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Nonlinear Dispersive Instabilities in Kelvin–Helmholtz Magnetohydrodynamic Flows

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Abstract

In this paper a weakly nonlinear theory of wave propagation in superposed fluids in the presence of magnetic fields is presented. The equations governing the evolution of the amplitude of the progressive waves are reported. The nonlinear evolution of Kelvin–Helmholtz instability (KHI) is examined in $2+1$ dimensions in the context of magnetohydrodynamics (MHD). We study the envelope properties of the $2+1$ dimensional wave packet. We converted the resulting nonlinear equation for the evolution of the wave packets in a $2+1$ dimensional nonlinear Schrödinger (NLS) equation by using the function transformation method into a sine–Gordon equation, which depends only on one function, ζ . We obtained rather general classes of solutions of the equation in ζ which leads to rather general soliton solutions of the $2+1$ dimensional NLS equation. This result contains interesting specific solutions such as N multiple solitons, propagational breathers and quadratic solitons.

1. Introduction

Under certain conditions the interface of two fluids in motion develops an instability known as the Kelvin–Helmholtz instability (KHI). This instability has many applications in astrophysics, space physics, plasma dynamics, gas and fluid flows. The development of this instability determines the dynamic processes in the hoses of spiral galaxies and in the jet bands in radiogalaxies [1], as well as the processes in the magnetic break zone [2] and at the layer boundaries of the stratified atmosphere of planets and cometary tails [3], sheared plasma flow such as the stability of the interface between the solar wind and the magnetosphere [4–6], interaction between the adjacent streams of different velocities in the solar wind [7] and laboratory experiments aiming to study the nonlinear saturated stage of the cross-field electrostatic KHI in a strongly magnetized plasma [8–13]. In application to plasma problems, the inhomogeneous plasma flux instability in crossed electric and magnetic fields [2,14–16] is an important particular case. In the geospace-magnetosphere, observations demonstrate that there are various nonlinear waves in the upper ionosphere [17] and auroral region [18]. Wu *et al.* [19–20] studied solitary waves and discussed the possibility of applying their results to nonlinear waves observed by the scientific satellite Freja in the upper ionosphere [21].

Michael [22] considered the stability of two inviscid, incompressible and perfectly conducting fluids, moving with uniform velocities parallel to their interface and in which parallel magnetic fields along the direction of streaming were present. He showed that, within the

framework of the linear theory, the magnetic fields have a stabilizing effect on the configuration. The linear theory governing the KHI in MHD when two superposed fluids move with uniform speeds was developed by Chandrasekhar [23]. Drazin [24], Nayfeh and Saric [25] and Weissman [26] investigated the nonlinear aspects of such instability in ideal fluids. In MHD, Kant and Malik [27] and Malik and Singh [28] studied the nonlinear evolution of wave packets for such a type of stability. They reported that for subcritical MHD flows where the velocity difference U is smaller than the critical velocity U_c , the dynamical equation yielding the amplitude is a nonlinear Schrödinger equation (NLS). Kant and Malik [27] also showed that a wave train of constant amplitude is unstable against modulations in the presence of a magnetic field. Using inverse-scattering transforms (IST), Zakharov and Shabat [29] obtained the solution of a NLS equation in $1+1$ dimensions. Their analysis reveals that smooth initial data lead to localized soliton solutions, and is related to the terminal state of the modulational instability. Moreover Khater *et al.* [30] used the Bäcklund transformations to obtain classes of exact solutions.

For postcritical values of U , the amplitude evolution for supercritical MHD flows was described by a nonlinear Klein–Gordon equation. From this equation, Malik and Singh [28] derived kink, antikink and solitary-wave solutions. If one neglects the spatial variation in amplitude then the Klein–Gordon equation gives rise to exponential decay, subcritical, supercritical and explosive instabilities. However, considerable effort has been devoted to the phenomenon of self-focusing and collapse; in nonlinear optics by Zakharov and Synakh [31], in plasma physics by Zakharov [32] and in two-layer Kelvin–Helmholtz flows by Gibbon and McGuinness [33]. Singh, Khosla and Malik [34] derived the dynamical equations governing the amplitudes of the fundamental mode and the second-harmonic mode, and they integrated these equations numerically by employing the fourth-order Runge–Kutta method. In conservative systems, solutions are found by direct integration, suitable transformation or substitution, or other *ad hoc* techniques. The original PDE could be solved also with more sophisticated methods such as the Hirota bilinear technique [35], truncated Painlevé expansion [36], direct algebra methods [37–38] and the like. Other PDEs are no longer that easy to solve. For instance, ingenious transformations are needed to obtain closed form solutions to the KdV–Burgers equation [39–40], despite the simplicity of this equation.

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In this paper, a function transformation is introduced and applied to obtain plane soliton solutions, N multiple soliton solutions, the propagational breathers and the quadratic solitons, which contain spherical, ellipsoidal and hyperboloidal solitons.

The paper is organized as follows: In Section 2, the basic equations and the boundary conditions governing the problem together with the multiple scale method are sketched. In Section 3, nonlinear focusing, self-focusing and the standard NLS equation in $2+1$ dimensions in elliptic and hyperbolic cases are briefly reported. In Section 4, we find general soliton solutions to the standard NLS equation in $2+1$ dimensions in elliptic and hyperbolic cases. We applied the function transformation method, then the resulting nonlinear equation for the evolution of the wave packets in $2+1$ dimensions, i.e. the NLS equation, is transformed into a sine–Gordon equation, which depends only on one function, ζ . We obtained a general solution of the equation in ζ which leads to a general soliton solution of the $2+1$ dimensions NLS equation. The latter contains some interesting specific solutions such as N multiple solitons, propagational breathers and quadratic solitons. In Section 5, we discuss the stability for the soliton solutions. Section 6 yields conclusions.

2. Problem formulation

Consider two superposed fluids separated by the interface $z = 0$. The fluids are semi-infinite, inviscid, incompressible and perfectly conducting with surface tension present at the boundary. A fluid with density ρ_1 occupies the half-space $z < 0$ whereas $z > 0$ (z is the vertical axis) is occupied by a fluid with density ρ_2 . The upper and lower fluids are moving with uniform velocities $U^{(1)}$ and $U^{(2)}$ along the x -axis. The system is subjected to a uniform tangential magnetic field $\mathbf{H}(H, 0, 0)$ permeating both fluids in the presence of a gravitational acceleration $\mathbf{g}(0, 0, -g)$. We examine the propagation of weakly nonlinear wave packets of small but finite amplitude, with principal direction along the x -axis, although modulation in both x and y directions is allowed. The basic equations describing this MHD flow are

$$\frac{\partial \mathbf{u}^{(i)}}{\partial t} + (\mathbf{u}^{(i)} \cdot \nabla) \mathbf{u}^{(i)} = -\nabla \Pi^{(i)} + \frac{1}{\rho^{(i)}} (\mathbf{h}^{(i)} \cdot \nabla) \mathbf{h}^{(i)} + \mathbf{g}, \quad (1)$$

$$\frac{\partial \mathbf{h}^{(i)}}{\partial t} = (\mathbf{h}^{(i)} \cdot \nabla) \mathbf{u}^{(i)} - (\mathbf{u}^{(i)} \cdot \nabla) \mathbf{h}^{(i)}, \quad (2)$$

$$\nabla \cdot \mathbf{u}^{(i)} = 0, \quad \nabla \cdot \mathbf{h}^{(i)} = 0. \quad (3)$$

Here $i = 1$ and 2 correspond to the lower and upper regions respectively. The fluid velocity is $\mathbf{u}^{(i)} = (u^{(i)}, v^{(i)}, w^{(i)})$, the magnetic field is $\mathbf{h}^{(i)}$ (including $\sqrt{\mu}$ with μ , the magnetic permeability, taken as constant) and the total pressure in region (i) is $\rho^{(i)} \Pi^{(i)} = P^{(i)} + \frac{1}{2} |\mathbf{h}^{(i)}|^2$, where $\frac{1}{2} |\mathbf{h}^{(i)}|^2$, $P^{(i)}$ and t stand as usual for the magnetic pressure, gas pressure and time respectively. In some papers [34] one has used for the magnetic field $(\rho^{(i)})^{1/2} \mathbf{h}^{(i)}$ instead. This avoids the factor $1/\rho^{(i)}$ in Eq. (1), but this factor has then to appear in the boundary condition related to the conservation of flux in Eq. (3). We have adapted for this aspect and hence our

expressions for the constants (see Eq. (11)) differ somewhat from theirs.

The boundary conditions at the free interface are:

(a) The kinematic boundary condition

$$\frac{\partial \eta}{\partial t} + (\mathbf{u}^{(i)} \cdot \nabla) \eta = w^{(i)} \quad \text{at } z = \eta, \quad (i = 1, 2). \quad (4)$$

(b) The continuity of the normal component of the magnetic field

$$\mathbf{n} \cdot \mathbf{h}^{(1)} = \mathbf{n} \cdot \mathbf{h}^{(2)},$$

where \mathbf{n} is a normal to the boundary and η denotes the elevation of the free surface above the unperturbed level $z = \eta(x, y, t)$; thus

$$h_z^{(1)} - h_x^{(1)} \frac{\partial \eta}{\partial x} - h_y^{(1)} \frac{\partial \eta}{\partial y} = h_z^{(2)} - h_x^{(2)} \frac{\partial \eta}{\partial x} - h_y^{(2)} \frac{\partial \eta}{\partial y}. \quad (5)$$

(c) The continuity of the normal stress at the perturbed surface $z = \eta(x, y, t)$ is

$$\begin{aligned} & \Pi^{(1)} - \rho \Pi^{(2)} - (1 - \rho) g \eta \\ &= \frac{T}{\rho_1} \left[\nabla^2 \eta (1 + (\nabla \eta)^2) - \frac{1}{2} \nabla \eta \cdot \nabla (\nabla \eta)^2 \right] \\ & \quad \times (1 + (\nabla \eta)^2)^{-3/2} \end{aligned}$$

where T is the coefficient of surface tension and $\rho = \rho_2/\rho_1$. This equation may be rewritten as

$$\begin{aligned} & \Pi^{(1)} - \rho \Pi^{(2)} - (1 - \rho) g \eta \\ &= \frac{T}{\rho_1} \left[\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \left(\frac{\partial \eta}{\partial y} \right)^2 \frac{\partial^2 \eta}{\partial x^2} + \left(\frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 \eta}{\partial y^2} \right. \\ & \quad \left. - 2 \frac{\partial^2 \eta}{\partial x \partial y} \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} \right] \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right]^{-3/2}. \quad (6) \end{aligned}$$

We wish to examine the nonlinear stability problem posed by Eqs. (1) to (6). In order to describe nonlinear interactions of small but finite-amplitude waves, we employ the multiple scales method [41]. To do this, we introduce the variables

$$x_n = \varepsilon^n x, \quad y_n = \varepsilon^n y, \quad t_n = \varepsilon^n t \quad (n = 0, 1, 2, \dots, N), \quad (7)$$

and expand the various perturbed quantities in the following series:

$$\begin{aligned} \eta(x, y, z, t) &= \sum_{n=0}^N \varepsilon^n \eta_n(x_0, x_1, \dots, x_n; y_0, y_1, \dots, y_n; \\ & \quad \times z; t_0, t_1, \dots, t_n) + O(\varepsilon^{N+1}), \quad (8) \end{aligned}$$

and similarly for $\mathbf{u}(x, y, z, t)$ and $\mathbf{h}(x, y, z, t)$. Here ε represents a small parameter characterizing the steepness ratio of the wave.

For the problem under investigation, it is sufficient to take $N = 3$ as far as the lowest significant order is concerned. Equations (1) to (6) require *a priori* information of the displacement $\eta(x, y, t)$ at the perturbed surface. To circumvent this problem, we use Taylor series expansions of various quantities about $z = 0$. On substituting expressions (8) and the corresponding derivatives into (1)–(3) and the transformed boundary conditions (4)–(6), we obtain on equating the coefficients of terms of equal powers in ε , a linear and successive nonlinear partial differential equations (PDEs) of various orders. The problem may then be solved for any order with the knowledge of the solutions of all the previous orders.

3. Nonlinear focusing

We shall now investigate the most dramatic consequence for the wave packets: nonlinear focusing. Self-focusing provides a mechanism by which a field of relatively small-amplitude waves produces a local region in which the amplitudes become unbounded in a finite time. The point where the field becomes singular is called the ‘focus’. Focusing is a potentially important mechanism in the redistribution of energy within the spectrum.

Consider the propagation of nonlinear wave packets of small but finite amplitudes propagating in the principal direction along the x -axis with modulation allowed in both the x and y directions.

Following Kant and Malik [27]; Malik and Singh [28]; Pusri and Malik [42]; Singh, Khosla and Malik [34]; and using the multiple scales method, (1)–(6) yield the linear dispersion relation

$$D(\omega, K) = -(\alpha_1\sigma_1^2 + \alpha_2\sigma_2^2) + l^2V_A^2 + K^3 + K = 0 \tag{9}$$

where

$$\left. \begin{aligned} \alpha_1 &= \frac{\rho_1}{\rho_1 + \rho_2}, & \alpha_2 &= \frac{\rho_2}{\rho_1 + \rho_2}, & V_A^2 &= \frac{2\mu H^2}{\rho_1 + \rho_2}, \\ \sigma_i &= lU^{(i)} - \omega, & K^2 &= l^2 + m^2, & i &= 1, 2. \end{aligned} \right\} \tag{10}$$

All the perturbed quantities are normalized with respect to the characteristic length $[T/(\rho_1 - \rho_2)g]^{1/2}$ and the characteristic speed $[(\rho_1 - \rho_2)gT/(\rho_1 + \rho_2)^2]^{1/4}$. The quantities l and m correspond to the components of the wave vector along the x and y directions respectively and V_A is the Alfvén wave speed corresponding to the averaged density. Equation (9) furnishes the critical velocity difference U_c at $K = K_c = 1$, as was developed by Chandrashekar [23]:

$$U_c = \left(\frac{2 + V_A^2}{\alpha_1\alpha_2} \right)^{1/2}. \tag{11}$$

In the dispersion relation (9), we observe the magnetic field having a stabilizing influence on the wave motion, like the surface tension. Since we are interested primarily in studying the amplitude modulation of the progressive wave, we assume that $U^{(1)} - U^{(2)} < U_c$ and proceed to obtain the second-order solution. This second-order solu-

tion is uniformly valid subject to the solvability condition and the non-secularity condition. For the second-order problem it reads:

$$-D_\omega \frac{\partial A}{\partial t_1} + D_l \frac{\partial A}{\partial x_1} + D_m \frac{\partial A}{\partial y_1} = 0, \tag{12}$$

where D_ω , D_l and D_m are the partial derivatives of the dispersion relation (9) with respect to frequency ω and wave numbers l and m , respectively. The amplitude A of the progressive wave is a function of the faster scales $(x_1, x_2; y_1, y_2; t_1, t_2)$. Introducing

$$V_l = -\frac{D_l}{D_\omega}, \quad V_m = -\frac{D_m}{D_\omega}, \tag{13}$$

and substituting into (12), we get

$$\frac{\partial A}{\partial t_1} + V_l \frac{\partial A}{\partial x_1} + V_m \frac{\partial A}{\partial y_1} = 0. \tag{14}$$

The quantities x_1 and y_1 are the fast space scales, while V_l and V_m represent a kind of group velocities along the x and y directions respectively, given by

$$D_\omega = 2(\alpha_1\sigma_1 + \alpha_2\sigma_2), \quad D_m = \frac{m}{K}(3K^2 + 1),$$

$$D_l = \frac{-2}{K}(\alpha_1U^{(1)}\sigma_1 + \alpha_2U^{(2)}\sigma_2) + \frac{l}{K}(3K^2 + 1) + \frac{2l}{K}V_A^2,$$

then we have

$$V_l = \frac{2(\alpha_1U^{(1)}\sigma_1 + \alpha_2U^{(2)}\sigma_2) - \frac{l}{K}(3K^2 + 1) - 2lV_A^2}{2(\alpha_1\sigma_1 + \alpha_2\sigma_2)}, \tag{15}$$

$$V_m = \frac{-m(3K^2 + 1)}{2K(\alpha_1\sigma_1 + \alpha_2\sigma_2)}. \tag{16}$$

It is interesting to remark here that, unlike ordinary gravity waves, for Kelvin–Helmholtz flows the group velocities along the x and y directions are not parallel to the wave velocities. Furthermore, the asymptotic expansions break down for $K^2 = \frac{1}{2}$ (the breakdown at $K^2 = \frac{1}{2}$ in Eq. (17) is due to Q_1), which corresponds to a second-harmonic resonance. At resonance, the fundamental and second-harmonic modes propagate with the same phase speed, resulting in energy transfer from one mode to another. It was observed that e.g. sea waves occurred at wind velocities far below U_c ($= 6.5$ m for wind and seawater). Kelvin attributed this to viscosity. However resonances like the one revealed by the second order theory are even stronger effects. In this section, we avoid such a resonance and assume that $K^2 \neq \frac{1}{2}$. In view of describing the amplitude modulation of the progressive waves at the surface of the fluid, we need to proceed to the third-order solution, which yields the following nonlinear equation for the evolution of wave packets in $2 + 1$ dimensions,

$$\begin{aligned} 2i \left(\frac{\partial A}{\partial t_2} + V_l \frac{\partial A}{\partial x_2} + V_m \frac{\partial A}{\partial y_2} \right) + P_1 \frac{\partial^2 A}{\partial x_1^2} \\ + 2P_2 \frac{\partial^2 A}{\partial x_1 \partial y_1} + P_3 \frac{\partial^2 A}{\partial y_1^2} = Q_1 |A|^2 A, \end{aligned} \tag{17}$$

where

$$P_1 = (\alpha_1\sigma_1 + \alpha_2\sigma_2)^{-1} \left\{ 2[\alpha_1(U^{(1)})^2 + \alpha_2(U^{(2)})^2] - K^{-1}(3K^2 + 1) + l^2K^{-3}(1 - 3K^2) - 4V_l(\alpha_1(U^{(1)})^2 + \alpha_2(U^{(2)})^2) - 2V_l^2 - 2V_A^2 \right\}, \quad (18a)$$

$$P_2 = \frac{lm}{K^3}(1 - 3K^2) - 2(\alpha_1U^{(1)} + \alpha_2U^{(2)})V_l - 2V_lV_m, \quad (18b)$$

$$P_3 = \frac{m^2}{K^3}(1 - 3K^2) - \frac{1}{K}(1 + 3K^2), \quad (18c)$$

$$Q_1 = \frac{1}{2}[4(2K^2 - 1)^{-1}(\alpha_1\sigma_1^2 - \alpha_2\sigma_2^2 - K^2V_A^2) - 4K^2 - K^4]. \quad (18d)$$

The PDE (18) is elliptic or hyperbolic, according to the sign of $P = (P_2^2 - P_1P_3)$,

$$P = (\alpha_1\sigma_1 + \alpha_2\sigma_2)^{-2} \times \left[\left(\frac{lm}{K^3}(1 - 3K^2) - 2(\alpha_1U^{(1)} + \alpha_2U^{(2)})V_l - 2V_lV_m \right)^2 - \left\{ 2[\alpha_1(U^{(1)})^2 + \alpha_2(U^{(2)})^2] - K^{-1}(3K^2 + 1) + l^2K^{-3}(1 - 3K^2) - 4V_l(\alpha_1(U^{(1)})^2 + \alpha_2(U^{(2)})^2) - 2V_l^2 - 2V_A^2 \right\} \left\{ \frac{m^2}{K^3}(1 - 3K^2) - \frac{1}{K}(1 + 3K^2) \right\} \right].$$

For the elliptic case, when P is negative, we introduce the transformations

$$\zeta_1 = \frac{x_1}{P_1^{1/2}}, \quad \eta_1 = \left(P_3 - \frac{P_2^2}{P_1} \right)^{-1/2} \left(y_1 - \frac{P_2}{P_1}x_1 \right). \quad (19)$$

Under such a transformation, (17) now reduces to a standard two-dimensional NLS equation. On rewriting (17) in the group-velocity reference frame, we obtain the elliptic equation

$$2i \frac{\partial A}{\partial t_2} + \frac{\partial^2 A}{\partial \zeta_1^2} + \frac{\partial^2 A}{\partial \eta_1^2} = \frac{Q_1}{P_1} [(\alpha_1\sigma_1 + \alpha_2\sigma_2)\rho_1]^{-1} |A|^2 A_0.$$

Putting

$$Q \equiv Q(l, m, U, V_A^2) = \frac{Q_1}{P_1} [(\alpha_1\sigma_1 + \alpha_2\sigma_2)\rho_1]^{-1}, \quad t_2 = \tau, \quad (20)$$

we obtain the elliptic equation as:

$$2i \frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial \zeta_1^2} + \frac{\partial^2 A}{\partial \eta_1^2} = Q |A|^2 A. \quad (21)$$

Equation (21) is the standard NLS equation in 2 + 1 dimensions. For the hyperbolic case, when P is positive, we

introduce the transformations

$$\zeta_1 = \frac{x_1}{P_1^{1/2}}, \quad \eta_1 = \left(\frac{P_2^2}{P_1} - P_3 \right)^{-1/2} \left(\frac{P_2}{P_1}x_1 - y_1 \right). \quad (22)$$

Proceeding as before, we obtain the hyperbolic equation

$$2i \frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial \zeta_1^2} - \frac{\partial^2 A}{\partial \eta_1^2} = Q |A|^2 A. \quad (23)$$

Equations (21) and (23) can be expressed in the form

$$2i \frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial \zeta_1^2} + \Delta_2 \frac{\partial^2 A}{\partial \eta_1^2} = Q |A|^2 A, \quad (24)$$

$$\text{with } \Delta_2 = \begin{cases} 1 & \text{when } P_2^2 - P_1P_3 < 0, \\ -1 & \text{when } P_2^2 - P_1P_3 > 0. \end{cases} \quad (25)$$

4. General soliton solutions of two-dimensional NLS equation

4.1. The elliptic case

Now we shall find classes of solutions of the standard two-dimensional NLS equation, by applying the function transformation method. We search for solutions of the elliptic Eq. (21) in the form

$$A = \theta(\tau, \zeta_1, \eta_1) \exp i(c_0\tau + c_1\zeta_1 + c_2\eta_1), \quad (26)$$

where $\theta(\tau, \zeta_1, \eta_1) = \theta^*(\tau, \zeta_1, \eta_1)$ and c_0, c_1, c_2 are real constants. By inserting (26) into (21), and putting $2c_0 + c_1^2 + c_2^2 = -C$ (constant), we get:

$$2i \frac{\partial \theta}{\partial \tau} + 2ic_1 \frac{\partial \theta}{\partial \zeta_1} + 2ic_2 \frac{\partial \theta}{\partial \eta_1} + \frac{\partial^2 \theta}{\partial \zeta_1^2} + \frac{\partial^2 \theta}{\partial \eta_1^2} = Q\theta^3 - C\theta. \quad (27)$$

Using the function transformation

$$\theta = \sqrt{\frac{C}{Q}} \sin\left(\frac{\varphi}{2}\right), \quad (28)$$

where φ is a function in τ, ζ_1 and η_1 , we have

$$2i \frac{\partial \theta}{\partial \tau} + 2ic_1 \frac{\partial \theta}{\partial \zeta_1} + 2ic_2 \frac{\partial \theta}{\partial \eta_1} = \sqrt{\frac{C}{Q}} \cos(\varphi/2) \left[i \frac{\partial \varphi}{\partial \tau} + ic_1 \frac{\partial \varphi}{\partial \zeta_1} + ic_2 \frac{\partial \varphi}{\partial \eta_1} \right], \quad (29a)$$

$$\frac{\partial^2 \theta}{\partial \zeta_1^2} = \frac{1}{2} \sqrt{\frac{C}{Q}} \left[\cos(\varphi/2) \frac{\partial^2 \varphi}{\partial \zeta_1^2} - \frac{1}{2} \sin(\varphi/2) \left(\frac{\partial \varphi}{\partial \zeta_1} \right)^2 \right], \quad (29b)$$

$$\frac{\partial^2 \theta}{\partial \eta_1^2} = \frac{1}{2} \sqrt{\frac{C}{Q}} \left[\cos(\varphi/2) \frac{\partial^2 \varphi}{\partial \eta_1^2} - \frac{1}{2} \sin(\varphi/2) \left(\frac{\partial \varphi}{\partial \eta_1} \right)^2 \right], \quad (29c)$$

$$Q\theta^3 - C\theta = Q \left(\frac{C}{Q} \right)^{3/2} \sin^3(\varphi/2) - C \sqrt{\frac{C}{Q}} \sin(\varphi/2) = -\frac{C}{2} \sqrt{\frac{C}{Q}} \cos(\varphi/2) \sin \varphi. \quad (29d)$$

By substituting from (29) in Eq. (27) one gets

$$i \frac{\partial \varphi}{\partial \tau} + ic_1 \frac{\partial \varphi}{\partial \zeta_1} + ic_2 \frac{\partial \varphi}{\partial \eta_1} + \frac{1}{2} \left(\frac{\partial^2 \varphi}{\partial \zeta_1^2} + \frac{\partial^2 \varphi}{\partial \eta_1^2} \right) - \frac{1}{4} \tan(\varphi/2) \left(\left(\frac{\partial \varphi}{\partial \zeta_1} \right)^2 + \left(\frac{\partial \varphi}{\partial \eta_1} \right)^2 \right) = -\frac{C}{2} \sin \varphi. \quad (30)$$

Setting $\varphi = \varphi(\zeta)$ which is a function of one variable ζ only (a function of the variables, see below), we easily see that

$$\frac{\partial^2 \varphi}{\partial \zeta_1^2} = \left(\frac{\partial \zeta}{\partial \zeta_1} \right)^2 \frac{d^2 \varphi}{d\zeta^2} + \frac{\partial^2 \zeta}{\partial \zeta_1^2} \frac{d\varphi}{d\zeta},$$

$$\frac{\partial^2 \varphi}{\partial \eta_1^2} = \left(\frac{\partial \zeta}{\partial \eta_1} \right)^2 \frac{d^2 \varphi}{d\zeta^2} + \frac{\partial^2 \zeta}{\partial \eta_1^2} \frac{d\varphi}{d\zeta}. \quad (31)$$

By substituting (31) into (30), we obtain

$$\left(i \frac{\partial \zeta}{\partial \tau} + ic_1 \frac{\partial \zeta}{\partial \zeta_1} + ic_2 \frac{\partial \zeta}{\partial \eta_1} + \frac{1}{2} \frac{\partial^2 \zeta}{\partial \zeta_1^2} + \frac{1}{2} \frac{\partial^2 \zeta}{\partial \eta_1^2} \right) \frac{d\varphi}{d\zeta} + \left[\left(\frac{\partial \zeta}{\partial \zeta_1} \right)^2 + \left(\frac{\partial \zeta}{\partial \eta_1} \right)^2 \right] \left[\frac{1}{2} \frac{d^2 \varphi}{d\zeta^2} - \frac{1}{4} \tan(\varphi/2) \left(\frac{d\varphi}{d\zeta} \right)^2 \right] = -\frac{C}{2} \sin \varphi. \quad (32)$$

Explicitly, some solutions of (32) obey the following system of equations:

$$i \frac{\partial \zeta}{\partial \tau} + ic_1 \frac{\partial \zeta}{\partial \zeta_1} + ic_2 \frac{\partial \zeta}{\partial \eta_1} = \frac{\partial^2 \zeta}{\partial \zeta_1^2} + \frac{\partial^2 \zeta}{\partial \eta_1^2} = 0,$$

$$\left(\frac{\partial \zeta}{\partial \zeta_1} \right)^2 + \left(\frac{\partial \zeta}{\partial \eta_1} \right)^2 = 1, \quad (33)$$

$$\frac{d^2 \varphi}{d\zeta^2} = -\frac{C}{2} \sin \varphi, \quad \frac{d\varphi}{d\zeta} = \sqrt{2C} \cos\left(\frac{\varphi}{2}\right). \quad (34)$$

Equation (34) is equivalent to a sine-Gordon equation, which has a well known soliton solution

$$\varphi = 4 \tan^{-1} \exp\left(\sqrt{\frac{C}{2}} \zeta + \zeta_0\right) - \pi, \quad \zeta_0 = \text{cst.} \quad (35)$$

From (35), (28) and (26), we obtain the corresponding soliton solution of the two-dimensional NLS equation (21) in the form

$$\theta = \sqrt{\frac{C}{Q}} \tanh\left(\sqrt{\frac{C}{2}} \zeta + \zeta_0\right),$$

$$A = \sqrt{\frac{C}{Q}} \tanh\left(\sqrt{\frac{C}{2}} \zeta + \zeta_0\right) e^{i(c_0\tau + c_1\zeta_1 + c_2\eta_1)}, \quad (36)$$

where ζ denotes a solution of (33). As (33) has a lot of solutions, (36) contains a wide variety of interesting solitons of the two-dimensional NLS equation.

Now we intend to find a rather general solution of (33) in the form

$$\zeta = F(\xi_j) + d_0\tau + d_1\zeta_1 + d_2\eta_1,$$

$$\xi_j = b_{j0}\tau + b_{j1}\zeta_1 + b_{j2}\eta_1 + \varepsilon_j,$$

$$d_0, d_1, d_2, b_{j0}, b_{j1}, b_{j2}, \varepsilon_j = \text{constants}, \quad (37)$$

where $F(\xi_j)$ denotes an arbitrary function of ξ_j . Combining (33) and (37) we easily obtain

$$i \frac{\partial \zeta}{\partial \tau} + ic_1 \frac{\partial \zeta}{\partial \zeta_1} + ic_2 \frac{\partial \zeta}{\partial \zeta_2} = i(b_{j0} + c_1b_{j1} + c_2b_{j2}) \frac{\partial F}{\partial \xi_j} + i(d_0 + c_1d_1 + c_2d_2), \quad (38a)$$

$$\frac{\partial \zeta}{\partial \zeta_1} = b_{j1} \frac{\partial F}{\partial \xi_j} + d_1, \quad \frac{\partial \zeta}{\partial \eta_1} = b_{j2} \frac{\partial F}{\partial \xi_j} + d_2,$$

$$\left(\frac{\partial \zeta}{\partial \zeta_1} \right)^2 + \left(\frac{\partial \zeta}{\partial \eta_1} \right)^2 = [(b_{j1})^2 + (b_{j2})^2] \left(\frac{\partial F}{\partial \xi_j} \right)^2 + (2d_1b_{j1} + 2d_2b_{j2}) \frac{\partial F}{\partial \xi_j} + d_1^2 + d_2^2, \quad (38b)$$

$$\frac{\partial^2 \zeta}{\partial \zeta_1^2} = b_{j1}b_{k1} \frac{\partial^2 F}{\partial \xi_j \partial \xi_k}, \quad \frac{\partial^2 \zeta}{\partial \eta_1^2} = b_{j2}b_{k2} \frac{\partial^2 F}{\partial \xi_j \partial \xi_k}. \quad (38c)$$

Requiring $F(\xi_j)$ to be arbitrary leads to the conditions

$$b_{j0} + c_1b_{j1} + c_2b_{j2} = 0, \quad d_0 + c_1d_1 + c_2d_2 = 0,$$

$$b_{j1}b_{k1} + b_{j2}b_{k2} = 0, \quad d_1^2 + d_2^2 = 1,$$

$$d_1b_{j1} + d_2b_{j2} = 0. \quad (39)$$

The general solution (37) yields for (36) and (35) a rather general soliton solution of the two-dimensional NLS and sine-Gordon equations. They contain some interesting specific solutions, such as plane solitons, N multiple solitons, propagational breathers and quadratic solitons. We will discuss them respectively below.

(a) Plane solitons

This is a simple case. By taking all b_{ji} in (39) to be equal to zero, all ξ_j become constant so that $F(\xi_j) = F(\varepsilon_j) = \text{constants}$. Then (35) and (36) take the form

$$\varphi = 4 \tan^{-1} \exp\left(\sqrt{\frac{C}{2}} \zeta + \zeta_0\right) - \pi, \quad \zeta_0 = \text{cst.} \quad (40)$$

$$A = \sqrt{\frac{C}{Q}} \tanh\left(\sqrt{\frac{C}{2}} \zeta + \zeta_0\right) e^{i(c_0\tau + c_1\zeta_1 + c_2\eta_1)}, \quad (41)$$

where now $\zeta = d_0\tau + d_1\zeta_1 + d_2\eta_1 + \text{constant}$. In this case, (40) and (41) denote hyperplanar soliton solutions of the two-dimensional NLS and sine-Gordon equations; this solution is stable.

(b) N multiple soliton solutions

Let us select $F(\xi_j)$ in the form

$$F(\xi_j) = \ln \sum_{j=1}^N \exp(b_{j0}\tau + b_{j1}\zeta_1 + b_{j2}\eta_1 + \varepsilon_j). \quad (42)$$

Thus (37) yields:

$$\zeta = \ln \left(\sum_{j=1}^N \exp[a_{j0}\tau + a_{j1}\zeta_1 + a_{j2}\eta_1 + \varepsilon_j] \right), \quad (43)$$

where $a_{j\alpha} = d_\alpha + b_{j\alpha}$, $\alpha = 0, 1, 2$. The conditions (39) are simplified to

$$a_{j0} + c_1 a_{j1} + c_2 a_{j2} = 0, \quad a_{j1} a_{k1} + a_{j2} a_{k2} = 1. \quad (44)$$

By direct calculation, we can easily prove that (43) is a solution of (33). Taking j equal to $1, 2, \dots, N$ respectively yields

$$\varphi = 4 \tan^{-1} \exp \left(\sqrt{\frac{C}{2}} \xi + \zeta_0 \right) - \pi, \quad (45)$$

$$A = \sqrt{\frac{C}{Q}} \tanh \left(\sqrt{\frac{C}{2}} \zeta + \zeta_0 \right) e^{i(c_0\tau + c_1\zeta_1 + c_2\eta_1)}, \quad (46)$$

where ζ is now given by (43). (45) and (46) are N multiple soliton solutions of the two-dimensional NLS and sine–Gordon equations.

(c) Propagational breathers

Considering $N = 2$ in (43) we get

$$\zeta = \ln \left(\frac{\exp(a_{10}\tau + a_{11}\zeta_1 + a_{12}\eta_1 + \varepsilon_1) + \exp(a_{20}\tau + a_{21}\zeta_1 + a_{22}\eta_1 + \varepsilon_2)}{\exp(a_{20}\tau + a_{21}\zeta_1 + a_{22}\eta_1 + \varepsilon_2)} \right), \quad (47)$$

which corresponds to a 2-soliton solution of the two-dimensional NLS equation. For the 2 + 1 dimensional case, we set

$$\begin{aligned} a_{10} &= -a_{20}, & a_{11} &= -a_{21}, & a_{12} &= a_{22}, \\ \varepsilon_1 &= \ln S, & \varepsilon_2 &= \ln S + i\pi. \end{aligned} \quad (48)$$

Then (47) becomes

$$\zeta = \ln[2S \exp(a_{12}\eta_1) \sinh(a_{10}\tau + a_{11}\zeta_1)] \quad (49)$$

and (44) gives the conditions

$$a_{11}^2 + a_{12}^2 = 1, \quad a_{10} + 2c_1 a_{11} = 0, \quad c_2 a_{12} = 0. \quad (50)$$

If a_{10} , a_{11} , S are some imaginary numbers such as

$$a_{10} = ia_0, \quad a_{11} = ia_1, \quad 2S = -iB \quad (51)$$

then (49) takes the form

$$\zeta = \ln[B \exp(a_{12}\eta_1) \sin(a_0\tau + a_1\zeta_1)]. \quad (52)$$

Summarizing again:

$$\varphi = 4 \tan^{-1} \exp \left(\sqrt{\frac{C}{2}} \zeta + \zeta_0 \right) - \pi, \quad (53)$$

$$A = \sqrt{\frac{C}{Q}} \tanh \left(\sqrt{\frac{C}{2}} \zeta + \zeta_0 \right) e^{i(c_0\tau + c_1\zeta_1 + c_2\eta_1)}, \quad (54)$$

where now ζ is given by (52). Thus (53) and (54) are a two dimensional breather which propagates in the ζ_1

direction and has a constant value on the straight line $a_{12}\eta_1 = \text{constant}$. This is an interesting result.

(d) Quadratic solitons

We take the solution (33) in the form

$$\zeta = (b_{j0}\tau + b_{j1}\zeta_1 + b_{j2}\eta_1 + \varepsilon_j)^2 + d_0\tau + d_1\zeta_1 + d_2\eta_1,$$

or

$$\begin{aligned} \zeta &= b_{j0}b_{j0}\tau^2 + b_{j1}b_{j1}\zeta_1^2 + b_{j2}b_{j2}\eta_1^2 + 2b_{j1}b_{j2}\zeta_1\eta_1 \\ &+ (2\varepsilon_j b_{j1} + 2b_{j0}b_{j1}\tau + d_1)\zeta_1 + (2\varepsilon_j b_{j2} + 2b_{j0}b_{j2}\tau + d_2)\eta_1 \\ &+ (2\varepsilon_j b_{j0} + d_0)\tau + \varepsilon_j \varepsilon_j, \quad j = 1, 2, \dots, N. \end{aligned} \quad (55)$$

It describes some general quadratic solutions at any definite time. These quadratic solitons include all specific ones such as:

(i) Circular soliton

By choosing the constants of (55) so that

$$b_{ji}b_{jk} = \delta_{ik}, \quad j = 1, 2, \dots, N, \quad (56)$$

(55) becomes

$$\begin{aligned} \zeta &= \tau^2 + \zeta_1^2 + \eta_1^2 + (2\varepsilon_j b_{j1} + d_1)\zeta_1 + (2\varepsilon_j b_{j2} + d_2)\eta_1 \\ &+ (2\varepsilon_j b_{j0} + d_0)\tau + \varepsilon_j \varepsilon_j. \end{aligned} \quad (57)$$

Thus we obtain a circular soliton with radius

$$\begin{aligned} R &= \left[\left(\varepsilon_j b_{ji} + \frac{d_i}{2} \right) \left(\varepsilon_k b_{ki} + \frac{d_i}{2} \right) \right. \\ &\quad \left. - \tau^2 - (2\varepsilon_j b_{j0} + d_0)\tau - \varepsilon_j \varepsilon_j \right]^{1/2}, \end{aligned} \quad (58)$$

and with center

$$G(-2\varepsilon_j b_{j1} - 2b_{j0}b_{j1}\tau - d_1, -2\varepsilon_j b_{j2} - 2b_{j0}b_{j2}\tau - d_2). \quad (59)$$

This moves along a space line. Thus (57), (35) and (36) give some circularly shaped solitons of the two-dimensional NLS and sine–Gordon equations.

(ii) Elliptical solitons

Under the conditions

$$\begin{aligned} b_{j1}b_{j1} &> 0, & b_{j2}b_{j2} &> 0, \\ b_{ji}b_{jk} &= 0, & i \neq k, & \quad j = 1, 2, \dots, N, \end{aligned} \quad (60)$$

then (55) becomes

$$\begin{aligned} \zeta &= b_{j0}b_{j0}\tau^2 + b_{j1}b_{j1}\zeta_1^2 + b_{j2}b_{j2}\eta_1^2 + (2\varepsilon_j b_{j1} + d_1)\zeta_1 \\ &+ (2\varepsilon_j b_{j2} + d_2)\eta_1 + (2\varepsilon_j b_{j0} + d_0)\tau + \varepsilon_j \varepsilon_j, \\ & \quad j = 1, 2, \dots, N. \end{aligned} \quad (61)$$

Equation (61) are ellipses and (35) and (36) are elliptical solitons.

(iii) Hyperbolic solitons

Let the constants satisfy

$$b_{j1}b_{j1} > 0, \quad b_{j2}b_{j2} < 0, \quad b_{ji}b_{jk} = 0, \quad i \neq k, \\ j = 1, 2, \dots, N. \tag{62}$$

From (55), we obtain hyperbolas and (35) and (36) give the corresponding hyperbolic shape solitons of two-dimensional NLS and sine-Gordon equations.

4.2. The hyperbolic case

We search now solutions of the hyperbolic equation (23) in the form

$$A = \Phi(\tau, \zeta_1, \eta_1) \exp i(n_0\tau + n_1\zeta_1 + n_2\eta_1), \tag{63}$$

where $\Phi(\tau, \zeta_1, \eta_1) = \Phi^*(\tau, \zeta_1, \eta_1)$ and n_0, n_1, n_2 are real constants.

By inserting (63) into (23), one obtains putting $2n_0 + n_1^2 - n_2^2 = -C'$ (constant)

$$2i \frac{\partial \Phi}{\partial \tau} + 2in_1 \frac{\partial \Phi}{\partial \zeta_1} - 2in_2 \frac{\partial \Phi}{\partial \eta_1} + \frac{\partial^2 \Phi}{\partial \zeta_1^2} - \frac{\partial^2 \Phi}{\partial \eta_1^2} = Q\Phi^3 - C'\Phi. \tag{64}$$

Let us make a function transformation

$$\Phi = \sqrt{\frac{C'}{Q}} \sin\left(\frac{\Psi}{2}\right), \tag{65}$$

where Ψ is a function in τ, ζ_1 and η_1 . Calculation of the terms in (64) yields

$$2i \frac{\partial \Phi}{\partial \tau} + 2in_1 \frac{\partial \Phi}{\partial \zeta_1} - 2in_2 \frac{\partial \Phi}{\partial \eta_1} \\ = \sqrt{\frac{C'}{Q}} \cos(\Psi/2) \left[i \frac{\partial \Psi}{\partial \tau} + in_1 \frac{\partial \Psi}{\partial \zeta_1} - in_2 \frac{\partial \Psi}{\partial \eta_1} \right], \tag{66a}$$

$$\frac{\partial^2 \Phi}{\partial \zeta_1^2} = \frac{1}{2} \sqrt{\frac{C'}{Q}} \left[\cos(\Psi/2) \frac{\partial^2 \Psi}{\partial \zeta_1^2} - \frac{1}{2} \sin(\Psi/2) \left(\frac{\partial \Psi}{\partial \zeta_1} \right)^2 \right], \tag{66b}$$

$$\frac{\partial^2 \Phi}{\partial \eta_1^2} = \frac{1}{2} \sqrt{\frac{C'}{Q}} \left[\cos(\Psi/2) \frac{\partial^2 \Psi}{\partial \eta_1^2} - \frac{1}{2} \sin(\Psi/2) \left(\frac{\partial \Psi}{\partial \eta_1} \right)^2 \right], \tag{66c}$$

$$Q\Phi^3 - C'\Phi = -\frac{C'}{2} \sqrt{\frac{C'}{Q}} \cos(\Psi/2) \sin \Psi. \tag{66d}$$

By substituting from (66) in Eq. (64), we get

$$i \frac{\partial \Psi}{\partial \tau} + in_1 \frac{\partial \Psi}{\partial \zeta_1} - in_2 \frac{\partial \Psi}{\partial \eta_1} + \frac{1}{2} \left(\frac{\partial^2 \Psi}{\partial \zeta_1^2} - \frac{\partial^2 \Psi}{\partial \eta_1^2} \right) \\ - \frac{1}{4} \tan(\Psi/2) \left(\left(\frac{\partial \Psi}{\partial \zeta_1} \right)^2 - \left(\frac{\partial \Psi}{\partial \eta_1} \right)^2 \right) = -\frac{C'}{2} \sin \Psi. \tag{67}$$

Setting $\Psi = \Psi(\zeta)$ which is a function of one function ζ only and by substituting from (31) into (67), we easily see that

$$\left(i \frac{\partial \zeta}{\partial \tau} + in_1 \frac{\partial \zeta}{\partial \zeta_1} - in_2 \frac{\partial \zeta}{\partial \eta_1} + \frac{1}{2} \frac{\partial^2 \zeta}{\partial \zeta_1^2} - \frac{1}{2} \frac{\partial^2 \zeta}{\partial \eta_1^2} \right) \frac{d\Psi}{d\zeta} \\ + \left[\left(\frac{\partial \zeta}{\partial \zeta_1} \right)^2 - \left(\frac{\partial \zeta}{\partial \eta_1} \right)^2 \right] \left[\frac{1}{2} \frac{d^2 \Psi}{d\zeta^2} - \frac{1}{4} \tan(\Psi/2) \left(\frac{d\Psi}{d\zeta} \right)^2 \right] \\ = -\frac{C'}{2} \sin \Psi. \tag{68}$$

Some solutions of (68) obey the following system of equations:

$$i \frac{\partial \zeta}{\partial \tau} + in_1 \frac{\partial \zeta}{\partial \zeta_1} - in_2 \frac{\partial \zeta}{\partial \eta_1} = \frac{\partial^2 \zeta}{\partial \zeta_1^2} - \frac{\partial^2 \zeta}{\partial \eta_1^2} = 0, \\ \left(\frac{\partial \zeta}{\partial \zeta_1} \right)^2 - \left(\frac{\partial \zeta}{\partial \eta_1} \right)^2 = 1, \tag{69}$$

$$\frac{d^2 \Psi}{d\zeta^2} = -\frac{C'}{2} \sin \Psi, \quad \frac{d\Psi}{d\zeta} = \sqrt{2C'} \cos\left(\frac{\Psi}{2}\right) \tag{70}$$

Equation (70) is equivalent to a sine-Gordon equation, which has the well known soliton solution

$$\Psi = 4 \tan^{-1} \exp\left(\sqrt{\frac{C'}{2}} \zeta + \zeta_0\right) - \pi, \quad \zeta_0 = \text{constant}. \tag{71}$$

Applying (71) and (65) to (63), we obtain a soliton solution of the two-dimensional NLS equation (24) in the form

$$\Phi = \sqrt{\frac{C'}{Q}} \tanh\left(\sqrt{\frac{C'}{2}} \zeta + \zeta_0\right), \\ A = \sqrt{\frac{C'}{Q}} \tanh\left(\sqrt{\frac{C'}{2}} \zeta + \zeta_0\right) e^{i(c_0\tau + c_1\zeta_1 + c_2\eta_1)}, \tag{72}$$

where ζ denotes a solution of (70). As (70) has a large amount of solutions, (72) contains a vast variety of interesting solitons of the two-dimensional NLS equation.

Now we search for a kind of general solution of (67) in the form

$$\zeta = F(\xi_j) + m_0\tau + m_1\zeta_1 + m_2\eta_1, \\ \xi_j = b_{j0}\tau + b_{j1}\zeta_1 + b_{j2}\eta_1 + \varepsilon_j, \\ m_0, m_1, m_2, b_{j0}, b_{j1}, b_{j2}, \varepsilon_j = \text{constants} \tag{73}$$

where $F(\xi_j)$ denotes an arbitrary function of ξ_j . Combining (70) and (73) we easily obtain

$$i \frac{\partial \zeta}{\partial \tau} + in_1 \frac{\partial \zeta}{\partial \zeta_1} - in_2 \frac{\partial \zeta}{\partial \zeta_2} = i(m_0 + n_1m_1 - n_2m_2) \\ + i(b_{j0} + n_1b_{j1} - n_2b_{j2}) \frac{\partial F}{\partial \xi_j}, \tag{74a}$$

$$\begin{aligned} & \left(\frac{\partial \zeta}{\partial \zeta_1}\right)^2 + \left(\frac{\partial \zeta}{\partial \eta_1}\right)^2 \\ &= \left[(b_{j1})^2 - (b_{j2})^2 \right] \left(\frac{\partial F}{\partial \xi_j}\right)^2 + (2m_1 b_{j1} - 2m_2 b_{j2}) \frac{\partial F}{\partial \xi_j} \\ &+ m_1^2 - m_2^2 = 1, \end{aligned} \quad (74b)$$

$$\frac{\partial^2 \zeta}{\partial \zeta_1^2} = b_{j1} b_{k1} \frac{\partial^2 F}{\partial \xi_j \partial \xi_k} = 0, \quad \frac{\partial^2 \zeta}{\partial \eta_1^2} = b_{j2} b_{k2} \frac{\partial^2 F}{\partial \xi_j \partial \xi_k} = 0. \quad (74c)$$

Requiring $F(\xi_j)$ to be arbitrary leads to the conditions

$$\begin{aligned} b_{j0} + n_1 b_{j1} - n_2 b_{j2} &= 0, & m_0 + n_1 m_1 - n_2 m_2 &= 0, \\ b_{j1} b_{k1} - b_{j2} b_{k2} &= 0, & m_1 m_1 - m_2 m_2 &= 1, \\ m_1 b_{j1} - m_2 b_{j2} &= 0. \end{aligned} \quad (75)$$

The general solution (73) yields for (72) and (71) a rather general soliton solution of the two-dimensional NLS and sine–Gordon equations. They contain some interesting specific solutions, such as plane solitons, N multiple solitons, propagational breathers and quadratic solitons. We will discuss them respectively below.

(a) The plane solitons

This is a simple case. By taking all b_{ji} in (75) to be equal to zero, $F(\xi_j)$ becomes a constant so that $b_{j0} = b_{j1} = b_{j2} = 0$, $F(\xi_j) = F(\varepsilon_j) = \text{constants}$.

Summarizing we have (71) and (72)

$$\Psi = 4 \tan^{-1} \exp\left(\sqrt{\frac{C'}{2}} \zeta + \zeta_0\right) - \pi, \quad \zeta_0 = \text{const.} \quad (76)$$

$$A = \sqrt{\frac{C}{Q}} \tanh\left(\sqrt{\frac{C}{2}} \zeta + \zeta_0\right) e^{i(c_0 \tau + c_1 \zeta_1 + c_2 \eta_1)}, \quad (77)$$

where $\zeta = m_0 \tau + m_1 \zeta_1 + m_2 \eta_1 + \text{constant}$. In this case, (76) and (77) represent a hyperplanar soliton solution of the two-dimensional NLS and sine–Gordon equations. This solution is stable.

(b) N multiple soliton solutions

Let us select $F(\xi_j)$ in the form

$$F(\xi_j) = \ln \sum_{j=1}^N \exp(b_{j0} \tau + b_{j1} \zeta_1 + b_{j2} \eta_1 + \varepsilon_j). \quad (78)$$

Application of (73) and (78) leads to

$$\zeta = \ln \left(\sum_{j=1}^N \exp[D_{j0} \tau + D_{j1} \zeta_1 + D_{j2} \eta_1 + \varepsilon_j] \right) \quad (79)$$

Here $D_{j\alpha} = m_\alpha + b_{j\alpha}$, $\alpha = 0, 1, 2$. The conditions (75) are simplified to

$$D_{j0} + n_1 D_{j1} - n_2 D_{j2} = 0, \quad D_{j1} D_{k1} - D_{j2} D_{k2} = 1. \quad (80)$$

By direct calculation, we can easily prove that (79) is a solution of (69). Taking j equal to $1, 2, \dots, N$,

respectively, (79), (71) and (72) gives N multiple soliton solutions of the two-dimensional NLS and sine–Gordon equations.

(c) Propagational breathers

Considering $N = 2$ in (79) we get

$$\zeta = \ln \left(\frac{\exp(D_{10} \tau + D_{11} \zeta_1 + D_{12} \eta_1 + \varepsilon_1) + \exp(D_{20} \tau + D_{21} \zeta_1 + D_{22} \eta_1 + \varepsilon_2)}{\exp(D_{20} \tau + D_{21} \zeta_1 + D_{22} \eta_1 + \varepsilon_2)} \right), \quad (81)$$

which corresponds to a 2-soliton of the two-dimensional NLS equation. For the $2 + 1$ dimensional case, we set

$$\begin{aligned} D_{10} &= -D_{20}, & D_{11} &= -D_{21}, & D_{12} &= D_{22}, \\ \varepsilon_1 &= \ln S, & \varepsilon_2 &= \ln S + i\pi. \end{aligned} \quad (82)$$

Then (73) becomes

$$\zeta = \ln[2S \exp(D_{12} \eta_1) \sinh(D_{10} \tau + D_{11} \zeta_1)]. \quad (83)$$

and (80) gives the conditions

$$D_{11}^2 - D_{12}^2 = 1, \quad D_{10} + 2n_1 D_{11} = 0, \quad n_2 D_{12} = 0. \quad (84)$$

Taking D_{10} , D_{11} , S as imaginary numbers

$$D_{10} = iD_0, \quad D_{11} = iD_1, \quad 2S = i\lambda, \quad (85)$$

then (49) takes the form

$$\zeta = D_{12} \eta_1 + \ln[\lambda \sin(D_0 \tau + D_1 \zeta_1)]. \quad (86)$$

Then (71), (72) and (86) are a two-dimensional breather which propagates in the ζ_1 direction and has a constant value on the straight line $D_{12} \eta_1 = \text{constant}$. This is an interesting result.

(d) Quadratic solitons

We take the solution (69) in the form:

$$\begin{aligned} \zeta &= (b_{j0} \tau + b_{j1} \zeta_1 + b_{j2} \eta_1 + \varepsilon_j)^2 + m_0 \tau + m_1 \zeta_1 + m_2 \eta_1, \\ j &= 1, 2, \dots, N. \end{aligned} \quad (87)$$

It describes some general quadratic surfaces at any definite time. These quadratic solutions include all specific ones such as:

(i) The circular soliton

By choosing the constants of (87) as in (56), (87) becomes

$$\begin{aligned} \zeta &= \tau^2 + \zeta_1^2 + \eta_1^2 + (2\varepsilon_j b_{j1} + m_1) \zeta_1 \\ &+ (2\varepsilon_j b_{j2} + m_2) \eta_1 + (2\varepsilon_j b_{j0} + m_0) \tau + \varepsilon_j \varepsilon_j, \end{aligned} \quad (88)$$

which moves along a space line. Thus (88), (71) and (72) give circular solitons which are solutions of the two-dimensional NLS and sine–Gordon equations.

(ii) Elliptical solitons
Under the conditions (60), (87) becomes

$$\begin{aligned} \zeta = & b_{j0}b_{j0}\tau^2 + b_{j1}b_{j1}\zeta_1^2 + b_{j2}b_{j2}\eta_1^2 + (2\varepsilon_j b_{j1} + m_1)\zeta_1 \\ & + (2\varepsilon_j b_{j2} + m_2)\eta_1 + (2\varepsilon_j b_{j0} + m_0)\tau + \varepsilon_j \varepsilon_j, \\ & j = 1, 2, \dots, N. \end{aligned} \tag{89}$$

Equation (89) represents ellipses and (71) and (72) are thus elliptical shaped solitons.

(iii) Hyperbolic solitons

Let the constants satisfy (62), then from (87), we obtain some hyperbolas and (71) and (72) give the corresponding hyperbolic shaped solitons of the two-dimensional NLS and sine-Gordon equations.

5. Soliton stability

Equation (24) admits two-dimensional soliton solutions traveling at any acute angle relative to the group velocity of the wave packet. The amplitude of the traveling wave depends on both η and ζ . There may be a crossover from one region to another. Now we shall discuss the stability of the solutions of (24). Following Zakharov [32] and Berge [43], Eq. (24) can be obtained by deriving the functional relation $\delta L/\delta A^* = 0$ from the Lagrangian

$$L = \frac{1}{2}i \iint \left(A^* \frac{\partial A}{\partial \tau} - A \frac{\partial A^*}{\partial \tau} \right) d\zeta_1 d\eta_1 - H, \tag{90}$$

where A^* denotes the complex conjugate function of A where H and N are two integrals of motion for the elliptic case $P < 0$ as:

$$N = \iint |A|^2 d\zeta_1 d\eta_1, \tag{91}$$

$$H = \frac{1}{2} \iint \left(\Delta_1 \left| \frac{\partial A}{\partial \zeta_1} \right|^2 + \Delta_2 \left| \frac{\partial A}{\partial \eta_1} \right|^2 + \frac{Q}{2} |A|^2 \right) d\zeta_1 d\eta_1, \tag{92}$$

where the normalization N is the wave action and H is the Hamiltonian. Furthermore, one can show by direct calculation that (91) and (92) in (24) yields the virial equation

$$\frac{\partial^2 I}{\partial \tau^2} = \frac{\partial^2}{\partial \tau^2} \iint (\zeta_1^2 + \eta_1^2) |A|^2 d\zeta_1 d\eta_1 = 4H. \tag{93}$$

Here I , the moment of inertia of the wave packet, is

$$I = \iint (\zeta_1^2 + \eta_1^2) |A|^2 d\zeta_1 d\eta_1, \tag{94}$$

since H is a constant of motion. Therefore (93) may be integrated twice, and making use of (94), we have

$$\langle I \rangle = \frac{2\tau^2}{N} H + \gamma_1 \tau + \gamma_2, \tag{95}$$

where the average value of the moment of inertia I is given by

$$\langle I \rangle = \frac{I}{N} \tag{96}$$

and γ_1 and γ_2 are constants of integration given by the initial data:

$$\gamma_1 = \left. \frac{\partial \langle I \rangle}{\partial \zeta_1} \right|_{\tau=0} \quad \text{and} \quad \gamma_2 = \langle I \rangle|_{\tau=0}. \tag{97}$$

It should be noted that γ_2 is always positive. A sufficient condition for the finite-time collapse of I to zero is $H < 0$. Here $|A| \rightarrow 0$ everywhere except at the collapse point. The value of $|A|$ tends to infinity at the collapse point at the same time, due to the invariance of N . This implies that the L^2 norm of the solution is preserved as the collapse of its support proceeds. H can only be negative when $Q < 0$. It is possible to find certain initial conditions rendering $Q < 0$, implying $H < 0$. In particular, if $\gamma_1(0) = 0$ then $\langle I \rangle \rightarrow 0$ as τ approaches the time of collapse, $\tau_0 = (\gamma_2 N / 2H)^{1/2}$, to a point in the (ζ_1, η_1) plane. If, however, $H > 0$ then an additional forcing of the initial data is required for collapse to occur. However $\langle I \rangle \rightarrow 0$ in a finite time requires $\gamma_1 < 0$ and $\gamma_1^2 \geq (8H/N)\gamma_2$. Following Berkshire and Gibbon [44], two important results on collapse are

- (a) if a collapse occurs then it must do so in a finite time;
- (b) a collapse cannot occur unless the total angular momentum is zero.

We now define the angular momentum J as

$$\begin{aligned} J = & \iint (\zeta_1, \eta_1) \\ & \times \left(\frac{\Delta_1}{2i} \left(A^* \frac{\partial A}{\partial \zeta_1} - A \frac{\partial A^*}{\partial \zeta_1} \right), \frac{\Delta_2}{2i} \left(A^* \frac{\partial A}{\partial \eta_1} - A \frac{\partial A^*}{\partial \eta_1} \right) \right) d\zeta_1 d\eta_1. \end{aligned} \tag{98}$$

Total wave collapse will take place provided the blow-up singularity does not occur at times earlier than the collapse predicted by the virial theorem. Rasmussen and Rypdal [45] and Berge [43], have shown that the virial theorem cannot be used to predict the time of blow up, and it fails to give information about whether or not $\langle I \rangle$ is concentrated at the singular point. In an actual physical case, the dissipative effects will inhibit the earlier collapse from occurring. However, Berge [43] has shown that collapse will not take place in the hyperbolic two-dimensional case. It is interesting to note here that the corresponding Eq. (92) for the Hamiltonian H does not contain any nonlinear term for the hyperbolic case. We should like to point out here that for collapse to occur in finite time, the group-velocity rate and the nonlinear interaction coefficient Q must change sign as the quantities ρ , l , m and V_A are varied.

6. Discussion and summary

It has been shown that the evolution of the amplitude of the progressive waves in superposed fluids, in the presence of a magnetic field, is governed by a 2 + 1 dimensional

NLS equation. It was demonstrated that the waves can be unstable against modulation [34]. In this paper, we examined the Kelvin–Helmholtz MHD flow in the subcritical regime, when the velocity difference U is less than the critical velocity U_c . The phenomenon of nonlinear internal resonance for the case where fundamental and second-harmonic modes travel with the same phase velocity is investigated. A special scaling is used to determine the nonlinear interaction coefficients. The basic equations governing the MHD flow are given; and the standard NLS equations in $2 + 1$ dimensions in elliptic and hyperbolic cases are rederived. We found soliton solutions to the standard NLS equation in $2 + 1$ dimensions. We applied the function transformation method, thus the two-dimensional NLS equation is transformed to a sine–Gordon equation, which depends only on one function, ζ . We obtained rather general classes of solutions of the equation in ζ which leads to rather general soliton solutions of NLS. The latter contains quite interesting specific solutions, such as N multiple solitons, propagational breathers and quadratic solitons. Wu *et al.* [19–20] studied the solitary waves and discussed the possibility for applying their results to nonlinear waves observed by the Freja scientific satellite in the upper ionosphere [21]. Finally, the stability of the solution of the standard NLS equation in $2 + 1$ dimensions is discussed. The present analysis was applied by us previously to nonlinear dispersive instabilities in Rayleigh–Taylor magnetohydrodynamic flows [46].

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