, y) \in \{0, 1\}$ for all $x, y \in G$. Moreover, m is symmetric and $m(e, e) = 0$.

Furthermore, set $H_0 = \{x \in G \mid m(x, x^{-1}) = 0\}$, then $[G : H_0] < \infty$ and $G' \subset H_0$.

Proof. (1) Take $\widetilde{G} = \mathbb{Z} \times G$ with multiplication defined by $(a, x)(b, y) = (a + b + k(x, y), xy)$ for all $a, b \in \mathbb{Z}$, $x, y \in G$, and consider the associated exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{i} \widetilde{G} \xrightarrow{\pi} G \to 0.$$

Choose $n \in \mathbb{N}$ such that nk is equivalent to 0. So there is a map $\gamma : G \to \mathbb{Z}$ such that $nk(x, y) = \gamma(x) + \gamma(y) - \gamma(xy)$ for all $x, y \in G$. Observe that $\gamma(e) = 0$. Now define $\phi : \widetilde{G} \to n^{-1}\mathbb{Z}$ by $\phi(a, x) = a + n^{-1}\gamma(x)$. It is clear that ϕ is a homomorphism of groups.

Given $x \in G$, choose $a \in \mathbb{Z}$ such that $0 \leqslant a + n^{-1}\gamma(x) < 1$. Further, let $s : G \to \widetilde{G}$ be the section of π given by $s(x) = (a, x)$, with the element a defined as above. Note that $s(e) = (0, e)$.

We define $m : G \times G \to \mathbb{Z}$ by $m(x, y) = \phi(s(x)) + \phi(s(y)) - \phi(s(xy))$. Clearly, m is a 2-cocycle equivalent to k, $m(e, e) = 0$ and $m(x, y) \in \{0, 1\}$.

(2) Let $\widetilde{H}$ be the kernel of ϕ. We need the following property: $\phi(s(x)) = \phi(s(\pi(t)x))$ for $t \in \widetilde{H}$, $x \in G$.

To prove this equality, observe that $ts(x)(s(\pi(t)x))^{-1} \in \ker \pi$. As a consequence, $\phi(ts(x)) - \phi(s(\pi(t)x)) \in \mathbb{Z}$ and $\phi(ts(x)) = \phi(s(x))$. But then $\phi(s(x)) - \phi(s(\pi(t)x)) = 0$ follows.

(3) We now show that $\pi(\widetilde{H}) = H_0$. It is easily seen that $x \in H_0$ implies $\phi(s(x)) = 0$, whence $s(x) \in \widetilde{H}$ and thus $x \in \pi(\widetilde{H})$. Conversely, let $x \in \pi(\widetilde{H})$. From (2) it then follows that $\phi(s(x)) = \phi(s(e)) = 0$. Note that $x^{-1} \in \pi(\widetilde{H})$, whence also $\phi(s(x^{-1})) = 0$. So we may conclude that $x \in H_0$.

Further, it is clear that $\widetilde{H}$ is a normal subgroup of $\widetilde{G}$ containing the commutator group $\widetilde{G}'$. Thus $H_0 = \pi(\widetilde{H})$ is a normal subgroup of G containing G'. Next, we show that $[G : H_0] < \infty$. Clearly $A = \{\phi(s(x)) | x \in G\}$ is a finite set, $\#A \leqslant n$. So there exist $g_1, \ldots, g_l \in G$ such that $A = \{\phi(s(g_1)), \ldots, \phi(s(g_l))\}$. Let $x \in G$, then $\phi(s(x)) = \phi(s(g_i))$ for some g_i, $1 \leqslant i \leqslant l$. This gives $s(g_i)^{-1}s(x) \in \widetilde{H}$ and thus $g_i^{-1}x \in \pi(\widetilde{H}) = H_0$.

(4) To conclude, we verify that m is symmetric. Let $x, y \in G$ and put $h = xyx^{-1}y^{-1}$. We know that $h \in H_0 = \pi(\widetilde{H})$ and, using (2), we get $\phi(s(xy)) = \phi(s(hyx)) = \phi(s(yx))$. It follows that m is symmetric. □

Example. In Proposition 1.3, let $G = \mathbb{Z}$ and $k = 0$. Consider $n \in \mathbb{N}$, $t \in \mathbb{N} \setminus \{0\}$ and $\gamma : \mathbb{Z} \to \mathbb{Z}$ with $\gamma(i) = -it$. Then $m : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by

$$m(i, j) = \left[\frac{it}{n}\right] + \left[\frac{jt}{n}\right] - \left[\frac{(i+j)t}{n}\right]$$

is a symmetric 2-cocycle with values in $\{0, 1\}$ ($[q]$ with $q \in \mathbb{Q}$ stands for the smallest integer $\geqslant q$).

As a consequence of Proposition 1.3, we give a result on $H^2(G, \mathbb{Z})_{\mathrm{tors}}$.

Let G be a group. Set $\mathrm{Ext}(G/G', \mathbb{Z}) = \{[f] \in H^2(G/G', \mathbb{Z}) \mid f \text{ is symmetric}\}$. Consider the restriction to $\mathrm{Ext}(G/G', \mathbb{Z})$ of the inflation map; this restriction $\tau : \mathrm{Ext}(G/G', \mathbb{Z}) \to H^2(G, \mathbb{Z})$ sends a class $[f]$ to the cohomology class represented by $\widetilde{f} : G \times G \to \mathbb{Z}$ given by $\widetilde{f}(x, y) = f(\bar{x}, \bar{y})$ ($\bar{x}, \bar{y}$ being images in G/G'). The map τ is an injective homomorphism of groups, see, e.g., [2, Chapter 2, Theorem 2.9].

Corollary 1.4. *Let G be any group, let τ be as above and let $H^2(G, \mathbb{Z})_{\mathrm{tors}}$ denote the torsion subgroup of $H^2(G, \mathbb{Z})$. Then $H^2(G, \mathbb{Z})_{\mathrm{tors}}$ is contained in the image of τ.*

Proof. Let $[k] \in H^2(G, \mathbb{Z})$ have finite order. Then by Proposition 1.3, there is a symmetric 2-cocycle $m \in [k]$ for which $\{x \in G \mid m(x, x^{-1}) = 0\}$ contains G'. Now we can define a 2-cocycle $m' : G/G' \times G/G' \to \mathbb{Z}$ by setting $m'(\bar{x}, \bar{y}) = m(x, y)$ for $x, y \in G$ ($\bar{x}, \bar{y}$ being images in G/G'). Indeed, if $\bar{x} = \bar{z}$ in G/G', then $m(xz^{-1}, z) + m(x, y) = m(xz^{-1}, zy) + m(z, y)$ yields $m(x, y) = m(z, y)$ by Note 1.2(3). Similarly for the second component. It is clear that m' is symmetric and $\tau([m']) = [m] = [k]$. □

2. The graded structure of $R *_\alpha G$

Throughout this section, R is a domain with quotient field K, G a group and $\alpha : G \times G \to R \setminus \{0\}$ a generalized 2-cocycle ($\alpha(e, e) = 1$). Let $\{u_x \mid x \in G\}$ denote a basis of $R *_\alpha G$. We set $H = \{x \in G \mid \alpha(x, x^{-1}) \text{ is invertible in } R\}$ and $S = R *_\alpha H$, $S \subset R *_\alpha G$. Note that S is a classical twisted group ring by Lemma 1.1. We need the following lemmas.

Lemma 2.1. *Suppose $G' \subset H$ and $R\alpha(x, y) = R\alpha(y, x)$ for $x, y \in G$. Then $u_x u_y = s u_y u_x = u_y u_x s'$ for some invertible homogeneous elements $s, s' \in S$.*

Proof. Since G/H is commutative, there is an element $h \in H$ such that $xy = hyx$ and $R\alpha(x, y) = R\alpha(y, x)$ implies that $\alpha(x, y) = r\alpha(y, x)$ for some invertible element $r \in R$. In $K *_\alpha G$ we have:

$$\begin{aligned} u_x u_y &= \alpha(x, y)u_{hyx} = \alpha(x, y)\alpha(h, yx)^{-1}\alpha(y, x)^{-1}u_h u_y u_x \\ &= r\alpha(h, yx)^{-1}u_h u_y u_x \end{aligned}$$

Similar reasoning for the second equality. □

We now focus on the graded ideals of $R *_\alpha G$. First, if L is a nonzero graded left ideal of $R *_\alpha G$, then $L \cap Ru_x \neq 0$ for all $x \in G$. Indeed, suppose $L \cap Ru_x = 0$ for some $x \in G$ and take $ru_g \in L$, $r \in R$. Then $u_{xg^{-1}}ru_g = r\alpha(xg^{-1}, g)u_x \in L \cap Ru_x$, whence $r = 0$, entailing $L = 0$.

Proposition 2.2. *Suppose $G' \subset H$ and $R\alpha(x, y) = R\alpha(y, x)$ for $x, y \in G$.*

(1) *Every graded left ideal of $R *_\alpha G$ is also a right ideal.*
(2) *In addition, suppose that R is a Dedekind domain and G/H is a torsion group. Then a proper graded ideal P of $R *_\alpha G$ which is a gr-prime ideal is gr-maximal.*

Proof. (1) This follows from Lemma 2.1.

(2) Put $A = R *_\alpha G$, let I be a proper graded ideal of A and assume that $P \subset I$. It is easily seen that $P \cap Ru_e$ is a prime ideal of $Ru_e \cong R$, and $P \cap Ru_e \neq 0$. Clearly, $I \cap Ru_e \neq Ru_e$. Consequently, $P \cap Ru_e = I \cap Ru_e$, because R is a Dedekind domain.

Take $ru_x \in I$, $r \in R$. Then $(ru_x)^n \in I \cap Ru_h$ for some $n \in \mathbb{N}$ and $h \in H$, whence $u_{h^{-1}}(ru_x)^n \in I \cap Ru_e$. Now $u_{h^{-1}}$ is invertible in A and, by (1), Aru_x is a graded two-sided ideal of A. We obtain $(Aru_x)^n \subset P$ and thus $ru_x \in P$. So $I = \bigoplus_{x \in G}(I \cap Ru_x) = P$. □

Proposition 2.3. *Assume that $[G : H] < \infty$ and let $e = g_1, \ldots, g_r$ be a set of right coset representatives of H in G.*

(1) *Let L be a graded left ideal of $R *_\alpha G$, then $L = \bigoplus_{i=1}^r L_i u_{g_i}$ where each L_i is a graded left ideal of $S = \bigoplus_{h \in H} Ru_h$.*
(2) *If R is Noetherian, then $R *_\alpha G$ is left gr-Noetherian.*

Proof. (1) Clearly, $L = \bigoplus_{x \in G} I_x u_x$ where each I_x is an ideal of R. Moreover, for $h \in H$ we have $I_{hg_i}u_{hg_i} = I_{hg_i}\alpha(h, g_i)^{-1}u_h u_{g_i} = I_{hg_i}u_h u_{g_i}$.

Put $L_i = \bigoplus_{h \in H} I_{hg_i}u_h$ $(i = 1, \ldots, r)$. Then $L = \bigoplus_{i=1}^r L_i u_{g_i}$ and we only have to show that $u_t L_i \subset L_i$ for $t \in H$.

Clearly, $u_t I_{hg_i}u_{hg_i} \subset I_{thg_i}u_{thg_i}$ and thus $u_t I_{hg_i}u_h u_{g_i} \subset I_{thg_i}u_{th}u_{g_i}$. Since u_{g_i} is invertible in $K *_\alpha G$, we obtain $u_t I_{hg_i}u_h \subset I_{thg_i}u_{th}$, proving the assertion.

(2) Since R is Noetherian and the ring S is strongly graded by H, S is left gr-Noetherian. Let L and L_i be as in (1), then each L_i is a finitely generated S-module, hence L is finitely generated over S and thus also over $R *_\alpha G$. □

Note that cocycles with the above properties exist, see, e.g., Section 1. We now investigate whether $R *_\alpha G$ is a gr-maximal order in $K *_\alpha G$. We need the following lemma.

Lemma 2.4. *Let G be a group. Let $[t]=[m]$ in $H^2(G,\mathbb{Z})$, so there is a map $l: G \to \mathbb{Z}$ such that*

$$t(x,y) = l(x) + l(y) - l(xy) + m(x,y) \quad \text{for } x, y \in G.$$

Suppose $t(e,e) = m(e,e) = 0$, $t(x,y) \in \mathbb{N}$ and $m(x,y) \in \{0,1\}$ for $x, y \in G$. Further, suppose that for each $x \in G$ there is a $k \in \mathbb{N}$ such that $l(x^k) = 0$. Then $l(x) \in \mathbb{N}$ for all $x \in G$.

Proof. Suppose there is a $y \in G$ such that $l(y) < 0$. By the hypothesis, $l(y^k) = 0$ for some $k \in \mathbb{N}$. The following equality is easily verified:

$$\sum_{i=1}^{k-1} t(y, y^i) = kl(y) - l(y^k) + \sum_{i=1}^{k-1} m(y, y^i).$$

It is easily seen that the right side of the equation is strictly negative and the left side is an element of $\mathbb{N}$, yielding a contradiction. □

We say that $R *_\alpha G$ is a *gr-maximal order* in $K *_\alpha G$ if the following holds: if B is a graded subring of $K *_\alpha G$ containing $R *_\alpha G$ and such that $aBb \subset R *_\alpha G$ for some regular homogeneous elements $a, b \in K *_\alpha G$, then $B = R *_\alpha G$.

The above condition $aBb \subset R *_\alpha G$ is equivalent to $rB \subset R *_\alpha G$ for some $r \in R \setminus \{0\}$, as is easily verified.

Theorem 2.5. *Let R be a Dedekind domain. As in Note* 1.2(1), *we associate with α cocycles $m_p : G \times G \to \mathbb{Z}$ (for each prime ideal $p \neq 0$ of R) and we assume that $m_p(x,y) \in \{0,1\}$ for all $x, y \in G$. Further, assume that for each $x \in G$ there is a $k \in \mathbb{N}$ such that $x^k \in H$. Then $R *_\alpha G$ is a gr-maximal order in $K *_\alpha G$.*

Proof. Let B be a graded subring of $K *_\alpha G$ containing $R *_\alpha G$ and such that $rB \subset R *_\alpha G$ for some $r \in R \setminus \{0\}$. Then $B = \bigoplus_{x \in G} I_x u_x$, where each I_x is a nonzero R-submodule of K (using $R *_\alpha G \subset B$). Moreover, $rB \subset R *_\alpha G$ implies $rI_x \subset R$, hence $I_x \subset Rr^{-1}$, and therefore I_x is a finitely generated R-module. So I_x is a fractional R-ideal.

Furthermore, $I_x I_y \alpha(x,y) \subset I_{xy}$ for all $x, y \in G$ and thus $I_{xy}^{-1} I_x I_y \alpha(x,y) \subset R$. Observe that this relation entails $I_e \subset R$, hence $I_e = R$.

If $I_x \neq R$, we may uniquely express I_x as a (finite) product of powers of distinct prime ideals p of R, say $I_x = \prod p^{l_p(x)}$ with $l_p(x) \in \mathbb{Z}$. If a nonzero prime ideal p does not occur, then we put $l_p(x) = 0$. Also, if $I_x = R$, set $l_p(x) = 0$ for all prime ideals p. Note that $l_p(x) \leqslant 0$ for all $x \in G$, because $R \subset I_x$.

Now, for each prime ideal p of R, define $t_p : G \times G \to \mathbb{Z}$ by setting

$$t_p(x,y) = l_p(x) + l_p(y) - l_p(xy) + m_p(x,y).$$

Clearly, t_p is a 2-cocycle equivalent to m_p, $t_p(e,e)=0$ and $t_p(x,y)\in\mathbb{N}$ by the above relation between the fractional R-ideals. Further, let $x\in G$, then there is a $k\in\mathbb{N}$ such that $x^k\in H$, whence $m_p(x^k,x^{-k})=0$ for all p. Consequently, $l_p(x^k)+l_p(x^{-k})\geqslant 0$. But $l_p(y)\leqslant 0$ for all $y\in G$, hence $l_p(x^k)=0$ for all p. From Lemma 2.4 it then follows that $l_p(x)\in\mathbb{N}$ for all $x\in G$. So $l_p(x)=0$ for all $x\in G$ and $B=R*_\alpha G$. □

Remarks 2.6. (1) Let R be a Dedekind domain and associate with α cocycles $m_p: G\times G\to\mathbb{Z}$, having values in $\mathbb{N}$ (for each nonzero prime ideal p of R). Put $H_p=\{x\in G\mid m_p(x,x^{-1})=0\}$. Let $x\in G\setminus H$ and let $p_1,\ldots,p_n$ be the distinct prime ideals occuring in the decomposition of $R\alpha(x,x^{-1})$. If there are $j_i\in\mathbb{N}$ such that $x^{j_i}\in H_{p_i}$, $i=1,\ldots,n$, then $x^j\in H$ with $j=j_1,\ldots,j_n$, because H_{p_i} is a subgroup of G, see 1.2(3).

(2) In Proposition 1.3 we have constructed 2-cocycles $m: G\times G\to\mathbb{Z}$ with $m(x,y)\in\{0,1\}$, $m(e,e)=0$ and such that $\{x\in G\mid m(x,x^{-1})=0\}$ is a subgroup of finite index. Now let R be a principal ideal domain. Then in view of (1) and Note 1.2(2) we can construct rings $R*_\alpha G$ satisfying the hypotheses of Theorem 2.5.

Note 2.7 (*torsion groups*). Let G be a torsion group.

(1) Let $[m]=[m']$ in $H^2(G,\mathbb{Z})$. Suppose $m(x,y)\in\{0,1\}$ and $m'(x,y)\in\{0,1\}$ for all $x,y\in G$, and $m(e,e)=m'(e,e)=0$. Then from Lemma 2.4 it follows that $m=m'$.

(2) Let R be a Dedekind domain, associate with α cocycles $m_p: G\times G\to\mathbb{Z}$ and assume that $m_p(x,y)\in\{0,1\}$ for all $x,y\in G$. Let B be a graded subring of $K*_\alpha G$ containing u_e and such that $rB\subset R*_\alpha G$ for some $r\in R\setminus\{0\}$. Moreover, suppose that B is an R-module and that $KB=K*_\alpha G$. Along the same lines as the proof of Theorem 2.5, we may show that $B\subset R*_\alpha G$.

Next, we show that under suitable hypotheses the graded ideals of $R*_\alpha G$ are invertible. We need the following lemma.

Lemma 2.8. *Let $k: G\times G\to\mathbb{Z}$ be a 2-cocycle with $k(e,e)=0$ and having values in $\mathbb{N}$. Suppose that $[G:H_0]<\infty$, where $H_0=\{x\in G\mid k(x,x^{-1})=0\}$.*

Then $[k]\in H^2(G,\mathbb{Z})$ has finite order. Moreover, if $k(x,y)\in\{0,1\}$ for all $x,y\in G$, then k is symmetric and $G'\subset H_0$.

Proof. Note that H_0 contains a subgroup N such that $N\lhd G$ and $[G:N]<\infty$. We define a 2-cocycle $k': G/N\times G/N\to\mathbb{Z}$ by setting $k'(\bar{x},\bar{y})=k(x,y)$ for $x,y\in G$ ($\bar{x},\bar{y}$ being images in G/N). By Note 1.2(3), k' is well defined. Since G/N is a finite group, $[k']\in H^2(G/N,\mathbb{Z})$ has finite order. Then it is easily seen that $[k]$ has finite order too.

Next, from Proposition 1.3 and Note 2.7(1) we deduce that k' is symmetric and $(G/N)'\subset\{\bar{x}\in G/N\mid k'(\bar{x},\bar{x}^{-1})=0\}=H_0/N$. It follows that k is symmetric and $G'\subset H_0$. □

As an immediate consequence of the above lemma, we obtain:

Corollary 2.9. *Let R be a Dedekind domain. As before, associate with α cocycles $m_p : G \times G \to \mathbb{Z}$ and assume that $m_p(x, y) \in \{0, 1\}$ for all $x, y \in G$. If $[G : H] < \infty$, then $G' \subset H$ and $R\alpha(x, y) = R\alpha(y, x)$ for all $x, y \in G$.*

Now let L be a nonzero graded left and right $R *_\alpha G$-submodule of $K *_\alpha G$ such that $aL \subset R *_\alpha G$ and $Lb \subset R *_\alpha G$ for some regular homogeneous elements $a, b \in K *_\alpha G$. Consequently, $rL \subset R *_\alpha G$ for some $r \in R \setminus \{0\}$. Further, it is easy to see that $L \cap Ru_x \neq 0$ for all $x \in G$. Moreover, if $G' \subset H$ and $R\alpha(x, y) = R\alpha(y, x)$ for $x, y \in G$, then Lemma 2.1 implies that a graded left $R *_\alpha G$-submodule of $K *_\alpha G$ is also a right $R *_\alpha G$-module.

The left order of L in $K *_\alpha G$ is defined as $O_l(L) = \{q \in K *_\alpha G \mid qL \subset L\}$ and the right order of L is defined as $O_r(L) = \{q \in K *_\alpha G \mid Lq \subset L\}$.

Lemma 2.10. *If $R *_\alpha G$ is a gr-maximal order in $K *_\alpha G$, then $O_l(L) = R *_\alpha G = O_r(L)$ with L as above.*

Proof. It is easily verified that $O_l(L)$ and $O_r(L)$ are graded rings containing $R *_\alpha G$. Further, $L \cap Ru_e \neq 0$ and we take $r'u_e \in L \cap Ru_e$, $r' \in R \setminus \{0\}$. Let now $q \in O_l(L)$, then $rqr' \in rL \subset R *_\alpha G$ with $r \in R \setminus \{0\}$. Similarly, $r'O_r(L)r \subset R *_\alpha G$. Since $R *_\alpha G$ is a gr-maximal order, $O_l(L) = R *_\alpha G = O_r(L)$ follows. □

Again, let L be as above. We consider $L^{-1} = \{q \in K *_\alpha G \mid LqL \subset L\}$. If $R *_\alpha G$ is a gr-maximal order, then

$$L^{-1} = \{q \in K *_\alpha G \mid Lq \subset R *_\alpha G\} = \{q \in K *_\alpha G \mid qL \subset R *_\alpha G\}.$$

It is easily seen that L^{-1} is a nonzero graded left and right $R *_\alpha G$-submodule of $K *_\alpha G$. Moreover, $r'L^{-1} \subset R *_\alpha G$ with $r'u_e \in L \cap Ru_e$ $(r' \neq 0)$.

Proposition 2.11. *Let R be a Dedekind domain, associate with α cocycles $m_p : G \times G \to \mathbb{Z}$ (for each nonzero prime ideal p of R) and assume that $m_p(x, y) \in \{0, 1\}$ for all $x, y \in G$ and all p. Further, suppose that $[G : H] < \infty$.*

Let L be a nonzero graded left and right $R *_\alpha G$-submodule of $K *_\alpha G$ such that $rL \subset R *_\alpha G$ for some $r \in R \setminus \{0\}$. Then $LL^{-1} = R *_\alpha G = L^{-1}L$.

Proof. First, note that $G' \subset H$ and $R\alpha(x, y) = R\alpha(y, x)$, see Corollary 2.9. Then one shows the following: if I is a nonzero graded ideal of $R *_\alpha G$ such that $I^{-1} = R *_\alpha G$, then $I = R *_\alpha G$. The proof is a graded version of the one in [8, Lemma 23.4]. This proof relies on Propositions 2.2(2) and 2.3(2) and Theorem 2.5. Next, analogous to [6, Proposition 4.4] one proves that $LL^{-1} = R *_\alpha G = L^{-1}L$. □

Along the same lines as [6, Proposition 4.5] one proves:

Proposition 2.12. *Keep the hypotheses of Proposition* 2.11 *and let L be as in* 2.11. *Then L is a projective left* (*and right*) *$R *_\alpha G$-module.*

In particular, $R *_\alpha G$ is left and right gr-hereditary.

Special case. Let R be a Dedekind domain. Associate with α cocycles $m_p : G \times G \to \mathbb{Z}$ and assume that $m_p(x, y) \in \{0, 1\}$ for all $x, y \in G$. Further, assume that α is symmetric and that $[G : H] < \infty$.

If G is a torsionfree abelian group, then $R *_\alpha G$ is an integrally closed domain. If, in addition, G is finitely generated, then $R *_\alpha G$ is Noetherian.

Indeed, from [3, Proposition II.1.4] it follows that $R *_\alpha G$ is a domain. We know that $R *_\alpha G$ is a gr-maximal order in $K *_\alpha G$, see Theorem 2.5. Then by [5, Theorem 3.3], $R *_\alpha G$ is a maximal order in its quotient field L, i.e., if B is a subring of L containing $R *_\alpha G$ and such that $aB \subset R *_\alpha G$ for some regular element $a \in L$, then $B = R *_\alpha G$. It follows that $R *_\alpha G$ is integrally closed.

Finally, since H is also finitely generated, $S = \bigoplus_{h \in H} Ru_h$ is Noetherian by [3, Theorem II.3.8]. Let $g_1, \ldots, g_r$ be a set of right coset representatives of H in G, then $R *_\alpha G = \bigoplus_i Su_{g_i}$, hence $R *_\alpha G$ is Noetherian.

3. Hereditary and maximal orders

Throughout this section, R is a discrete valuation ring with maximal ideal (π) and quotient field K. Further, G is a finitely generated group containing a central subgroup E of finite index. We consider a generalized 2-cocycle $\alpha : G \times G \to R \setminus \{0\}$ ($\alpha(e, e) = 1$), associate with α a cocycle $m : G \times G \to \mathbb{Z}$ and assume that $m(x, y) \in \{0, 1\}$ for all $x, y \in G$. Also we assume that α is symmetric. Again, set $H = \{x \in G \mid \alpha(x, x^{-1}) \text{ is invertible in } R\}$ and assume that $[G : H] < \infty$. Observe that $G' \subset H$ (see 2.9).

Since G is finitely generated and $[G : E] < \infty$, E is finitely generated too. Let F be the free direct summand of E; then again $[G : F] < \infty$. Now put $N = F \cap H$ and $T = R *_\alpha N$. Clearly, N is a normal central subgroup of G and $[G : N] < \infty$. Further, T is a commutative classical twisted group ring and a Noetherian integrally closed domain, see the example at the end of Section 2. We denote the quotient field of T by L.

Put $A = R *_\alpha G$ and consider a set of right coset representatives of N in G, say $\{e = x_1, x_2, \ldots, x_n\}$. Then $A = \bigoplus_{i=1}^n Tu_{x_i}$ and $tu_{x_i} = u_{x_i}t$ for $t \in T$. Indeed, for $n \in N$ we have: $u_n u_{x_i} = \alpha(n, x_i)u_{nx_i} = \alpha(x_i, n)u_{x_i n} = u_{x_i}u_n$, and $\alpha(n, x_i)$ is invertible in R. Furthermore, $x_i x_j = nx_k$ for some $n \in N$ and

$$u_{x_i}u_{x_j} = \alpha(x_i, x_j)u_{nx_k} = \alpha(x_i, x_j)\alpha(n, x_k)^{-1}u_n u_{x_k}.$$

Now we define $\beta : G/N \times G/N \to T \setminus \{0\}$ as follows: if $\bar{x}, \bar{y} \in G/N$, say $x = n'x_i$, $y = n''x_j$ with $n', n'' \in N$, then set $\beta(\bar{x}, \bar{y}) = \alpha(x_i, x_j)\alpha(n, x_k)^{-1}u_n$, where $x_i x_j = nx_k$ with $n \in N$. The associativity of A implies that β is a generalized 2-cocycle. From the above discussion we deduce that $A \cong T *_\beta G/N$ as rings and G/N is a finite group.

Further, $T\beta(\bar{x}_i, \bar{x}_j) = T\alpha(x_i, x_j)$, because $\alpha(n, x_k)^{-1}u_n$ is invertible in T. Note also that $T\alpha(x_i, x_j) = T\alpha(nx_i, n'x_j)$ for $n, n' \in N$, use Lemma 1.1 and the fact that $N \lhd G$. Moreover, $T\alpha(x, y) = (T\pi)^{m(x,y)}$ for $x, y \in G$ (with $(T\pi)^0 = T$). Since $T\pi \neq T$, $T\pi \cap R = R\pi$. Then it is easily seen that $T\pi$ is a prime ideal of T, using the fact that N is an ordered group, see [3, Proposition II. 1.4].

Theorem 3.1. *Keep the above notation and hypotheses and let p be a height 1 prime ideal of T. If $|G/N|$ is invertible in T_p, i.e., $|G/N| \notin p$, then $A_p \cong T_p *_\beta G/N$ is a left and right hereditary ring.*

Proof. The ring T_p is a discrete valuation ring, because T is a Noetherian integrally closed domain. As $|G/N|$ is invertible in T_p, $L *_\beta G/N$ is a separable L-algebra. We now distinguish two cases: $p \cap R = 0$ and $p \cap R \neq 0$.

(1) If $p \cap R = 0$, then $K \subset T_p$, hence $\alpha(x, y)$ is invertible in T_p for all $x, y \in G$. So $\beta(\bar{x}, \bar{y})$ is invertible in T_p, in other words, $T_p *_\beta G/N$ is a classical twisted group ring. Since $T_p *_\beta G/N$ is gr-hereditary and $|G/N|$ is invertible in T_p, we may conclude that $T_p *_\beta G/N$ is hereditary, see Proposition 2.12 and [4, Theorem 4.10].

(2) If $p \cap R \neq 0$, then $p \cap R = (\pi)$, hence $T\pi \subset p$. But $T\pi$ is a nonzero prime ideal of T and thus $T\pi = p$. Consequently, the maximal ideal of T_p is $T_p\pi$. Further, from the introductory remarks we deduce that $T_p\beta(\bar{x}, \bar{y}) = (T_p\pi)^{m(x,y)}$, and $m(x, y) \in \{0, 1\}$.

Thus we may conclude that $T_p *_\beta G/N$ is gr-hereditary, see Proposition 2.12. From the invertibility of $|G/N|$ in T_p it then follows that $T_p *_\beta G/N$ is hereditary, see [4, Theorem 4.10]. □

In the remainder of this section we focus on maximal orders. Fix notation as follows: p is a height 1 prime ideal of T, $L_p = T_p/pT_p$ and $\overline{H} = H/N$.

We know that $T\beta(\bar{x}, \bar{y}) = T\alpha(x, y)$ for $\bar{x}, \bar{y} \in G/N$. So if $\bar{x}, \bar{y} \in \overline{H}$, then $\beta(\bar{x}, \bar{y})$ is invertible in T_p, whence $\beta(\bar{x}, \bar{y}) \notin pT_p$. Then define $\tilde{\beta} : \overline{H} \times \overline{H} \to L_p \setminus \{0\}$ by setting $\tilde{\beta}(\bar{x}, \bar{y}) = \beta(\bar{x}, \bar{y}) + pT_p$. Clearly, $\tilde{\beta}$ is a 2-cocycle and we consider the classical twisted group ring $L_p *_{\tilde{\beta}} \overline{H}$.

Now we construct an automorphism of $L_p *_{\tilde{\beta}} \overline{H}$. We consider the following surjective ring homomorphism: $\phi : T_p *_\beta \overline{H} \to L_p *_{\tilde{\beta}} \overline{H} : \sum t_{\bar{h}} u_{\bar{h}} \mapsto \sum \tilde{t}_{\bar{h}} u_{\bar{h}}$, where $\bar{h} \in \overline{H}$, $t_{\bar{h}} \in T_p$ and $\tilde{t}_{\bar{h}} = t_{\bar{h}} + pT_p$.

Further, for $\bar{g} \in G/N$, $\bar{h} \in \overline{H}$ we have:

$$u_{\bar{g}} u_{\bar{h}} (u_{\bar{g}})^{-1} = \beta\big(\bar{g}, \bar{h}\big)\beta\big(\bar{g}\bar{h}\bar{g}^{-1}, \bar{g}\big)^{-1} u_{\bar{g}\bar{h}\bar{g}^{-1}}$$

in $L *_\beta G/N$. We deduce that $u_{\bar{g}} u_{\bar{h}} (u_{\bar{g}})^{-1} \in T *_\beta \overline{H}$. Then we may define $\sigma_{\bar{g}} : T_p *_\beta \overline{H} \to T_p *_\beta \overline{H}$ by $\sigma_{\bar{g}}(u_{\bar{h}}) = u_{\bar{g}} u_{\bar{h}} (u_{\bar{g}})^{-1}$, extending T_p-linearly. Obviously, $\sigma_{\bar{g}}$ is an isomorphism of rings.

Finally, define $\tilde{\sigma}_{\bar{g}} : L_p *_{\tilde{\beta}} \overline{H} \to L_p *_{\tilde{\beta}} \overline{H}$ by setting $\tilde{\sigma}_{\bar{g}}(\phi(s)) = \phi(\sigma_{\bar{g}}(s))$ with $s \in T_p *_\beta \overline{H}$. It is easily verified that $\tilde{\sigma}_{\bar{g}}$ is an isomorphism of rings.

Theorem 3.2. *Keep the above notation and hypotheses. Moreover, assume $|G/N| \notin p$ for all height one prime ideals p of T. Then $A \cong T *_\beta G/N$ is a maximal T-order in $L *_\beta G/N$ if and only if for the height one prime ideal p of T, satisfying $p \cap R \neq 0$, the following holds: for each central idempotent η of $L_p *_{\tilde{\beta}} \overline{H}$, $\tilde{\sigma}_{\bar{g}_i}(\eta) = \eta$, where the g_i are right coset representatives of H in G.*

Proof. Note that $T *_\beta G/N$ is a finitely generated free T-module, hence it is a reflexive (divisorial) T-module. Therefore, $T *_\beta G/N$ is a maximal T-order in $L *_\beta G/N$ if and only if $T_p *_\beta G/N$ is a maximal T_p-order in $L *_\beta G/N$ for all height one prime ideals p of T, see [8, Theorem 11.4].

If $p \cap R = 0$, then $K \subset T_p$ and thus $T_p *_\beta G/N$ is a classical twisted group ring. Then the invertibility of G/N in T_p implies that $T_p *_\beta G/N$ is a maximal T_p-order, see [6, Theorem 5.8].

If $p \cap R \neq 0$, then $T_p\pi$ is the maximal ideal of T_p and $T_p\beta(\bar{x}, \bar{y}) = (T_p\pi)^{m(x,y)}$ with $m(x, y) \in \{0, 1\}$ (as in the proof of Theorem 3.1). Also observe that $\{\bar{x} \in G/N \mid \beta(\bar{x}, \bar{x}^{-1})$ invertible in $T_p\} = \{\bar{x} \in G/N \mid m(x, x^{-1}) = 0\} = \overline{H}$. Then according to [6, Theorem 5.8 and Proposition 5.11], $T_p *_\beta G/N$ is a maximal order if and only if the central idempotents of $L_p *_{\tilde{\beta}} \overline{H}$ satisfy the above property. □

Remark. The preceding results hold when R is just a Dedekind domain. The proofs have to be slightly modified; one has to include a local-global approach with respect to localization at height one prime ideals of R.

4. Central separable algebras

First we focus on classical twisted group rings. Let R be a commutative ring, let G be a group and $\alpha : G \times G \to U(R)$ a 2-cocycle such that $\alpha(e, e) = 1$ ($U(R)$ denotes the group of units). Recall that an element $g \in G$ is *α-G-regular* or α-regular if $\alpha(g, x) = \alpha(x, g)$ for all $x \in G$ such that $gx = xg$. It is known that the inverse of an α-regular element is again α-regular. Now let $Z(G)$ denote the center of G and define $Z(G)_{\mathrm{reg}} = \{g \in Z(G) \mid g$ is α-G-regular$\}$. It is easily verified that $Z(G)_{\mathrm{reg}}$ is a group.

Proposition 4.1. *Keep the above notation. If $Z(G)_{\mathrm{reg}}$ has finite index in G and $[G : Z(G)_{\mathrm{reg}}]$ is invertible in R, then $R *_\alpha G$ is an Azumaya algebra, i.e., separable over its center.*

Proof. Set $E = Z(G)_{\mathrm{reg}}$ and $T = R *_\alpha E$; T is a commutative ring. Set $[G : E] = n$ and let $\{e = x_1, x_2, \ldots, x_n\}$ be a set of right coset representatives of E in G. Then $R *_\alpha G = \bigoplus_{i=1}^n T u_{x_i}$ and $t u_{x_i} = u_{x_i} t$ for $t \in T$. Further, $x_i x_j = z x_k$ for some $z \in E$ and

$$u_{x_i} u_{x_j} = \alpha(x_i, x_j) u_{z x_k} = \alpha(x_i, x_j)\alpha(z, x_k)^{-1} u_z u_{x_k}.$$

Now define $\beta : G/E \times G/E \to U(T)$ as follows: if $\bar{x}, \bar{y} \in G/E$, say $x = z' x_i$, $y = z'' x_j$ with $z', z'' \in E$, then set $\beta(\bar{x}, \bar{y}) = \alpha(x_i, x_j)\alpha(z, x_k)^{-1} u_z$, where $x_i x_j = z x_k$ with $z \in E$. Then β is a 2-cocycle and $R *_\alpha G \cong T *_\beta G/E$ as rings. Since n is invertible in R, $T *_\beta G/E$ is separable over T, as is well known. It then follows that $T *_\beta G/E$ is separable over its center. □

Now let R be a domain with quotient field K, let G be a group and consider a generalized 2-cocycle $\alpha : G \times G \to R \setminus \{0\}$ (with $\alpha(e, e) = 1$). Again, put $H = \{x \in G \mid$

$\alpha(x, x^{-1})$ invertible in $R\}$. Suppose $[G : H] < \infty$ and let $\{e = g_1, \ldots, g_r\}$ be a set of right coset representatives of H in G. Further, set $A = R *_\alpha G$, $S = R *_\alpha H$ and let $Z(-)$ stand for the center.

As before, we define α-G-regular elements and we can consider the group $Z(G)_{\text{reg}} = \{g \in Z(G) \mid g \text{ is } \alpha\text{-}G\text{-regular}\}$.

Theorem 4.2. *Keep the above notation, assume that $Z(G)_{\text{reg}}$ has finite index in G and $[G : Z(G)_{\text{reg}}]$ is invertible in R. Further, assume $[G : H] < \infty$, $Z(S) \subset Z(A)$ and $u_{g_i} \in Am$, $i = 2, \ldots, r$, for all maximal ideals m of $Z(A)$, satisfying $m \cap R \neq 0$. Then $A = R *_\alpha G$ is an Azumaya algebra.*

Proof. Set $C = Z(A)$. First we show that A is finitely generated over C. Put $N = H \cap Z(G)_{\text{reg}}$, $T = R *_\alpha N$ and consider a set of right coset representatives of N in G, say $\{e = x_1, \ldots, x_n\}$. Then $A = \bigoplus_{i=1}^n T u_{x_i}$, hence A is finitely generated over T. But then A is finitely generated over C, because $T \subset C$.

Now A is separable over C if and only if A/Am is separable over C/m for all maximal ideals m of C. Note that $Am \neq A$ (indeed, A is finitely generated over C, hence A is integral over C, and thus there is a prime ideal of A lying over m). As $Am \neq A$, $Am \cap C = m$ and $C + Am/Am \cong C/m$.

We distinguish two cases: $m \cap R = 0$ and $m \cap R \neq 0$.

(1) Suppose $m \cap R = 0$. Consider $i : R \to A/Am$ sending $r \in R$ to $r + Am$. It is clear that i is an injective ring homomorphism. Since C/m is a field, $i(r)$ is invertible in A/Am for all $r \neq 0$.

Now consider $\phi : K *_\alpha G \to A/Am : \sum r_x (s_x)^{-1} u_x \mapsto \sum i(r_x)(i(s_x))^{-1} \bar{u}_x$, where $r_x, s_x \in R$, $s_x \neq 0$ and $\bar{u}_x = u_x + Am$. Clearly, ϕ is a surjective ring homomorphism. By Proposition 4.1, the classical twisted group ring $K *_\alpha G$ is separable over its center. Then from [1, §II, Proposition 1.11] we deduce that A/Am is separable over $\phi(Z(K *_\alpha G))$ and the latter is equal to $Z(A/Am)$. Of course, $C + Am/Am \subset Z(A/Am)$. On the other hand, let $s \in R \setminus \{0\}$ and $c \in C$, then $\phi(s^{-1}c) = i(s)^{-1}\phi(c) \in C + Am/Am$. This entails that $\phi(Z(K *_\alpha G)) \subset C + Am/Am$.

(2) Suppose $m \cap R \neq 0$. Consider $\psi : S \to A/Am : \sum r_h u_h \mapsto \sum \bar{r}_h \bar{u}_h$, where $r_h \in R$, $\bar{r}_h = r_h + Am$ and $\bar{u}_h = u_h + Am$. The map ψ is a surjective ring homomorphism, because $A = \bigoplus_{i=1}^r S u_{g_i}$ and $u_{g_i} \in Am$.

Set $Z(H)_{\text{reg}} = \{h \in Z(H) \mid h \text{ is } \alpha\text{-}H\text{-regular}\}$; $H \cap Z(G)_{\text{reg}} \subset Z(H)_{\text{reg}}$. Now $[H : H \cap Z(G)_{\text{reg}}]$ divides $[G : Z(G)_{\text{reg}}]$ (being finite) and $[H : Z(H)_{\text{reg}}]$ divides $[H : H \cap Z(G)_{\text{reg}}]$. Therefore $[H : Z(H)_{\text{reg}}]$ is finite and invertible in R. So in view of Proposition 4.1, S is separable over its center. As a consequence, A/Am is separable over $\psi(Z(S))$ and $\psi(Z(S)) = Z(A/Am)$. Moreover, $Z(S) \subset C$ implies that $\psi(Z(S)) \subset C + Am/Am$. $\square$

Probably some of the conditions in Theorem 4.2 can be improved. In this context, we shall make some observations about the center and about the condition $u_{g_i} \in Am$.

Remarks 4.3. Keep the above notation and suppose R is a discrete valuation ring with maximal ideal (π). Suppose $[G:H]<\infty$, $G'\subset H$ and $R\alpha(x,y)=R\alpha(y,x)$ for all $x,y\in G$. Then we have:

(1) If P is a semiprime ideal of $A=R*_\alpha G$ such that $P\cap R\neq 0$, then $u_{g_i}\in P$, $i=2,\dots,r$. Indeed, $u_h(u_{g_i})^n\in Ru_e$ for some $n\in\mathbb{N}$ and $h\in H$. But u_{g_i} is not invertible in A and thus $u_h(u_{g_i})^n\in\pi Ru_e$. By Proposition 2.2, $Au_{g_i}=u_{g_i}A$, hence $(Au_{g_i})^n\subset\pi A$. Moreover, $P\cap R$ is a nonzero semiprime ideal of R, hence $P\cap R=(\pi)$, and thus $\pi A\subset P$. So $Au_{g_i}\subset P$.

(2) If $A=R*_\alpha G$ is an Azumaya algebra, then $u_{g_i}\in Am$, $i=2,\dots,r$, for all maximal ideals m of $Z(A)$, satisfying $m\cap R\neq 0$. Indeed, since A is an Azumaya algebra, Am is a prime ideal of A and $Am\cap R=m\cap R$.

(3) Suppose $G=HZ(G)_{\mathrm{reg}}$ and $Z(G)_{\mathrm{reg}}$ has finite index in G. So there is a set $\{e=g_1,\dots,g_r\}$ of right coset representatives of H in G with $g_i\in Z(G)_{\mathrm{reg}}$. In this case, $u_{g_i}\in Am$, $i=2,\dots,r$, for all maximal ideals m of $Z(A)$, satisfying $m\cap R\neq 0$. Indeed, since A is integral over its center, there is a prime ideal P of A lying over m. By (1), $u_{g_i}\in P\cap Z(A)=m$.

Note 4.4 (*center of* $R*_\alpha G$). Let R be a domain with quotient field K, let G be a group, consider a generalized 2-cocycle $\alpha: G\times G\to R\setminus\{0\}$ (with $\alpha(e,e)=1$) and put $H=\{x\in G\mid \alpha(x,x^{-1})$ invertible in $R\}$. We assume that $[G:Z(G)]<\infty$. Note that $[G:Z(G)]<\infty$ implies that the commutator subgroup G' is finite. If G is finitely generated, then the converse also holds (see, e.g., [9, Chapter 8]).

(1) Suppose $G'\subset H$ and $R\alpha(x,y)=R\alpha(y,x)$ for all $x,y\in G$.

For $x,g\in G$, we set $f_\alpha(x,g)=\alpha(x,g)\alpha(xgx^{-1},x)^{-1}$ in K. We observe that $u_xu_gu_x^{-1}=f_\alpha(x,g)u_{xgx^{-1}}$ in $K*_\alpha G$. Lemma 2.1 yields that $u_xu_gu_x^{-1}\in R*_\alpha G$. So $f_\alpha(x,g)\in R$ and $f_\alpha(x,g)$ is invertible in R by [7, Lemma 2.1].

For $g\in G$, put $C(g)=C_G(g)=\{x\in G\mid xg=gx\}$. From $[G:Z(G)]<\infty$ it follows that $[G:C(g)]<\infty$. Now let $g\in G$ be α-G-regular and let $x_1,\dots,x_k$ be a set of left coset representatives of $C(g)$ in G. We define

$$v_g=\sum_{i=1}^{k} f_\alpha(x_i,g)u_{x_igx_i^{-1}}.$$

It is easily verified that $xgx^{-1}=ygy^{-1}$ implies $f_\alpha(x,g)=f_\alpha(y,g)$ $(x,y\in G)$, hence this definition is independent of the choice of the representatives x_i. Also, for all $x\in G$, xgx^{-1} is again α-G-regular.

Consider the conjugacy classes of G consisting of α-G-regular elements of G, and choose an element in each class, say $\{g_i\mid i\in I\}$. As in [7, Proposition 2.4], we obtain that $\{v_{g_i}\mid i\in I\}$ forms an R-basis for the center of $R*_\alpha G$.

(2) It is clear that $[G:Z(G)]<\infty$ implies $[H:H\cap Z(G)]<\infty$, hence $[H:Z(H)]<\infty$. Further, note that $f_\alpha(x,h)$ is an invertible element of R for all $x,h\in H$ by Lemma 1.1. So as in (1), we obtain an R-basis for the center of $R*_\alpha H$.

(3) For some relations between the centers of $R *_\alpha H$ and $R *_\alpha G$, we refer to [7, Propositions 2.6 and 2.8]. Furthermore, assume that $H \triangleleft G$ and that each α-H-regular element of H is α-G-regular (of course this holds whenever α is symmetric). If for each α-H-regular $h \in H$, we have $G = HC_G(h)$, then $Z(R *_\alpha H) \subset Z(R *_\alpha G)$.

References

[1] F. DeMeyer, E. Ingraham, Separable Algebras over Commutative Rings, Lecture Notes in Math., vol. 181, Springer, New York, 1971.
[2] G. Karpilovsky, Projective Representations of Finite Groups, Dekker, New York, 1985.
[3] C. Năstăsescu, F. Van Oystaeyen, Graded Ring Theory, Math. Library, vol. 28, North-Holland, Amsterdam, 1982.
[4] C. Năstăsescu, Group rings of graded rings: Applications, J. Pure Appl. Algebra 33 (1984) 313–335.
[5] C. Năstăsescu, E. Nauwelaerts, F. Van Oystaeyen, Arithmetically graded rings revisited, Comm. Algebra 14 (1986) 1991–2017.
[6] E. Nauwelaerts, Generalized 2-cocycles of finite groups and maximal orders, J. Algebra 210 (1998) 225–241.
[7] E. Nauwelaerts, F. Van Oystaeyen, Generalized twisted group rings of finite groups, Comm. Algebra 27 (1999) 3685–3701.
[8] I. Reiner, Maximal Orders, Academic Press, New York, 1975.
[9] L. Rowen, Ring Theory, vol. II, Academic Press, New York, 1988.

Further reading

[10] K.S. Brown, Cohomology of Groups, Grad. Texts in Math., vol. 87, Springer, New York, 1982.