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ScienceDirect
Journal of Algebra 305 (2006) 292–320
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The Goldie Theorem for H-semiprime algebras ☆

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Received 5 September 2005

Available online 28 July 2006

Communicated by J.T. Stafford

Abstract

The main result states that, under certain assumptions about a Hopf algebra H, every H-semiprime right Noetherian H-module algebra has a quasi-Frobenius classical right quotient ring. Another question treated in the paper is concerned with the extension of H-module structures to quotient rings. These results have an application to the semiprimeness problem for smash product algebras $A \# H$.

Keywords: The Goldie Theorem; Quotient rings; Hopf algebras; Module algebras; H-semiprime algebras; Smash products

Introduction

Let H be a Hopf algebra over a commutative ring k. An H-module algebra A is called H-*semiprime* if A contains no nonzero nilpotent H-stable ideals. We say that A is H-*simple* if A has no nonzero proper H-stable ideals and A is H-*semisimple* if it is isomorphic to a direct product of finitely many H-simple H-module algebras. This paper is devoted to the proof of the following main result:

☆ Both authors acknowledge support of the ESF program on Noncommutative Geometry. The first author thanks the Free University of Brussels VUB and the Mathematics Research Center of Warwick University for hospitality.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.06.030

Theorem 0.1. *Suppose H is admissible. Then every H-semiprime right Noetherian H-module algebra A has a quasi-Frobenius classical right quotient ring Q. Moreover, Q is an H-semisimple H-module algebra.*

The precise definition of admissible Hopf algebras given in Section 3 is somewhat technical. Its main purpose consists in selecting a sufficiently large class of Hopf algebras (see Proposition 4.4). When k is a field all finite-dimensional and all cocommutative Hopf algebras are admissible; any Hopf algebra with a bijective antipode generated by its admissible Hopf subalgebras is itself admissible, and so too is any Hopf algebra whose coradical is contained in an admissible Hopf subalgebra. Whether every Hopf algebra with a bijective antipode over a field is admissible remains unanswered.

The classical theorem proved by Goldie [13] characterizes rings having a semisimple Artinian classical quotient ring. In particular, the quotient ring Q in Theorem 0.1 is semisimple if and only if A is semiprime. In some cases the H-semiprime module algebras happen to be semiprime. This is obviously so when H is a group algebra. Another such case is given in Theorem 0.5(ii). Certainly one cannot expect this to hold in general. If k is a field and $\dim H < \infty$, then H^* is always an H-simple H-module algebra with respect to the left hit action, but H^* is not necessarily semisimple.

Dealing with the nonsemiprime module algebras presents a major challenge. As was discovered by L. Small [37], a right Noetherian ring R has a right Artinian classical right quotient ring if and only if every element of R, regular modulo the prime radical of R, is regular in R. Under the hypotheses of Theorem 0.1 Small's condition does not seem to be directly verifiable except when H is pointed. Although initial steps in the proof are based on modification of classical arguments, the existence of regular elements in right ideals of A turns out to be a very subtle question.

We employ the general localization technique [12]. Consider the largest filter $\mathcal{E}_H$ of essential right ideals of A such that the action of H on A is continuous with respect to corresponding topology. If all right ideals in $\mathcal{E}_H$ have zero left annihilators then A will be termed *right H-nonsingular*. If this is the case, $\mathcal{E}_H$ turns out to be a Gabriel topology. The corresponding localization $E_H(A)$ of A may be viewed as an H-analog of Johnson's quotient ring of a nonsingular ring and is always contained in the maximal quotient ring $Q_{\max}(A)$ constructed by Utumi [43].

The next result is one of steps in the proof of Theorem 0.1. To simplify the language we use the notion "*quasi-Frobenius algebra*" in the sense "quasi-Frobenius as a ring." Such rings are characterized by the property that they are Artinian and selfinjective on both sides.

Theorem 0.2. *Let H be admissible. If A is right H-nonsingular and right Noetherian then $E_H(A)$ is quasi-Frobenius and H-semisimple.*

In the proof of this theorem one first verifies that $E_H(A)$ is semiprimary, and then one has to deal with semiprimary H-module algebras (a ring R is *semiprimary* if its Jacobson radical $\mathrm{Jac}(R)$ is nilpotent and the factor ring $R/\mathrm{Jac}(R)$ is semisimple Artinian).

Theorem 0.3. *Let H be admissible. If A is semiprimary and H-semiprime then A is quasi-Frobenius and H-semisimple.*

When H is *finitely projective* (which means "finitely generated and projective as a k-module") this theorem can be deduced from the results in [36]. However, we provide an independent proof even in this case. It can be used to show more directly that the right coideal subalgebras of finite-dimensional Hopf algebras are quasi-Frobenius. This question was investigated by Masuoka [21] in an attempt to generalize the Nichols–Zoeller freeness theorem [30]. In the special case where H is the universal enveloping algebra of a Lie algebra Theorem 0.3 improves a result of Block [4, Corollary 8.3] (see also [39, Chapter 3]). None of the hypotheses in Theorems 0.1–0.3 can be omitted as one checks by looking at the case $H = 0$.

The H-module algebras for a finitely projective H may be interpreted as comodule algebras over the dual Hopf algebra H^*. If H^* is a group algebra, such a structure on A can be described in terms of gradings [7, Proposition 1.3]. Taking in Theorem 0.1 $k = \mathbb{Z}$, the ring of integers, we get

Corollary 0.4. *Let Γ be a finite group. Then every graded-semiprime Γ-graded right Noetherian ring has a quasi-Frobenius classical right quotient ring Q, the grading extends to Q, and Q is a direct product of finitely many graded-simple rings.*

It is interesting to compare this corollary with the existing results. The first version of Goldie's Theorem for graded rings was concerned with semiprime $\mathbb{Z}$-graded rings satisfying graded Goldie conditions [28, Proposition 9.2.3]. Recently Goodearl and Stafford [16] proved that, for an arbitrary abelian grading group, any graded-prime graded-Goldie ring can be localized with respect to an Ore set of homogeneous regular elements, and the quotient ring is graded-simple graded-Artinian (this is reproduced in [29, Theorem 8.4.5]). The latter theorem has an application in the classification of prime ideals of quantum group algebras (see [5, Chapter II.3]). Using finite duals of Hopf algebras the existence of Q can be generalized to some infinite groups Γ, but Q may fail to be Γ-graded.

The quotient rings in Theorems 0.1, 0.2 are H-module algebras. The possibility of extension of the H-module structure to various quotient rings was investigated in several articles. The main obstacle is that it may be not clear whether the action of H on A is continuous with respect to a filter of one-sided or two-sided ideals under consideration. For instance, Montgomery and Schneider verified the necessary conditions for Ore localizations, but only when H is pointed [25, Corollary 3.14]. The problem was also considered for Martindale quotient rings [9,24,25] and for maximal quotient rings [20].

We prove in Theorem 2.2 that the action always extends to Artinian classical quotient rings using a surprisingly simple observation that the regular elements of A induce regular elements of certain convolution algebras. Theorem 8.4 solves this problem for Martindale quotient rings under hypotheses similar to those in Theorem 0.1. Under the hypotheses of Theorem 0.2, $Q_{\max}(A) = E_H(A)$ by Proposition 6.5, so that $Q_{\max}(A)$ has an H-module structure.

The question about the semiprimeness of the smash product $A \# H$ has its origin in Fisher and Montgomery's work on skew group rings [11] and was raised by Cohen and Fischman for a semiprime A and a semisimple H [8]. A positive answer has been known under restrictions on either H (duals of group algebras, pointed cocommutative, semisolvable, cosemisimple triangular H) [7], [6], [25], [20] or the type of action (inner or X_H-inner) [3,25]. In a recent paper Linchenko, Montgomery and Small [19] discovered a connection between the semiprimeness of $A \# H$ and the invariance of the Jacobson radicals of H^*-module algebras. They proved that $A \# H$ with a semisimple H is semiprimitive under several assumptions about A. Especially the conclusion is true when A is H-semiprime, satisfies a polynomial identity and is either affine or algebraic over the ground field k, $\operatorname{char} k = 0$; when $\operatorname{char} k > 0$ further restrictions are needed.

A generalization of the latter results to Galois extensions of algebras appears in [27]. Our new result involves the Noetherian hypothesis:

Theorem 0.5. *Suppose that k is a field,* $\dim H < \infty$ *and A is H-semiprime right Noetherian.*

(i) *If H is semisimple then $A \# H$ is semiprime.*
(ii) *If H is cosemisimple then A is semiprime.*

A stronger formulation in Theorem 8.3 actually allows an arbitrary commutative ring k. If A is semiprime right Goldie and H is semisimple then $A \# H$ is still semiprime. The proof of this fact proposed by Rumynin [34] used the extension of H-module structures to quotient rings which was not correctly proved at that time.

Let us fix some notation. For a ring R denote by ${}_R\mathcal{M}$ and $\mathcal{M}_R$ the categories of left and right R-modules, by $\mathcal{E}(R)$ the set of all essential right ideals of R, by $\operatorname{lann}_R X$ and $\operatorname{rann}_R X$ the left and right annihilators of a subset X in R, by $\operatorname{udim} V$ the uniform dimension (Goldie rank) of a right module or a right ideal V (see, e.g., [22, Chapter 2]). If T is an overring of R let $T_R \in \mathcal{M}_R$ denote T regarded as a module with respect to right multiplications. Unless preceded by the prefix "left" or "right," an ideal is to be understood as a two-sided ideal.

Two assumptions about H will not be repeated in the statements of results. In Sections 2–8 H is assumed to have a family $\mathcal{F}$ of coalgebras which serve as substitutes for finite-dimensional coalgebras in case when k is a field. Properties of $\mathcal{F}$ are axiomatized in Section 2. In Sections 5–8 the antipode S of H is assumed to be bijective. Basic facts and definitions from Hopf algebra theory can be looked up in [23,40].

1. Convolution structures

Given a k-algebra A and a k-coalgebra C, let $[C, A] = \operatorname{Hom}_k(C, A)$ denote the convolution algebra and $\mathcal{M}^C$ the category of right C-comodules. If $U \in \mathcal{M}^C$ and $V \in \mathcal{M}_A$, then $[U, V] = \operatorname{Hom}_k(U, V)$ will be regarded as a right $[C, A]$-module with respect to the convolution action. For $\xi \in [C, A]$ and $\eta \in [U, V]$ one has

$$(\eta\xi)(u) = \sum_{(u)} \eta(u_{(0)})\xi(u_{(1)}), \quad u \in U,$$

where $u \mapsto \sum_{(u)} u_{(0)} \otimes u_{(1)}$ denotes the comodule structure map $U \to U \otimes C$ (all tensor products are taken over k unless specified otherwise). In the special case where $U = C$ is a comodule with respect to the comultiplication $\Delta : C \to C \otimes C$ and $V = A$ is a module with respect to right multiplications, the formula above gives the multiplication in $[C, A]$.

Assume further that A is a left H-module algebra. For $a \in A$ define $\delta_a, \tau_a \in [H, A]$ by the rules

$$\delta_a(h) = \varepsilon(h)a, \qquad \tau_a(h) = ha$$

where $h \in H$ and $\varepsilon : H \to k$ is the counit. The same notation will be used to denote the elements of $[C, A]$ obtained from δ_a, τ_a above by restricting them to a subcoalgebra $C \subset H$. The compatibility of the H-module structure with the multiplication in A means precisely that the map $\tau : A \to [H, A]$, $a \mapsto \tau_a$, is a homomorphism of unital algebras. So too is the map $\delta : A \to [H, A]$,

$a \mapsto \delta_a$. In fact δ is a special case of τ obtained by equipping an arbitrary algebra A with the trivial H-module structure. Every right $[H, A]$-module may be regarded as a right A-module either via δ or via τ. In particular,

$$(\eta\delta_a)(u) = \eta(u)a, \qquad (\eta\tau_a)(u) = \sum_{(u)} \eta(u_{(0)})(u_1 a)$$

for $\eta \in [U, V]$ and $u \in U$ when $U \in \mathcal{M}^H$ and $V \in \mathcal{M}_A$. There is also a right $[H, A]$-module structure on $U \otimes V$ defined by the rule

$$(u \otimes v)\xi = \sum_{(u)} u_{(0)} \otimes v\xi(Su_{(1)}), \quad u \in U,\ v \in V.$$

So

$$(u \otimes v)\delta_a = u \otimes va, \qquad (u \otimes v)\tau_a = \sum_{(u)} u_{(0)} \otimes v\big((Su_{(1)})a\big).$$

In the sequel we will tacitly use τ when considering $[U, V]$ and $U \otimes V$ as modules over A. If the coaction of H on U is trivial, so that $\sum_{(u)} u_{(0)} \otimes u_{(1)} = u \otimes 1$ for all $u \in U$, the two A-module structures defined on either $[U, V]$ or $U \otimes V$ coincide. In general we denote by U_{triv} the H-comodule which has the same underlying k-module as U but the trivial coaction of H. The A-module structures in $[U_{\text{triv}}, V]$ and $U_{\text{triv}} \otimes V$ are those that derive from δ.

Denote by ${}_H\mathcal{M}_A$ the category whose objects are right A-modules equipped with a left H-module structure such that $h(va) = \sum_{(h)} (h_{(1)}v)(h_{(2)}a)$ for all $h \in H$, $v \in M$, $a \in A$. The morphisms in ${}_H\mathcal{M}_A$ are maps respecting both module structures. It is possible to identify ${}_H\mathcal{M}_A$ with the category of left modules over the smash product algebra $A^{\text{op}} \# H^{\text{cop}}$ where the pair A^{op}, H^{cop} is obtained from A, H by taking the opposite multiplication in A and comultiplication in H. Define

$$\theta_v \in [H, M], \quad \theta_v(h) = hv.$$

The compatibility of the two module structures on M means precisely that $\theta_{va} = \theta_v\tau_a$ for all $v \in M$ and $a \in A$. This may be also expressed by saying that the map $\theta : M \to [H, M]$, $v \mapsto \theta_v$, is an $\mathcal{M}_A$-morphism.

Lemma 1.1. *Let $U \in \mathcal{M}^H$ and $V, W \in \mathcal{M}_A$.*

(i) $\operatorname{Hom}_A(U \otimes V, W) \cong \operatorname{Hom}_A(V, [U, W])$ *canonically.*
(ii) *$U \otimes V$ is projective in $\mathcal{M}_A$ whenever so is V and U is projective in $\mathcal{M}_k$.*
(iii) *$[U, W]$ is injective in $\mathcal{M}_A$ whenever so is W and U is flat in $\mathcal{M}_k$.*
(iv) *$U \otimes V$ is finitely generated in $\mathcal{M}_A$ whenever so are V in $\mathcal{M}_A$ and U in $\mathcal{M}_k$.*
(v) *$H \otimes V$ and $[H, W]$ are objects of ${}_H\mathcal{M}_A$ in a natural way.*

Proof. (i) Under the canonical bijection $\mathrm{Hom}_k(U \otimes V, W) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(U, W))$ the A-module homomorphisms $V \to [U, W]$ correspond precisely to the k-linear maps $\varphi : U \otimes V \to W$ such that

$$\varphi(u \otimes va) = \sum_{(u)} \varphi(u_{(0)} \otimes v)(u_{(1)}a)$$

for all $u \in U$, $v \in V$ and $a \in A$. This is equivalent to another identity defining the A-module homomorphisms $U \otimes V \to W$:

$$\varphi\Big(\sum_{(u)} u_{(0)} \otimes v\big((Su_{(1)})a\big)\Big) = \varphi(u \otimes v)a.$$

(ii) In view of (i) the functor $\mathrm{Hom}_A(U \otimes V, ?)$ is exact since $\mathrm{Hom}_A(V, ?)$ and $[U, ?]$ are exact.

(iii) This follows from the exactness of functors $\mathrm{Hom}_A(?, W)$ and $U \otimes ?$.

(iv) Clearly $U \otimes V$ is a factor module of $U \otimes F$ where F is a finitely generated free A-module. It suffices to prove the finite generatedness for $U \otimes F$, and this reduces to the case where $F \cong A$ in $\mathcal{M}_A$. Now $U \otimes A \cong U_{\mathrm{triv}} \otimes A$ by Lemma 1.2 below. The latter A-module is generated by the set $X \otimes 1$ where X is any finite set generating U as a k-module.

(v) It is straightforward to check that the A-module structures already in use are compatible with the H-module structures defined by the rules

$$h(u \otimes v) = hu \otimes v, \qquad (h\eta)(u) = \eta(uh),$$

where $h, u \in H$, $v \in V$ and $\eta \in [H, W]$. □

Lemma 1.2. *Let $U \in \mathcal{M}^H$ and $M \in {}_H\mathcal{M}_A$.*

(i) *$U \otimes M \cong U_{\mathrm{triv}} \otimes M$ in $\mathcal{M}_A$.*
(ii) *If S is bijective then $[U, M] \cong [U_{\mathrm{triv}}, M]$ in $\mathcal{M}_A$.*
(iii) *The map $H \otimes M \to M$, $h \otimes v \mapsto hv$, is a morphism in ${}_H\mathcal{M}_A$.*

Proof. (i) The rule $u \otimes v \mapsto \sum_{(u)} u_{(0)} \otimes u_{(1)}v$ defines a k-linear transformation Φ of $U \otimes M$ which has inverse $u \otimes v \mapsto \sum_{(u)} u_{(0)} \otimes S(u_{(1)})v$ and intertwines the two A-module structures. Indeed,

$$\Phi\Big(\sum_{(u)} u_{(0)} \otimes v\big((Su_{(1)})a\big)\Big) = \sum_{(u)} u_{(0)} \otimes (u_{(1)}v)a,$$

that is, $\Phi((u \otimes v)\tau_a) = \Phi(u \otimes v)\delta_a$ for all $u \in U$, $v \in M$ and $a \in A$.

(ii) We have to find a bijective k-linear transformation Ψ of $[U, M]$ such that $\Psi(\eta\delta_a) = \Psi(\eta)\tau_a$ for all $\eta \in [U, M]$ and $a \in A$. It is defined, together with its inverse, by the formulas

$$\Psi(\eta)(u) = \sum_{(u)} u_{(1)}\eta(u_{(0)}), \qquad \Psi^{-1}(\eta)(u) = \sum_{(u)} S^{-1}(u_{(1)})\eta(u_{(0)})$$

for $\eta \in [U, M]$ and $u \in U$. Now

$$\Psi(\eta\delta_a)(u) = \sum_{(u)} u_{(1)}\big(\eta(u_{(0)})a\big) = \sum_{(u)} \big(u_{(1)}\eta(u_{(0)})\big)(u_{(2)}a) = \big(\Psi(\eta)\tau_a\big)(u).$$

(iii) The map considered here clearly commutes with the actions of H. This map is also a morphism in $\mathcal{M}_A$ as it corresponds, under the bijection of Lemma 1.1(i), to the $\mathcal{M}_A$-morphism $\theta : M \to [H, M]$ defined earlier. □

The *trace ideal* of a right module V over a ring R is an ideal of R defined as

$$T_V = \sum_{f \in \operatorname{Hom}_R(V,R)} f(V).$$

Lemma 1.3. *Suppose that $\mathcal{X}$ is a set of right A-modules such that for each $V \in \mathcal{X}$ the A-module $H \otimes V$ is a sum of its submodules isomorphic to epimorphic images of modules from $\mathcal{X}$. Then $I = \sum_{V \in \mathcal{X}} T_V$ is an H-stable ideal of A. In particular, the ideal T_M is H-stable whenever $M \in {}_H\mathcal{M}_A$.*

Proof. Clearly I is an ideal since each T_V is an ideal. If $f : V \to A$ is a morphism in $\mathcal{M}_A$ then so also is the composite

$$g : H \otimes V \xrightarrow{\mathrm{id} \otimes f} H \otimes A \to A,$$

where the second map is afforded by the H-module structure on A. This follows from Lemma 1.2(iii). If $\pi : W \to H \otimes V$ is any $\mathcal{M}_A$-morphism with $W \in \mathcal{X}$ then $g(\pi(W)) \subset I$ by definition of I. The assumption about $H \otimes V$ entails $\operatorname{Im} g \subset I$, that is, $Hf(V) \subset I$. Letting f and V vary, we deduce that I is stable under H.

If $M \in {}_H\mathcal{M}_A$, then the set $\mathcal{X} = \{M\}$ satisfies the hypothesis of the lemma. Indeed, $H \otimes M \cong H_{\mathrm{triv}} \otimes M$ by Lemma 1.2(i). The A-module $H_{\mathrm{triv}} \otimes M$ is the sum of its submodules $h \otimes M$ taken for different $h \in H$. Each of these submodules is an epimorphic image of M. □

A ring R is called *semiperfect* if $R/\operatorname{Jac}(R)$ is semisimple Artinian and all idempotents in $R/\operatorname{Jac}(R)$ are liftable to idempotents in R. If R is semiperfect then all finitely generated R-modules admit a projective cover [33, Theorem 2.8.40]. Any semiprimary ring is semiperfect. Moreover, a semiprimary ring is left and right perfect in the sense of Bass [1], that is, the projective covers exist for arbitrary modules. Denote by $\operatorname{Max} R$ the finite set of maximal ideals of a semiperfect ring R. If $V \in \mathcal{M}_R$ is finitely generated, then the R-module V/VP is semisimple of finite length for any $P \in \operatorname{Max} R$ since the factor ring R/P is simple Artinian. One may define

$$r_P(V) = \frac{\operatorname{length} V/VP}{\operatorname{length} R/P}.$$

Lemma 1.4. *Suppose A is semiperfect and $M \in {}_H\mathcal{M}_A$ is finitely generated over A. Let $m = \max\{r_P(M) \mid P \in \operatorname{Max} A\}$ and $\Omega = \{P \in \operatorname{Max} A \mid r_P(M) < m\}$. For each $P \in \operatorname{Max} A$ denote by F_P the projective cover in $\mathcal{M}_A$ of a simple A/P-module. Suppose also that H coincides with the sum of its right coideals which are finitely generated projective in $\mathcal{M}_k$. Then $I = \sum_{P \in \Omega} T_{F_P}$ is an H-stable ideal of A.*

Proof. We will verify that the set $\mathcal{X} = \{F_P \mid P \in \Omega\}$ satisfies the hypothesis of Lemma 1.3. Each $r_P(M)$ is a nonnegative rational number. We can find an integer $t > 0$ such that $r_P(M^t) = r_P(M)t \in \mathbb{Z}$ for all $P \in \operatorname{Max} A$. Replacing M with M^t, the direct sum of t copies of M, we may assume that $r_P(M) \in \mathbb{Z}$ for all P. This does not affect the set Ω. Now $m \in \mathbb{Z}$ as well.

Every projective module $F \in \mathcal{M}_A$ is a direct sum of indecomposable projectives F_P with $P \in \operatorname{Max} A$. Denote by $e_P(F)$ the multiplicity with which F_P occurs in the decomposition of F. Clearly $e_P(F) = \operatorname{length} F/FP$ and $e_P(A) = \operatorname{length} A/P$ in particular. Assume now that F stands for the projective cover of M in $\mathcal{M}_A$. Then $F/FP \cong M/MP$, and so

$$e_P(F) = \operatorname{length} M/MP = e_P(A)r_P(M) \leqslant e_P(A)m = e_P\big(A^m\big)$$

for each P. This shows that F is a direct summand of A^m. Hence $A^m \cong F \oplus G$ where $G \in \mathcal{M}_A$ is projective with $e_P(G) = e_P(A^m) - e_P(F)$. It is clear from the inequality above that $e_P(G) > 0$ if and only if $P \in \Omega$.

Let $U \in \mathcal{M}^H$, and suppose that U is finitely generated projective in $\mathcal{M}_k$. For each prime ideal $\mathfrak{p}$ of k one defines $\operatorname{rk}_{\mathfrak{p}}(U)$, the rank of U at $\mathfrak{p}$, as the dimension of the vector space $U \otimes \kappa(\mathfrak{p})$ over the field of fractions $\kappa(\mathfrak{p})$ of the commutative domain $k/\mathfrak{p}$. Take now $\mathfrak{p} = \{\lambda \in k \mid \lambda A \subset P\}$ where $P \in \operatorname{Max} A$. If $\lambda \in k \setminus \mathfrak{p}$ then $\lambda A + P = A$ since P is a maximal ideal of A. Consequently A/P is an algebra over $\kappa(\mathfrak{p})$. We claim that there is an equality

$$r_P(U \otimes M) = \operatorname{rk}_{\mathfrak{p}}(U) r_P(M).$$

Using Lemma 1.2(i), we get

$$(U \otimes M)/(U \otimes M)P \cong U_{\mathrm{triv}} \otimes (M/MP) \cong \big(U \otimes \kappa(\mathfrak{p})\big) \otimes_{\kappa(\mathfrak{p})} M/MP$$

in $\mathcal{M}_A$. The A-module on the right-hand side is a direct sum of $\operatorname{rk}_{\mathfrak{p}}(U)$ copies of M/MP, whence the desired formula. As $A \in {}_H\mathcal{M}_A$, this formula also applies with A replacing M. Hence $r_P(U \otimes A) = \operatorname{rk}_{\mathfrak{p}}(U)$.

The A-module $U \otimes M$ is an epimorphic image of $U \otimes F$, and $U \otimes F$ is projective in $\mathcal{M}_A$ by Lemma 1.1. It follows that the projective cover of $U \otimes M$ in $\mathcal{M}_A$ is a direct summand of $U \otimes F$, whence for any $P \in \operatorname{Max} A$ with $r_P(M) = m$ one has

$$e_P(U \otimes F) \geqslant e_P(A)r_P(U \otimes M) = e_P(A) \operatorname{rk}_{\mathfrak{p}}(U)m.$$

Now $U \otimes A^m$ is also projective in $\mathcal{M}_A$ and

$$e_P\big(U \otimes A^m\big) = e_P(A)r_P\big(U \otimes A^m\big) = e_P(A) \operatorname{rk}_{\mathfrak{p}}(U)m.$$

We see that $e_P(U \otimes F) \geqslant e_P(U \otimes A^m)$ whenever $P \notin \Omega$. On the other hand, $U \otimes A^m \cong (U \otimes F) \oplus (U \otimes G)$, and therefore

$$e_P\big(U \otimes A^m\big) = e_P(U \otimes F) + e_P(U \otimes G).$$

Hence $e_P(U \otimes G) = 0$ whenever $P \notin \Omega$. In other words the projective A-module $U \otimes G$ is a direct sum of modules F_P with $P \in \Omega$. The same is true for any $U \otimes F_Q$ with $Q \in \Omega$ because F_Q is a direct summand of G for such Q.

The assumption about H implies that the A-module $H \otimes F_Q$ is a sum of images of the $U \otimes F_Q$'s with U a right coideal of H, finitely generated and projective in $\mathcal{M}_k$. From the description of $U \otimes F_Q$ above we deduce that $H \otimes F_Q$, when $Q \in \Omega$, is a sum of epimorphic images of modules F_P with $P \in \Omega$. Thus Lemma 1.3 may be applied. □

2. Extension of module structures to classical quotient rings

A *subcoalgebra* $C \subset H$ may be defined as a k-submodule such that the k-linear maps $C^{\otimes n} \to H^{\otimes n}$ induced by the inclusion $C \to H$ are injective for $n = 2, 3$ and $\Delta(C) \subset C \otimes C$. The injectivity of maps is automatic when both C and H are flat in $\mathcal{M}_k$. We will need a family $\mathcal{F}$ of subcoalgebras satisfying two conditions:

$(\mathcal{F})_1$ *each $C \in \mathcal{F}$ is finitely generated projective in $\mathcal{M}_k$,*
$(\mathcal{F})_2$ *every finite subset of H is contained in some $C \in \mathcal{F}$.*

Such a family $\mathcal{F}$ is directed by inclusion: given $C', C'' \in \mathcal{F}$ there exists $C \in \mathcal{F}$ containing both C' and C''. Thus the existence of $\mathcal{F}$ implies that $H \cong \varinjlim_{\mathcal{F}} C$ is flat in $\mathcal{M}_k$. In particular $U = H$ can be used in Lemma 1.1(iii). Also, H satisfies the hypothesis of Lemma 1.4 since each $C \in \mathcal{F}$ may be regarded as a right coideal.

If H is finitely projective, then $\mathcal{F} = \{H\}$ obviously satisfies $(\mathcal{F})_1$ and $(\mathcal{F})_2$. If k is a field and H any Hopf algebra, then $\mathcal{F}$ may be taken to be the family of all finite-dimensional subcoalgebras. In the sequel we assume that $\mathcal{F}$ is given.

For each subcoalgebra $C \subset H$ the two algebra homomorphisms $\delta, \tau : A \to [C, A]$ can be defined as in Section 1. An element x is called *left regular* (respectively *right regular*) in a ring R if $\operatorname{lann}_R x = 0$ (respectively $\operatorname{rann}_R x = 0$). If both annihilators are zero then x is *regular*.

Lemma 2.1. *Let $C \in \mathcal{F}$ and $D = [C, k]$.*

(i) *If S is bijective then there exists a k-linear bijective transformation Ψ of $[C, A]$ such that $\Psi(\xi\delta_a) = \Psi(\xi)\tau_a$ for all $a \in A$ and $\xi \in [C, A]$.*
(ii) *There exists a k-linear bijective transformation Φ of $[C, A]$ such that $\Phi(\delta_a\xi) = \tau_a\Phi(\xi)$ for all $a \in A$ and $\xi \in [C, A]$.*
(iii) *If a is right regular in A then so are both δ_a and τ_a in $[C, A]$.*
(iv) *$[C, A] \cong A \otimes D$ as algebras and $[C, A] = \tau(A)D$.*

Proof. (i) This is a special case of Lemma 1.2(ii) applied with $U = C$ and $M = A$.

(ii) Put $\Phi(\xi)(c) = \sum_{(c)} c_{(1)}\xi(c_{(2)})$ and $\Phi^{-1}(\xi)(c) = \sum_{(c)} S(c_{(1)})\xi(c_{(2)})$ where $c \in C$ and $\xi \in [C, A]$. The identity in (ii) is verified by evaluating its left- and right-hand sides at c to $\sum_{(c)} c_{(1)}a \cdot c_{(2)}\xi(c_{(3)})$.

(iii) It follows from (ii) that τ_a is right regular if and only if so is δ_a. Now $(\delta_a\xi)(c) = a\xi(c)$ for each $c \in C$. If a is right regular then the equality $\delta_a\xi = 0$ entails $\xi = 0$.

(iv) We may view D as a subalgebra of $[C, A]$. Since $(\xi\delta_a)(c) = \xi(c)a = (\delta_a\xi)(c)$ for all $\xi \in D$, $a \in A$ and $c \in C$, the assignment $a \otimes \xi \mapsto \delta_a\xi$ defines an algebra homomorphism $A \otimes D \to [C, A]$. It is bijective by $(\mathcal{F})_1$. Note that $\Phi(\xi) = \xi$, and thus $\Phi(\delta_a\xi) = \tau_a\xi$, when $\xi \in D$. The second assertion in (iv) follows now from (ii). □

Theorem 2.2. *Suppose that A has a right Artinian classical right quotient ring Q. Then the H-module structure on A has a unique extension to Q with respect to which Q becomes a left H-module algebra.*

Proof. Denote by U the set of all regular elements of A. The hypotheses mean that A is contained in Q as a subring, all elements of U are invertible in Q, and each element of Q can be presented as au^{-1} for suitable $u \in U$ and $a \in A$. Note that the center of A is contained in the center of Q, and so Q is an algebra over k. By the universality property of classical quotient rings [22, Lemma 2.1.4] there exists at most one algebra homomorphism $\varphi: Q \to [H, Q]$ rendering commutative the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \tau\ } & [H, A] \\ {\scriptstyle \text{can.}}\big\downarrow & & \big\downarrow{\scriptstyle \text{can.}} \\ Q & \xrightarrow{\ \varphi\ } & [H, Q], \end{array}$$

and such a φ does exist provided that τ_u is invertible in $[H, Q]$ for each $u \in U$. In view of the ring isomorphism $[H, Q] \cong \varprojlim_{C \in \mathcal{F}} [C, Q]$ the latter condition is equivalent to τ_u being invertible in $[C, Q]$ for each $C \in \mathcal{F}$. Let us fix C. By Lemma 2.1(iv) $[C, Q] \cong Q \otimes D$ where $D = [C, k]$ (the H-module structure is not needed here). Since C is finitely generated projective in $\mathcal{M}_k$, so is D, whence $[C, Q]$ is finitely generated in $\mathcal{M}_Q$. As Q is a right Artinian ring, so is $[C, Q]$.

Recall that an element of a right Artinian ring is invertible if and only if it is right regular [22, Proposition 3.1.1]. We are led therefore to checking that τ_u is right regular in $[C, Q]$ for each $u \in U$. Suppose that $\tau_u \xi = 0$ for some $\xi \in [C, Q]$. Picking finitely many generators $c_1, \ldots, c_n$ of C as a k-module, we can find $t \in U$ such that $\xi(c_i)t \in A$ for each i (common denominator property [22, Proposition 2.1.16]). Then $\xi(C)t \subset A$, that is, $\xi\delta_t \in [C, A]$ (we view $[C, A]$ as a subring of $[C, Q]$). Since $\tau_u \xi \delta_t = 0$, Lemma 2.1(iii) entails $\xi\delta_t = 0$. This may be rewritten as $\xi(C)t = 0$, and the desired conclusion $\xi = 0$ is immediate because t is invertible in Q.

Since φ is a homomorphism of unital algebras, the formula $hq = \varphi(q)(h)$ for $h \in H$ and $q \in Q$ defines a measuring of H on Q, that is, $h1_Q = \varepsilon(h)1_Q$ and $h(xy) = \sum_{(h)} h_{(1)}x \cdot h_{(2)}y$ for all $h \in H$ and $x, y \in Q$ [40, Proposition 7.0.1]. The condition that this measuring is a left H-module structure can be expressed by means of two equalities $\pi \circ \varphi = \mathrm{id}_Q$ and $\alpha \circ \varphi = \beta \circ \varphi$ where

$$\pi : [H, Q] \to Q \quad \text{and} \quad \alpha, \beta : [H, Q] \to [H \otimes H, Q]$$

are defined by the formulas

$$\pi(\xi) = \xi(1), \qquad \alpha(\xi)(g \otimes h) = \xi(gh), \qquad \beta(\xi)(g \otimes h) = g\xi(h)$$

for $\xi \in [H, Q]$ and $g, h \in H$. When restricted to A, the equalities above do hold since the measuring on Q extends the original H-module structure on A. The equalities will have to be fulfilled on the whole Q by the uniqueness of extensions of homomorphisms as soon as we make certain that π, α, β are all algebra homomorphisms. For π and α this follows from the functoriality of convolution algebras since these maps are induced by coalgebra homomorphisms $k \to H$ and

$H \otimes H \to H$. The equality $\beta(\xi\eta) = \beta(\xi)\beta(\eta)$ for $\xi, \eta \in [H, Q]$ is verified straightforwardly by computing the values at $g \otimes h$ of the two maps in the formula as

$$g\left(\sum_{(h)} \xi(h_{(1)})\eta(h_{(2)})\right) = \sum_{(g)}\sum_{(h)} g_{(1)}\xi(h_{(1)}) \cdot g_{(2)}\eta(h_{(2)}). \qquad \square$$

3. The semiprimary case

This section will provide a proof of Theorem 0.3. First comes the definition of admissible Hopf algebras.

Definition. A Hopf algebra H will be called *right admissible* if H has a family of subcoalgebras satisfying conditions $(\mathcal{F})_1$, $(\mathcal{F})_2$ and if for every semiprimary left H-module algebra B, every $M \in {}_H\mathcal{M}_B$ and every $P \in \operatorname{Max} B$ satisfying $MP = M$ there exists an H-stable ideal I of B such that $I \subset P$ and $MI = M$. If both H and H^{cop} are right admissible Hopf algebras then H will be called *admissible*.

If H is admissible then it has a bijective antipode S since H^{cop} is requested to be a Hopf algebra. In this case H^{cop} is also admissible. Furthermore, A^{op} is a left H^{cop}-module algebra which is semiprimary whenever A is semiprimary; A^{op} is H^{cop}-semiprime whenever A is H-semiprime. Any statement proved for A and H under such hypotheses can be translated into a statement about A^{op} and H^{cop}. In particular, the right conditions in Theorems 0.1, 0.2 and other results may be replaced with their left versions.

We mention two simple facts concerning semiprimary rings which will be used in the proofs without comment. If R is semiprimary, then every nonzero R-module contains a simple submodule and a maximal submodule. Every homomorphic image $\pi(R)$ of a semiprimary ring R is itself semiprimary; in fact the Jacobson radical of $\pi(R)$ coincides with $\pi(\operatorname{Jac}(R))$.

Lemma 3.1. *Suppose that R is a ring and V a right R-module having a projective cover F. Denote by T_F the trace ideal of F.*

(i) *$I = T_F$ is the smallest ideal of R such that $VI = V$.*
(ii) *If R is a k-algebra and D another k-algebra which is finitely generated projective as a k-module, then $J = T_F \otimes D$ is the smallest ideal of $R \otimes D$ satisfying $(V \otimes D)J = V \otimes D$.*

Proof. (i) There exists an epimorphism $\pi : F \to V$ in $\mathcal{M}_R$ whose kernel is a superfluous submodule of F so that $\pi(G) \neq V$ for every proper submodule $G \subset F$. It follows that the equality $VI = V$ for any ideal I of R is equivalent to $FI = F$. If $FI = F$, then $g(F)I = g(F)$ for each $\mathcal{M}_R$-morphism $g : F \to R$; hence $T_F I = T_F$. In this case $T_F \subset I$. Conversely, $FT_F = F$ by the dual basis property of projective modules [33, Proposition 3.4.20].

(ii) First note that $\operatorname{Ker}\pi \otimes X$ is a superfluous R-submodule of $F \otimes X$ for every finitely generated projective k-module X. Indeed, when $X = Y \oplus Z$ in $\mathcal{M}_k$ the previous assertion holds for X if and only if it holds for both Y and Z (see [33, Lemma 3.4.39]). This reduces the verification to the case $X = k$ where the assertion is clear. We may now take $X = D$. In this case $\pi \otimes \mathrm{id} : F \otimes D \to V \otimes D$ is an epimorphism in $\mathcal{M}_{R\otimes D}$, and we have just observed that its kernel is superfluous even as an R-submodule of $F \otimes D$. Thus $F \otimes D$ is a projective cover of $V \otimes D$ in $\mathcal{M}_{R\otimes D}$.

For any two finitely generated projective k-modules X, Y the canonical map

$$\mathrm{Hom}_R(F, R) \otimes \mathrm{Hom}_k(X, Y) \to \mathrm{Hom}_R(F \otimes X, R \otimes Y)$$

is bijective. Indeed, since we deal here with a natural transformation of two additive functors in X and Y, it suffices to check the bijectivity in the case $X = Y = k$ which presents no difficulty. Taking $X = Y = D$, we see that every $\mathcal{M}_R$-morphism $F \otimes D \to R \otimes D$ has images in $T_F \otimes D$. Conversely, $g(F) \otimes D$ is the image of the $\mathcal{M}_{R\otimes D}$-morphism $g \otimes \mathrm{id} : F \otimes D \to R \otimes D$ for any $\mathcal{M}_R$-morphism $g : F \to R$. Thus the trace ideal of the right $R \otimes D$-module $F \otimes D$ coincides with $T_F \otimes D$. The proof is completed by applying (i). □

Lemma 3.2. *Suppose that R is a semiperfect ring and $\Lambda \subset \mathrm{Max}\, R$ any subset.*

(i) *There exists a smallest ideal I of R such that $\Lambda = \{P \in \mathrm{Max}\, R \mid I \subset P\}$. In fact $I = \sum_{P\in\Omega} T_{F_P}$ where F_P is the projective cover in $\mathcal{M}_R$ of a simple R/P-module and $\Omega = \mathrm{Max}\, R \setminus \Lambda$.*
(ii) *If R is a left H-module algebra which contains an H-stable ideal K satisfying $\Lambda = \{P \in \mathrm{Max}\, R \mid K \subset P\}$ then there exists a smallest element in the set of such ideals.*

Proof. (i) Denote by V the direct sum of a full set of pairwise nonisomorphic simple right R-modules with annihilators in Ω. Then $F = \bigoplus_{P\in\Omega} F_P$ is a projective cover of V and $T_F = \sum_{P\in\Omega} T_{F_P}$. For an ideal I of R one has $VI = V$ if and only if $I \not\subset P$ for each $P \in \Omega$. By Lemma 3.1 there exists a smallest I with this property. For each $P \in \Lambda$ one has $VP = V$, whence $I \subset P$. Thus I is the required ideal.

(ii) Let I be as in (i). Denote by K the smallest H-stable ideal of R which contains I (in fact $K = R(HI)$ since HI is a right ideal by Lemma 1.2(iii)). If J is any H-stable ideal of R such that $\Lambda = \{P \in \mathrm{Max}\, R \mid J \subset P\}$, then $I \subset J$, and it follows that $K \subset J$. Hence K satisfies the required conditions. □

Lemma 3.3. *If A is semiprimary and H-semiprime then A has a minimal nonzero H-stable ideal.*

Proof. Take a maximal subset $\Lambda \subset \mathrm{Max}\, A$ such that $\Lambda \neq \mathrm{Max}\, A$ and there exists an H-stable ideal K of A satisfying $\Lambda = \{P \in \mathrm{Max}\, A \mid K \subset P\}$. By Lemma 3.2 we may assume that K is minimal with respect to the previous property. If now T is any H-stable ideal A properly contained in K then $T \subset P$ for all $P \in \mathrm{Max}\, A$ by the maximality of Λ. In this case $T \subset \mathrm{Jac}(A)$, and so T is nilpotent, yielding $T = 0$. Thus K is the desired ideal. □

Lemma 3.4. *Suppose that A is semiprimary and H-semiprime. If H is right admissible, then A is right selfinjective and H-semisimple.*

Proof. Let K denote a minimal nonzero H-stable ideal of A (Lemma 3.3). Since A is H-semiprime, K is not nilpotent. Then $K \not\subset P$ for some $P \in \mathrm{Max}\, A$, whence $K + P = A$ by the maximality of P. We may view $M = A/K$ as an object of ${}_H\mathcal{M}_A$. As H is right admissible and $MP = P$, there exists an H-stable ideal I of A such that $I \subset P$ and $MI = M$. This implies that $I \neq A$ and $K + I = A$, whence $K \not\subset I$, and therefore $K \cap I = 0$ by the minimality of K. Now $A \cong A/I \times A/K$ as H-module algebras. Since K is mapped bijectively onto A/I, every

H-stable ideal of A/I has the form $(J+I)/I$ where J is an H-stable ideal of A contained in K. The minimality of K shows that A/I is H-simple. The algebra A/K has fewer maximal ideals than A. Proceeding by induction on the cardinality of the set Max A, we deduce that A is a direct product of finitely many H-simple algebras. Each of the latter certainly remains semiprimary.

To prove the selfinjectivity, we may assume that A is H-simple. Pick any nonzero injective $E \in \mathcal{M}_A$. By Lemma 1.1 $M = [H, E]$ is injective in $\mathcal{M}_A$ and is an object of ${}_H\mathcal{M}_A$. Since k is an $\mathcal{M}_k$-direct summand of H, the evaluation at $1 \in H$ yields a surjection $M \to E$. Hence $M \neq 0$. If I is any H-stable ideal of A then either $I = 0$ or $I = A$, and the equality $MI = M$ holds only when $I = A$. Since H is right admissible, we conclude that $MP \neq M$ for every $P \in \operatorname{Max} A$. This means that every simple right A-module is an epimorphic image of M. However, the semiprimary algebra A contains a simple right ideal. Consequently $\operatorname{Hom}_A(M, A) \neq 0$. Then the trace ideal T_M is nonzero as well. By Lemma 1.3 T_M is H-stable, yielding $T_M = A$. Hence there exist finitely many $\mathcal{M}_A$-morphisms $f_1, \ldots, f_n : M \to A$ such that $\sum f_i(M) = A$. They determine an epimorphism $M^n \to A$ in $\mathcal{M}_A$ which has to split. Since M^n is injective in $\mathcal{M}_A$, so is A. □

Proof of Theorem 0.3. By Lemma 3.4 A is right selfinjective and H-semisimple. Now we may replace A, H with A^{op}, H^{cop} and conclude that A^{op} is right selfinjective too, whence A is left selfinjective. Every left perfect left and right selfinjective ring is known to be quasi-Frobenius [17,31]. Since semiprimary rings are left perfect, A is quasi-Frobenius. □

4. Admissible Hopf algebras

In this section we aim to find interesting classes of admissible Hopf algebras. Let us start with auxiliary lemmas.

Lemma 4.1. *With the hypotheses and notation as in Lemma* 1.4 *we have*:

(i) *For* $P \in \operatorname{Max} A$ *the inclusion* $I \subset P$ *is equivalent to* $P \notin \Omega$.
(ii) $I \neq A$ *is always true, and* $I \neq 0$ *if and only if* $\Omega \neq \emptyset$.
(iii) *If* $M \neq 0$ *then* Ω *contains all* $P \in \operatorname{Max} A$ *such that* $MP = M$.

Proof. Part (i) follows from Lemma 3.2(i) applied with $R = A$ and $\Lambda = \operatorname{Max} R \setminus \Omega$. By definition of Ω in Lemma 1.4 the set Λ consists of those $P \in \operatorname{Max} A$ for which $r_P(M)$ attains the maximum value m. Therefore $\Lambda \neq \emptyset$. By (i) $I \subset P$ for any $P \in \Lambda$; hence $I \neq A$. Since the indecomposable projectives F_P are $\mathcal{M}_A$-direct summands of A, it holds $T_{F_P} \neq 0$ for all $P \in \operatorname{Max} A$, and (ii) is clear. If $M \neq 0$ then $M/MP \neq 0$ for at least one $P \in \operatorname{Max} A$ by Nakayama's Lemma, so that $m > 0$. If now P is such that $M = MP$ then $r_P(M) = 0$, yielding $P \in \Omega$. □

Lemma 4.2. *Suppose that A is semiprimary and H-semiprime. Suppose also that every nonzero H-stable right ideal of A contains a nonzero H-stable finitely generated right ideal of A. Then A is H-semisimple. If there exists $P \in \operatorname{Max} A$ containing no nonzero H-stable ideals of A then A is H-simple.*

Proof. Let $L = \operatorname{lann}_A K$ where K is a minimal nonzero H-stable ideal of A (Lemma 3.3). We first prove that the equality $L = 0$ implies $K = A$, so that A is H-simple in this case. Put $\Omega_0 = \{P \in \operatorname{Max} A \mid K \subset P\}$. Every semiprimary ring satisfies DCC on finitely generated right ideals [38, Proposition VIII.5.5]. In particular, K contains a minimal nonzero H-stable finitely generated right ideal M of A. The hypotheses of the lemma ensure that M is minimal in the set of all nonzero H-stable right ideals. We may regard M as an object of ${}_H\mathcal{M}_A$. Let Ω and I be defined as in Lemma 1.4, and let $\Lambda = \operatorname{Max} A \setminus \Omega$.

Now $M^2 \neq 0$ by the H-semiprimeness of A; hence $M^2 = M$ by the minimality of M. If $P \in \Omega_0$ then $MP = M$ since $M \subset K \subset P$. Lemma 4.1 yields $\Omega_0 \subset \Omega$ and $\Lambda = \{P \in \operatorname{Max} A \mid I \subset P\}$. The ideal IK is H-stable and $IK \subset P$ for all $P \in \Lambda \cup \Omega_0$. On the other hand, $K \not\subset P$ when $P \in \Lambda$. It follows that IK is properly contained in K, and therefore $IK = 0$ by the minimality of K. This yields $I \subset L$, and the assumption $L = 0$ implies $I = 0$. By Lemma 4.1 $\Omega = \emptyset$; hence $\Omega_0 = \emptyset$ too. The last equality is only possible when $K = A$.

In general L is an H-stable ideal of A [9, Corollary 2]. Clearly $L \neq A$ since $K \neq 0$. We may view A/L as a semiprimary left H-module algebra. Let T be any H-stable ideal of A. If $T^2 \subset L$ then $(TK)^3 \subset T^2K = 0$, and the H-semiprimeness of A yields $TK = 0$, that is $T \subset L$. This shows that A/L is H-semiprime. If $TK \subset L$ then $(TK)^3 \subset TK^2 = 0$, and we deduce similarly that $T \subset L$. It follows that $\pi(K)$ has zero left annihilator in A/L where $\pi : A \to A/L$ is the canonical projection. In particular $\pi(K) \neq 0$. Every H-stable ideal of A/L contained in $\pi(K)$ is equal to $\pi(J)$ for some H-stable ideal J of A contained in K. The minimality of K implies that $\pi(K)$ is a minimal nonzero H-stable ideal of A/L.

The previous step in the proof applied to A/L in place of A gives $\pi(K) = A/L$. This means that $K + L = A$. Moreover, $K \cap L = 0$ by the minimality of K. It follows that $A \cong A/K \times A/L$ as H-module algebras. Every $P \in \operatorname{Max} A$ contains exactly one of the ideals K, L. Therefore the set $\operatorname{Max} A/K$ has smaller cardinality than $\operatorname{Max} A$. Proceeding by induction we may assume that A/K is a direct product of finitely many H-simple algebras. The same holds then for A since A/L is H-simple. Finally, if both K and L are nonzero then every $P \in \operatorname{Max} A$ contains a nonzero H-stable ideal. This proves the final assertion of the lemma. □

Lemma 4.3. *Suppose that A is semiprimary. Let $M \in {}_H\mathcal{M}_A$, and let I be the smallest ideal of A such that $MI = M$. If H is right admissible then I is H-stable.*

Proof. Put $\Lambda = \{P \in \operatorname{Max} A \mid I \subset P\}$. If $P \in \Lambda$, then $MP = M$, and since A is right admissible, there exists an H-stable ideal P_H of A such that $P_H \subset P$ and $MP_H = M$. Taking the product of these ideals P_H for different $P \in \Lambda$, we get an H-stable ideal K of A such that $MK = M$ and $K \subset J$ where $J = \bigcap_{P\in\Lambda} P$. Clearly $I \subset K$ by the minimality of I. Now J/I is the Jacobson radical of the semiprimary ring A/I, and so J/I is nilpotent. Hence $K^n \subset J^n \subset I$ for some integer $n > 0$. Since $MK^n = M$, we get $K^n = I$ by the minimality of I. Clearly K^n is H-stable. □

Proposition 4.4. *H is admissible in each of the following cases*:

(a) *H is finitely projective,*
(b) *H is cocommutative,*
(c) *H is generated by its admissible Hopf subalgebras and S is bijective,*
(d) *$H = \bigcup H_i$ where $H_0 \subset H_1 \subset \cdots$ is a wedge filtration with H_0 an admissible Hopf subalgebra.*

Proof. If H satisfies any of the assumptions (a)–(d), then so does H^{cop}. Note also that S is bijective, so that H^{cop} is a Hopf algebra. In case (a) this follows from [32, Proposition 4]. In case (b) $S^2 = \mathrm{id}$ [40]. In case (d) S is bijective on H_0 since H_0 is admissible; the inclusion map

$H_0 \to H$ is then an invertible element in $[H_0^{\mathrm{cop}}, H]$, and the identity map $H \to H$ is invertible in $[H^{\mathrm{cop}}, H]$ by [42, Lemma 14]. It suffices therefore to check that H is right admissible. Let A be an arbitrary semiprimary left H-module algebra and $M \in {}_H\mathcal{M}_A$. Suppose that $MP = M$ for some $P \in \operatorname{Max} A$.

(a) Let P_H be the largest H-stable ideal of A contained in P. Suppose that $MP_H \neq M$. Replacing A, P, M with A/P_H, P/P_H, M/MP_H, respectively, we may assume that $M \neq 0$ and P contains no nonzero H-stable ideals of A. Then the product of any two nonzero H-stable ideals cannot be contained in P, showing that A is H-semiprime. Also, $(Hx)A$ is an H-stable finitely generated right ideal of A for any $x \in A$. Thus the hypotheses of Lemma 4.2 are fulfilled, and we conclude that A is H-simple.

Since M contains a maximal A-submodule, there exists an epimorphism $M \to V$ in $\mathcal{M}_A$ for some simple $V \in \mathcal{M}_A$. By Lemma 1.1 $H \otimes V \in {}_H\mathcal{M}_A$. Furthermore, $H \otimes V$ is an epimorphic image of $H \otimes M$, and $H \otimes M \cong H_{\mathrm{triv}} \otimes M$ in $\mathcal{M}_A$ by Lemma 1.2. It follows that $(H \otimes M)P = H \otimes M$ and $(H \otimes V)P = H \otimes V$. We have $H \otimes V \neq 0$ since there is a surjection $\varepsilon \otimes \mathrm{id} : H \otimes V \to V$. By Lemma 1.1 $H \otimes V$ is finitely generated in $\mathcal{M}_A$. We may replace M with $H \otimes V$ and apply Lemma 1.4 to obtain an H-stable ideal I of A which satisfies $0 \neq I \neq A$ in view of Lemma 4.1. This contradicts the H-simplicity of A. Thus $MP_H = M$.

In the proof of (b), (c), (d) we assume that I is the smallest ideal of A such that $MI = M$ (see Lemma 3.1). Then $I \subset P$, and we have to show that I is H-stable.

(b) We have $[U, M]I = [U, M]$ for every $U \in \mathcal{M}^H$, finitely generated and projective in $\mathcal{M}_k$. Lemma 1.2(ii) reduces the verification of this equality to the case when H coacts trivially on U. Since $[k_{\mathrm{triv}}, M] \cong M$ in $\mathcal{M}_A$ and U is a direct summand of a finitely generated free k-module, the right A-module $[U_{\mathrm{triv}}, M]$ is a direct summand of M^n for some integer $n > 0$. The desired equality follows now from $MI = M$.

Let $C \in \mathcal{F}$ and $D = [C, k]$. By Lemma 2.1 $[C, A] \cong A \otimes D$. We have similarly $[C, M] \cong M \otimes D$. Under these identifications the convolution action of $[C, A]$ on $[C, M]$ is expressed as

$$(v \otimes \eta)(a \otimes \xi) = va \otimes \eta\xi \quad \text{where } v \in M,\ a \in A,\ \xi, \eta \in D.$$

By Lemma 3.1 there exists a smallest ideal J of $[C, A]$ such that $[C, M]J = [C, M]$, and moreover $J = [C, I] \cong I \otimes D$. For $a \in I$ the element $a \otimes 1 \in I \otimes D$ corresponds to $\delta_a \in [C, I]$.

We claim that $J \subset K$ where we put $K = [C, A]\tau(I)$. In view of the minimality condition for J it suffices to check that K is an ideal of $[C, A]$ and that $[C, M]K = [C, M]$. The last equality does hold in view of the observation at the beginning of the proof applied with $U = C$ (in fact A operates in $[C, M]$ via $\tau : A \to [C, A]$). By definition K is a left ideal of $[C, A]$. It is also clear that $K\tau(A) \subset K$ since $\tau(I)\tau(A) \subset \tau(IA)$ and $IA = I$. Since C is cocommutative, D is contained in the center of $[C, A]$. Hence $KD \subset K$ as well, and it follows from Lemma 2.1(iv) that $K[C, A] \subset K$, as required.

Let Ψ be the bijective transformation of $[C, A]$ from Lemma 2.1(i). Recall that $\Psi(\xi\delta_a) = \Psi(\xi)\tau_a$ for all $\xi \in [C, A]$ and $a \in A$. Since J and K are generated as left ideals of $[C, A]$, respectively, by the elements δ_a and τ_a with $a \in I$, we arrive at $\Psi(J) = K$. Hence $\Psi^{-1}(K) = J$, and the inclusion $J \subset K$ entails $\Psi^{-1}(J) \subset J$. The explicit formula for Ψ^{-1} given in the proof of Lemma 1.2(ii) shows that

$$\Psi^{-1}(\delta_a)(c) = \sum_{(c)} S^{-1}(c_{(2)})\delta_a(c_{(1)}) = \sum_{(c)} S^{-1}(c_{(2)})\varepsilon(c_{(1)})a = S^{-1}(c)a$$

for all $c \in C$. If $a \in I$ then $\delta_a \in J$, whence $\Psi^{-1}(\delta_a) \in J$. Thus $S^{-1}(C)a \subset I$. Now $H = \bigcup_{C \in \mathcal{F}} S^{-1}(C)$ by $(\mathcal{F})_2$, and it follows that $HI \subset I$.

(c) By Lemma 4.3 I is stable under every admissible Hopf subalgebra of H. As H is generated by those subalgebras, I is H-stable.

(d) The hypothesis here means that $H_i = H_0 \wedge H_{i-1}$ for all $i > 0$ where the wedge operation is as defined in [40, Chapter 9]. Thus

$$H_i = \{h \in H \mid \Delta(h) \in \operatorname{Im}(H_0 \otimes H \to H \otimes H) + \operatorname{Im}(H \otimes H_{i-1} \to H \otimes H)\}.$$

By Lemma 4.3 I is H_0-stable. We check by induction that $H_i I \subset I$ for all $i > 0$. Suppose that $H_{i-1} I \subset I$ and $h \in H_i$. Then

$$h(ab) = \sum_{(h)} h_{(1)} a \cdot h_{(2)} b \in (H_0 I)A + A(H_{i-1} I) \subset I$$

for all $a, b \in I$. This shows that $H_i I^2 \subset I$. Finally, $I^2 = I$. This follows from the minimality of I since $MI^2 = M$. □

5. Equivariant version of Goldie's topology

Denote by $\mathcal{E}_H(A)$ the set of all right ideals I of A such that for each $h \in H$ one has $hJ \subset I$ for a suitable $J \in \mathcal{E}(A)$. Taking $h = 1$ in this definition, we deduce that $\mathcal{E}_H(A) \subset \mathcal{E}(A)$. We omit the indication of A from the notations $\mathcal{E}(A)$, $\mathcal{E}_H(A)$ when no ambiguity arises. For any right ideal I of A and a subcoalgebra $C \subset H$ put

$$I_C = \tau^{-1}([C, I]) = \{x \in A \mid Cx \subset I\}.$$

Since $\tau : A \to [C, A]$ is an algebra homomorphism and $[C, I]$ is a right ideal of $[C, A]$, it is clear that I_C is a right ideal of A.

Lemma 5.1. *A right ideal I of A is in $\mathcal{E}_H$ if and only if $I_C \in \mathcal{E}$ for each $C \in \mathcal{F}$. Moreover, $I_C \in \mathcal{E}_H$ whenever $I \in \mathcal{E}_H$.*

Proof. Suppose that $I \in \mathcal{E}_H$. Given $C \in \mathcal{F}$ and $h \in H$, we have $C = kX$ for a suitable finite subset X. The subset $Xh \subset H$ is finite as well. Since $\mathcal{E}$ is closed under finite intersections of right ideals, there exists $J \in \mathcal{E}$ such that $gJ \subset I$ for all $g \in Xh$. In this case $ChJ \subset I$, that is, $hJ \subset I_C$. This establishes the inclusion $I_C \in \mathcal{E}_H \subset \mathcal{E}$. The converse is clear since every element of H is contained in some $C \in \mathcal{F}$. □

It is immediate from the definition that I_H is the largest H-stable right ideal of A contained in I. If H is finitely projective, then $H \in \mathcal{F}$. In this case Lemma 5.1 says that $I \in \mathcal{E}_H$ if and only if $I_H \in \mathcal{E}$.

Recall that a *Gabriel topology* on a ring R is any set $\mathcal{G}$ of right ideals of R satisfying the four conditions listed below where I, J are assumed to be right ideals of R and we use the notation $(I : a) = \{x \in R \mid ax \in I\}$:

(T1) If $J \in \mathcal{G}$ and $J \subset I$ then $I \in \mathcal{G}$.
(T2) If $I, J \in \mathcal{G}$ then $I \cap J \in \mathcal{G}$.

(T3) If $I \in \mathcal{G}$ then $(I:a) \in \mathcal{G}$ for each $a \in R$.
(T4) If $J \in \mathcal{G}$ and $(I:a) \in \mathcal{G}$ for all $a \in J$ then $I \in \mathcal{G}$.

In fact (T2), (T3) suffice to make R into a topological ring with respect to a topology in which $\mathcal{G}$ is a neighborhood base at 0.

From now on we have to assume that S is bijective. When $H = k$ is the trivial Hopf algebra, one has $\mathcal{E}_H = \mathcal{E}$. The elements $x \in A$ such that $\operatorname{rann}_A x \in \mathcal{E}$ constitute an ideal $\operatorname{Sing} A$ of A called the right singular ideal. By analogy we define the *right H-singular ideal*

$$\operatorname{Sing}_H A = \{x \in A \mid \operatorname{rann}_A x \in \mathcal{E}_H\}$$

and say that A is *right H-nonsingular* if $\operatorname{Sing}_H A = 0$. As will shortly follow from (T1), A is right H-nonsingular if and only if $xI \neq 0$ for every $0 \neq x \in A$ and $I \in \mathcal{E}_H$.

More generally, for each $V \in \mathcal{M}_A$ denote by $\operatorname{Sing}_H V \subset V$ the subset consisting of all elements annihilated by a right ideal in $\mathcal{E}_H$. It will follow from (T2), (T3), that $\operatorname{Sing}_H V$ is a submodule of V.

Proposition 5.2.

(i) *$\mathcal{E}_H$ always satisfies* (T1)–(T3).
(ii) *All elements of H operate continuously on A with respect to the $\mathcal{E}_H$-topology.*
(iii) $\operatorname{Sing}_H A$ *is the largest H-stable ideal of A contained in* $\operatorname{Sing} A$.
(iv) *If A is right H-nonsingular then $\mathcal{E}_H$ satisfies* (T4).

Proof. (i) First, it is well known that $\mathcal{E}$ satisfies (T1)–(T3). Consequently for any $I \in \mathcal{E}$ and any finitely generated k-submodule $V \subset A$ there exists $K \in \mathcal{E}$ such that $VK \subset I$. Indeed, we may take $K = (I:e_1) \cap \cdots \cap (I:e_n)$ where $e_1, \ldots, e_n$ is any system of generators for V.

Obviously $\mathcal{E}_H$ satisfies (T1). Suppose $I, J \in \mathcal{E}_H$, $a \in A$ and $C \in \mathcal{F}$. By Lemma 5.1 $I_C, J_C \in \mathcal{E}$. We have $CL \subset I \cap J$ where $L = I_C \cap J_C \in \mathcal{E}$. This verifies (T2). As $V = S^{-1}(C)a \subset A$ is a finitely generated k-submodule, there exists $K \in \mathcal{E}$ such that $VK \subset I_C$. Then

$$a \cdot cK = \sum_{(c)} c_{(2)}\bigl(S^{-1}(c_{(1)})a \cdot K\bigr) \subset C(VK) \subset CI_C \subset I$$

for all $c \in C$. This yields $CK \subset (I:a)$, proving (T3).

(ii) Given $h \in H$ and $I \in \mathcal{E}_H$ we can find $J \in \mathcal{E}_H$ such that $hJ \subset I$. This follows from Lemma 5.1 since $h \in C$ for some $C \in \mathcal{F}$.

(iii) That $\operatorname{Sing}_H A$ is an ideal follows from (T1)–(T3). Since $\mathcal{E}_H \subset \mathcal{E}$, one has $\operatorname{Sing}_H A \subset \operatorname{Sing} A$. Let $x \in A$ and $I = \operatorname{rann}_A x$. We claim that $\operatorname{rann}_A Cx = I_{S(C)}$ for $C \in \mathcal{F}$. If $a \in I_{S(C)}$ then $S(C)a \subset I$, thus

$$cx \cdot a = \sum_{(c)} c_{(1)}\bigl(x \cdot S(c_{(2)})a\bigr) = 0$$

for all $c \in C$. Conversely, if $a \in \operatorname{rann}_A Cx$ and $c \in C$ then

$$x \cdot S(c)a = \sum_{(c)} S(c_{(1)})(c_{(2)}x \cdot a) = 0,$$

so that $S(C)a \subset I$. Note that $\operatorname{rann}_A Cx = \bigcap_{y \in Y} \operatorname{rann}_A y$ where Y is any finite set of generators for the k-module Cx. It follows that $Cx \subset \operatorname{Sing} A$ if and only if $I_{S(C)} \in \mathcal{E}$. Hence $Hx \subset \operatorname{Sing} A$ if and only if the previous inclusions hold for all $C \in \mathcal{F}$, that is, if and only if $I \in \mathcal{E}_H$ (we have to apply here Lemma 5.1 replacing $\mathcal{F}$ with the family of coalgebras $\{S(C) \mid C \in \mathcal{F}\}$ which enjoys the same properties). This shows that $\operatorname{Sing}_H A$ is stable under action of H and contains every H-stable k-submodule of $\operatorname{Sing} A$.

(iv) Suppose that $\operatorname{Sing}_H A = 0$ and I, J satisfy the hypothesis of (T4) for $\mathcal{G} = \mathcal{E}_H$. Let $C \in \mathcal{F}$ and $0 \neq x \in A$. Since $J_C \in \mathcal{E}$ by Lemma 5.1, there exists $a \in A$ such that $0 \neq y \in J_C$ for $y = xa$. Put $K = (I : e_1) \cap \cdots \cap (I : e_n)$ where $e_1, \ldots, e_n$ is any system of generators for the k-submodule $Cy \subset J$. We have $(I : e_i) \in \mathcal{E}_H$ for each $i = 1, \ldots, n$ since $e_i \in J$. Then $K \in \mathcal{E}_H$ as well. On the other hand, $(Cy)K \subset I$ by the definition of J, and it follows that

$$c(yK_C) = \sum_{(c)} c_{(1)}y \cdot c_{(2)}K_C \subset (Cy)K \subset I$$

for all $c \in C$, whence $yK_C \subset I_C$. Note that $yK_C \subset xA$ and $yK_C \neq 0$ since $K_C \in \mathcal{E}_H$ by Lemma 5.1 and $\operatorname{Sing}_H A = 0$. We conclude that $I_C \cap xA \neq 0$. Hence $I_C \in \mathcal{E}$ for any C, and $I \in \mathcal{E}_H$ by Lemma 5.1. □

Lemma 5.3. *If A is H-semiprime and satisfies ACC on right annihilators, then any $I \in \mathcal{E}_H$ has zero left and right annihilators. In particular,* $\operatorname{Sing}_H A = 0$.

Proof. The ideal $\operatorname{Sing} A$ is nilpotent by [22, Lemma 2.3.4]. Hence $\operatorname{Sing}_H A$ is an H-stable nilpotent ideal in view of Proposition 5.2, and the H-semiprimeness of A yields $\operatorname{Sing}_H A = 0$. This means that $\operatorname{lann}_A I = 0$ for any $I \in \mathcal{E}_H$.

Pick now $I \in \mathcal{E}_H$ such that $K = \operatorname{rann}_A I$ is maximal possible. Note that K is an ideal of A. If $J \in \mathcal{E}_H$ then $I \cap J \in \mathcal{E}_H$ and $K \subset \operatorname{rann}_A(I \cap J)$. It follows that $\operatorname{rann}_A(I \cap J) = K$ by the choice of I. This proves that $\operatorname{rann}_A J \subset K$ for any $J \in \mathcal{E}_H$. Given any $h \in H$, we can find $C \in \mathcal{F}$ such that $h \in C$. Then

$$I_C \cdot S(h)K = \sum_{(h)} S(h_{(1)})(h_{(2)}I_C \cdot K) \subset H(IK) = 0.$$

We deduce that $S(h)K \subset \operatorname{rann}_A I_C \subset K$ since $I_C \in \mathcal{E}_H$ by Lemma 5.1. Since S is bijective, K is an H-stable ideal. Then the ideal $L = \operatorname{lann}_A K$ is H-stable too. Since $(K \cap L)^2 = 0$, we get $K \cap L = 0$ by the H-semiprimeness of A. However, $L \in \mathcal{E}$ since $I \subset L$. It follows that $K = 0$. □

Lemma 5.4. *Suppose that R is a subring of a ring T such that* $\operatorname{udim} T_R < \infty$. *Then* $xT + \operatorname{rann}_T x$ *is an essential submodule of T_R for any $x \in T$ with* $\operatorname{rann}_T x = \operatorname{rann}_T x^2$.

Proof. This is a straightforward generalization of the well-known special case with $T = R$. Suppose that $V \subset T_R$ is a submodule such that $V \cap (xT + \operatorname{rann}_T x) = 0$. Then any relation $\sum_{i=0}^n x^i v_i = 0$ with an integer $n \geqslant 0$ and $v_0, \ldots, v_n \in V$ implies $v_i = 0$ for all i (proceed by induction on i observing that $v_i \in xT + \operatorname{rann}_T x$ once it is known already that $v_j = 0$ for all $j < i$). In other words, $\sum_{i=0}^\infty x^i V$ is a right R-submodule of T isomorphic to a direct sum of infinitely many copies of V. The finiteness of the uniform dimension entails $V = 0$. □

Lemma 5.5. *Suppose that* $\operatorname{udim} A_A < \infty$. *Then* $uA + \operatorname{rann}_A u \in \mathcal{E}_H$ *for any* $u \in A$ *with* $\operatorname{rann}_A u = \operatorname{rann}_A u^2$. *In particular,* $uA \in \mathcal{E}_H$ *whenever* u *is right regular in* A.

Proof. Put $I = uA + \operatorname{rann}_A u$. We want to apply Lemma 5.4 with $T = [C, A]$ and $R = \tau(A)$ where $C \in \mathcal{F}$ and $\tau : A \to T$ is the ring homomorphism defined in Section 1. If $U \in \mathcal{M}^H$ is finitely generated in $\mathcal{M}_k$ then $[U, A] \in \mathcal{M}_A$ has finite uniform dimension. Indeed, $[U, A] \cong [U_{\mathrm{triv}}, A]$ in $\mathcal{M}_A$ by Lemma 1.2(ii); hence any epimorphism $k^n \to U$ in $\mathcal{M}_k$ for some integer $n > 0$ induces a monomorphism $[U, A] \to [k^n_{\mathrm{triv}}, A] \cong A^n$ in $\mathcal{M}_A$. By [22, Corollary 2.2.10] $\operatorname{udim} A^n = n(\operatorname{udim} A) < \infty$, proving the claim. Taking $U = C$, we conclude that $\operatorname{udim} T_R < \infty$.

Let $x = \delta_u \in T$. The functor $\operatorname{Hom}_k(C, ?)$ from $\mathcal{M}_k$ to $\mathcal{M}_k$ takes the left multiplication by u on A to the left multiplication by x on $T = \operatorname{Hom}_k(C, A)$. As this functor is exact by projectivity of C in $\mathcal{M}_k$, the image and kernel of the left multiplication by x may be identified with $[C, uA]$ and $[C, \operatorname{rann}_A u]$, respectively. Hence

$$xT + \operatorname{rann}_T x = [C, I].$$

Since $x^2 = \delta_{u^2}$, we deduce that $\operatorname{rann}_T x^2 = [C, \operatorname{rann}_A u^2] = \operatorname{rann}_T x$ in a similar way. The hypotheses of Lemma 5.4 thus hold, and $[C, I]$ is then an essential right R-submodule of T.

Let $0 \neq b \in A$. If $\tau_b = 0$ in T then $b \in I_C$. If $\tau_b \neq 0$ then $\tau_b R \cap [C, I] \neq 0$; since $\tau_b R = \tau(bA)$, there exists $a \in bA$ such that $0 \neq \tau_a \in [C, I]$. In the latter case $0 \neq a \in I_C$. Thus $I_C \cap bA \neq 0$ in any case, and so $I_C \in \mathcal{E}$. Lemma 5.1 completes the proof. □

6. The quotient ring

Assume in this section that $\operatorname{Sing}_H A = 0$. Then $\mathcal{E}_H(A)$ is a Gabriel topology, and one can use all notions and constructions available for those topologies.

Every Gabriel topology $\mathcal{G}$ on a ring R corresponds to a hereditary torsion theory on $\mathcal{M}_R$ [38, Theorem VI.5.1]. The *$\mathcal{G}$-torsion submodule* of a right R-module M consists of all its elements annihilated by a right ideal in $\mathcal{G}$. One says that M is *$\mathcal{G}$-torsion* (respectively *$\mathcal{G}$-torsion-free*) if the $\mathcal{G}$-torsion submodule of M coincides with M (respectively with 0). We are concerned only with the case when R is $\mathcal{G}$-torsion-free. Under this assumption the *localization* of R with respect to $\mathcal{G}$ is a ring defined to be

$$R_{\mathcal{G}} = \varinjlim_{I \in \mathcal{G}} \operatorname{Hom}_R(I, R).$$

The product of elements represented by $\mathcal{M}_R$-morphisms $\alpha : I \to R$ and $\beta : J \to R$ with $I, J \in \mathcal{G}$ is represented by $\alpha \circ \beta|_K$ where $K \in \mathcal{G}$ is such that $K \subset J$ and $\beta(K) \subset I$.

There is also a different description of $R_{\mathcal{G}}$. Denote by $E(R)$ the injective hull of R in $\mathcal{M}_R$ and by $Q_{\max}(R) \subset E(R)$ the R-submodule consisting of all elements $x \in E(R)$ such that $\varphi(x) = 0$ for every $\varphi \in \operatorname{End}_R E(R)$ vanishing on R. According to [15, Theorem 2.29] there is a unique ring structure on $Q_{\max}(R)$ compatible with its R-module structure. This ring is called the *maximal right quotient ring* of R. Put

$$E_{\mathcal{G}}(R) = \{x \in E(R) \mid \text{there exists } I \in \mathcal{G} \text{ such that } xI \subset R\}.$$

Thus $R \subset E_{\mathcal{G}}(R) \subset E(R)$ and $E_{\mathcal{G}}(R)/R$ is precisely the $\mathcal{G}$-torsion submodule of the right R-module $E(R)/R$. (In fact, $E_{\mathcal{G}}(R)$ is the $\mathcal{G}$-injective hull of R [38, Proposition IX.2.2].) We mention several standard properties of quotient rings.

Lemma 6.1. *Suppose that R is $\mathcal{G}$-torsion-free and $Q = E_{\mathcal{G}}(R)$. Then*:

(i) *The right R-modules $E(R)$ and Q are $\mathcal{G}$-torsion-free.*
(ii) *Q is a subring of $Q_{\max}(R)$ isomorphic to $R_{\mathcal{G}}$.*
(iii) *A right ideal K of Q belongs to $\mathcal{E}(Q)$ if and only if $K \cap R \in \mathcal{E}(R)$.*
(iv) *For an annihilator right ideal K of Q and $y \in Q$ one has $y \in K$ if and only if $yR \cap R \subset K$.*
(v) *If R is right Noetherian then Q is right Goldie.*
(vi) *The set $\mathcal{G}^e$ of right ideals I of Q such that $I \cap R \in \mathcal{G}$ is a Gabriel topology, and Q coincides with its own localization with respect to $\mathcal{G}^e$.*
(vii) *The center of R is contained in the center of $Q_{\max}(R)$.*

Proof. (i) Every essential extension of a $\mathcal{G}$-torsion-free module is itself $\mathcal{G}$-torsion-free. Clearly $E(R)$ and Q are essential extensions of R in $\mathcal{M}_R$.

(ii) We have $\mathrm{Hom}_R(Q/R, E(R)) = 0$ since Q/R is $\mathcal{G}$-torsion. This implies that $\varphi \in \mathrm{End}_R\, E(R)$ vanishes on Q whenever $\varphi(R) = 0$. Hence $Q \subset Q_{\max}(R)$. Suppose that $x, y \in Q$. There exists $J \in \mathcal{G}$ such that $yJ \subset R$, and we have $xyJ \subset xR \subset Q$. This shows that the R-module $(xyR + Q)/Q$ is $\mathcal{G}$-torsion. So also is $(xyR + Q)/R$ since the class of torsion modules is closed under extensions. Hence $xy \in Q$.

Note that $Q = \bigcup_{I \in \mathcal{G}} E_I$ where $E_I = \{x \in E(R) \mid xI \subset R\}$. There is a compatible family of additive bijections $\theta_I : E_I \to \mathrm{Hom}_R(I, R)$ defined by the rule $\theta_I(x)(a) = xa$ for $x \in E_I$ and $a \in I$. Passing to the limit over $I \in \mathcal{G}$, we get an isomorphism of additive groups $\theta : Q \to R_{\mathcal{G}}$. Let $x, y \in Q$. There exist $I, J \in \mathcal{G}$ such that xI and yJ are both contained in R. Put $K = \{a \in J \mid ya \in I\}$. Then $xyK \subset R$ and $K \in \mathcal{G}$. Now $\theta_I(x) \circ \theta_J(y)$ is defined on K and agrees with $\theta_K(xy)$ since both maps are given by the rule $a \mapsto xya$ for $a \in K$. This proves that θ is a ring isomorphism.

(iii) This is contained in [15, Proposition 2.32(a)].

(iv) We have $K = \mathrm{rann}_Q\, X$ for some subset $X \subset Q$. There exists $I \in \mathcal{G}$ such that $yI \subset R$. Suppose that $yR \cap R \subset K$. Then $yI \subset K$ and $XyI = 0$. It now follows from (i) that $Xy = 0$, that is, $y \in K$.

(v) It follows from (iv) that any two right annihilators in Q coincide if and only if so do their intersections with R. Therefore there is an injection of the set of annihilator right ideals of Q into the set of all right ideals of R. If R is right Noetherian, then Q has to satisfy the ACC on right annihilators. Since Q is an essential extension of R in $\mathcal{M}_R$, one has $\mathrm{udim}\, Q_Q \leqslant \mathrm{udim}\, Q_R = \mathrm{udim}\, R_R < \infty$ under the Noetherian hypothesis.

(vi) See [38, Chapter X, §2].

(vii) If $z \in R$ is a central element, then the assignment $q \mapsto zq - qz$ defines an $\mathcal{M}_R$-endomorphism φ of $Q_{\max}(R)$ such that $\varphi(R) = 0$. We can extend φ to an endomorphism of $E(R)$, and the definition of $Q_{\max}(R)$ ensures that $\varphi = 0$ on $Q_{\max}(R)$. Hence $zq = qz$ for all $q \in Q_{\max}(R)$. $\square$

The $\mathcal{E}_H(A)$-torsion submodule of $V \in \mathcal{M}_A$ is nothing else than $\mathrm{Sing}_H V$. Our assumptions about A thus mean that A is $\mathcal{E}_H(A)$-torsion-free. Hence Lemma 6.1 may be applied with $R = A$ and $\mathcal{G} = \mathcal{E}_H(A)$. Put

$$E_H(A) = E_{\mathcal{E}_H(A)}(A) \subset Q_{\max}(A) \subset E(A).$$

By (ii) and (vii) of Lemma 6.1 $E_H(A)$ is a k-subalgebra of $Q_{\max}(A)$. In the sequel we denote $Q = E_H(A)$ for short.

Lemma 6.2. *The quotient ring Q is a right H-nonsingular left H-module algebra with respect to a module structure extending that on A. For a right ideal I of Q one has $I \in \mathcal{E}_H(Q)$ if and only if $I \cap A \in \mathcal{E}_H(A)$.*

Proof. Since H acts on A continuously in the $\mathcal{E}_H(A)$-topology, Montgomery and Schneider's result [25, Theorem 3.13] ensures the extension of the H-module structure to Q. Denoting $J = I \cap A$, we have $J_C = I_C \cap A$ for each $C \in \mathcal{F}$. By Lemma 6.1(iii) $I_C \in \mathcal{E}(Q)$ if and only if $J_C \in \mathcal{E}(A)$. The final assertion of Lemma 6.2 follows from Lemma 5.1. We conclude that the right H-singular ideal $\mathrm{Sing}_H Q$ coincides with the $\mathcal{E}_H(A)$-torsion submodule of Q. By Lemma 6.1(i) $\mathrm{Sing}_H Q = 0$. □

Lemma 6.3. *If $I \in \mathcal{E}_H(Q)$ then each $\mathcal{M}_Q$-morphism $I \to Q$ is induced by a left multiplication in Q.*

Proof. Lemma 6.2 shows that $\mathcal{G}^e = \mathcal{E}_H(Q)$ for $\mathcal{G} = \mathcal{E}_H(A)$. By Lemma 6.1(vi) Q coincides with its own localization with respect to $\mathcal{E}_H(Q)$. This means that the canonical map $Q \to \mathrm{Hom}_Q(I, Q)$ is bijective. □

Lemma 6.4. *If A is right Noetherian then Q is semiprimary right Goldie.*

Proof. Lemma 6.1(v) shows that Q is right Goldie. So Q satisfies the ACC on right annihilators and $\mathrm{udim}\, Q_Q < \infty$. The remaining part will be done in two steps.

Step 1. For any $u \in Q$ satisfying $\mathrm{rann}_Q u = \mathrm{rann}_Q u^2$ there exists an idempotent $e \in Q$ such that u is an invertible element of the ring eQe with unity e.

The right ideal $I = uQ + \mathrm{rann}_Q u$ is in $\mathcal{E}_H(Q)$ by Lemma 5.5. Note that the sum $uQ + \mathrm{rann}_Q u$ is direct by the assumption on u. Given any element $x \in Q$ such that $\mathrm{rann}_Q u \subset \mathrm{rann}_Q x$, the formula $ua + b \mapsto xa$ where $a \in Q$ and $b \in \mathrm{rann}_Q u$ defines therefore an $\mathcal{M}_Q$-morphism $\varphi : I \to Q$. By Lemma 6.3 there exists $y \in Q$ such that $\varphi(ua + b) = y(ua + b)$ for all a, b as above. This means that $yu = x$ and $\mathrm{rann}_Q u \subset \mathrm{rann}_Q y$.

We first apply this observation for $x = u$. It shows that there exists $e \in Q$ such that $eu = u$ and $\mathrm{rann}_Q u \subset \mathrm{rann}_Q e$. We have now $(qe - q)u = 0$ for any $q \in Q$. If $\mathrm{rann}_Q u \subset \mathrm{rann}_Q q$ then $(qe - q)I = 0$; hence $qe = q$ since $\mathrm{Sing}_H Q = 0$. In particular, $e^2 = e$ and $ue = u$. This shows also that $u \in eQe$.

Now take $x = e$. By the argument above we can find $v \in Q$ such that $vu = e$ and $\mathrm{rann}_Q u \subset \mathrm{rann}_Q v$. As has been noted already $ve = v$. Now $(ev - v)u = e^2 - e = 0$ and $(uv - e)u = ue - u = 0$. It follows that $(ev - v)I = 0$ and $(uv - e)I = 0$, whence $ev = v$ and $uv = e$. Thus v is the inverse of u in eQe.

Step 2. The ring Q is semiprimary.

Nil subrings (and even nil multiplicatively closed subsets) of a right Goldie ring are nilpotent [18,35]. In particular, this shows that there exists a largest nil right ideal N of Q. Since N is nilpotent, so too is QN, whence N is an ideal.

We will prove now that every right ideal I of Q has the form $aQ + K$ where $a \in Q$ is an idempotent and K a nil right ideal of Q, so that $K \subset N$. Since $\operatorname{udim} Q_Q < \infty$, we may proceed by induction on $\operatorname{udim} I$. If I is nil we may take $a = 0$ and $K = I$. Suppose that I contains a nonnilpotent element x. Since the ascending chain of right annihilators $\operatorname{rann}_Q x \subset \operatorname{rann}_Q x^2 \subset \cdots$ terminates at some step, we can find an integer $n > 0$ such that $u = x^n$ satisfies $\operatorname{rann}_Q u = \operatorname{rann}_Q u^2$. Let $e \in Q$ be as in Step 1. Then $eQ = uQ \subset I$. In particular, $e \neq 0$ since $u \neq 0$. As $Q = eQ \oplus (1 - e)Q$, we have $I = eQ \oplus J$ where $J = I \cap (1 - e)Q$. Now $\operatorname{udim} J < \operatorname{udim} I$. By induction hypothesis $J = bQ + K$ with an idempotent $b \in Q$ and a nil right ideal K. We get $I = eQ + bQ + K$. Since $b \in (1 - e)Q$, the right ideal bQ is a direct summand of $(1 - e)Q$. Then $eQ + bQ$ is a direct summand of Q, and so is generated by an idempotent $a \in Q$.

We can conclude that all right ideals of the factor ring Q/N are generated by idempotents. This implies that Q/N is semisimple Artinian. It is also clear that $\operatorname{Jac}(Q) = N$. □

Proof of Theorem 0.2. Denote by J the largest H-stable nilpotent ideal of Q. Then the H-module algebra Q/J is H-semiprime. Since Q is semiprimary by Lemma 6.4, so is Q/J. We may view $L = \operatorname{lann}_Q J$ as a right Q/J-module. Since J is nilpotent, each nonzero right ideal of Q contains a nonzero element annihilated by right multiplications of elements in J, that is, $L \in \mathcal{E}(Q)$. Moreover, $L \in \mathcal{E}_H(Q)$ because L is H-stable. Theorem 0.3 yields $Q/J \cong \prod_{I \in \Omega} Q/I$ where Ω is the finite set of maximal H-stable ideals of Q. It follows that L decomposes as a direct sum of right Q/I-modules, that is, $L = \bigoplus_{I \in \Omega} L_I$ where $L_I = \operatorname{lann}_Q I$.

Each L_I is an H-stable ideal of Q. Therefore $L_I L_K \subset L_I \cap L_K = 0$ for any two distinct ideals $I, K \in \Omega$. If $L_I \subset I$ for some I, then $L_I^2 \subset L_I I = 0$ as well, whence $L \subset \operatorname{rann}_Q L_I$. The latter inclusion implies that $L_I = 0$ since $\operatorname{Sing}_H Q = 0$ by Lemma 6.2.

Suppose now that $V = L_I \cap I \neq 0$ for some I. Then $L_I \not\subset I$ by the preceding observation, and so $I + L_I = Q$ by the maximality of I. We may regard L_I as a right Q/I-module. As $L_I/V \cong Q/I$ is free in $\mathcal{M}_{Q/I}$, we have $L_I = V \oplus F$ where F is a right ideal of Q such that $F \cong Q/I$ in $\mathcal{M}_Q$. By Theorem 0.3 the semiprimary H-simple algebra Q/I is quasi-Frobenius. Hence F is a cogenerator in $\mathcal{M}_{Q/I}$ [38, Proposition XIV.2.3], so that every nonzero right Q/I-module has a nonzero homomorphism into F. In particular, there exists $0 \neq \varphi \in \operatorname{Hom}_Q(V, F)$. Since V is a direct summand of L, we can extend φ to an $\mathcal{M}_Q$-morphism $L \to Q$. Lemma 6.3 says that φ is induced by a left multiplication, and therefore every ideal of Q contained in L has to be stable under φ. On the other hand, $\varphi(V) \not\subset V$ by construction although V is an ideal.

This contradiction proves that $L_I \cap I = 0$ for each $I \in \Omega$. Therefore $IL_I = 0$ as well. Now $JL = 0$ since $J \subset I$ for each $I \in \Omega$. We conclude that $J = 0$ since $\operatorname{Sing}_H Q = 0$. In other words, Q is H-semiprime. An application of Theorem 0.3 completes the proof. □

A consequence of $E_H(A)$ being quasi-Frobenius is that the Gabriel topology $\mathcal{E}_H(A)$ is *perfect*. There are several equivalent characterizations of perfect topologies (see [14, Theorem 4.3] or [38, Proposition XI.3.4]). One of them is stated in part (i) of the next proposition.

Proposition 6.5. *Suppose that A is right H-nonsingular and $Q = E_H(A)$ is quasi-Frobenius. Then*:

(i) *$IQ = Q$ for every $I \in \mathcal{E}_H(A)$.*
(ii) *Q is flat as a left A-module.*
(iii) *$Q = Q_{\max}(A) = E(A)$ and Q is H-semisimple.*
(iv) *Each right ideal of Q is of the form KQ for some right ideal K of A.*

Proof. (i) Each right ideal of Q is a right annihilator [38, Chapter XIV, §3]. This applies to $J = IQ$, yielding $J = \operatorname{rann}_Q L$ where $L = \operatorname{lann}_Q I$. As $\operatorname{Sing}_H Q_A = 0$, we get $L = 0$, whence $J = Q$.

(ii) This is a general property of perfect right localizations [38, Chapter XI, §2].

(iii) Since Q is right selfinjective and Q is flat as a left A-module, Q is injective in $\mathcal{M}_A$. But Q is also an essential extension of A in $\mathcal{M}_A$, so $Q = E(A)$. Then all inclusions $Q \subset Q_{\max}(A) \subset E(A)$ have to be equalities. To show the H-semisimplicity we need only to verify that Q satisfies the hypotheses of Lemma 4.2. Since Q is right Artinian, Q is semiprimary and all right ideals of Q are finitely generated.

Let us check that Q is H-semiprime as well. Suppose that J is an H-stable nilpotent ideal of Q. Then $L = \operatorname{lann}_Q J$ is an H-stable ideal of Q. Moreover, $L \in \mathcal{E}(Q)$ since J is nilpotent; hence $L \in \mathcal{E}_H(Q)$ since L is H-stable. The equality $\operatorname{Sing}_H Q = 0$ entails $\operatorname{lann}_Q L = 0$. But then $L = Q$ since L is a right annihilator in Q. This implies that $J = \operatorname{rann}_Q L = 0$.

(iv) It suffices to consider principal right ideals of Q. Given $x \in Q$, there exists $I \in \mathcal{E}_H(A)$ such that $xI \subset A$. Part (i) entails that $x \in KQ$ where $K = xI$ is a right ideal of A. As $K \subset xQ$, one has $xQ = KQ$. (See [14, Proposition 4.6].) □

7. Finding regular elements

In this section we take the final step in completing the proof of Theorem 0.1. This theorem follows immediately from the next result in conjunction with Theorem 0.2 proved in Section 6.

Proposition 7.1. *Suppose that A is H-semiprime right Goldie and $E_H(A)$ is quasi-Frobenius. Then $E_H(A)$ is the classical right quotient ring of A.*

By Lemma 5.3 A is right H-nonsingular, so that the definition of $Q = E_H(A)$ makes sense. We proceed in a series of lemmas. Since Q is quasi-Frobenius, the assignments $L \mapsto \operatorname{rann}_Q L$ and $K \mapsto \operatorname{lann}_Q K$ provide mutually inverse bijections between the sets of left and right ideals of Q. This fact will be used several times in the proofs below.

Lemma 7.2. *$QI = Q$ for every $I \in \mathcal{E}_H(A)$.*

Proof. Put $L = QI$ and $K = \operatorname{rann}_Q L$, so that $L = \operatorname{lann}_Q K$. It is clear that $K \cap A \subset \operatorname{rann}_A I$ where $\operatorname{rann}_A I = 0$ by Lemma 5.3. Since Q is an essential extension of A in $\mathcal{M}_A$, we get $K = 0$, whence $L = Q$. □

Lemma 7.3. *$QT = TQ$ for any ideal T of A.*

Proof. Suppose that there exists an ideal T of A such that $QT \not\subset TQ$. Since Q is right Artinian, we may choose such a T with the additional property that TQ is minimal among all right ideals of Q having this form. There exists $I \in \mathcal{E}_H(A)$ such that $ITQ \neq TQ$. Indeed, $xTQ \not\subset TQ$ for a suitable $x \in Q$, while $xITQ \subset ATQ \subset TQ$ for any $I \in \mathcal{E}_H(A)$ such that $xI \subset A$.

Having already chosen T, we now pick an $I \in \mathcal{E}_H(A)$ such that $ITQ \neq TQ$ and the right ideal ITQ of Q is minimal with respect to the previous condition. If $J \in \mathcal{E}_H(A)$ is arbitrary then $(I \cap J)TQ = ITQ$ by the minimality assumption, whence $ITQ \subset JTQ$. Given any $a \in A$, we take $J \in \mathcal{E}_H(A)$ such that $aJ \subset I$ (axiom (T3) of Gabriel topologies). Then $aITQ \subset aJTQ \subset ITQ$, and consequently $AITQ = ITQ$.

So the ideal $V = AIT$ of A satisfies $VQ = ITQ \neq TQ$. On the other hand, $VQ \subset TQ$ because $V \subset T$. The choice of T ensures that VQ has to be an ideal of Q. Then so is $L = \operatorname{lann}_Q VQ$. By the correspondence between left and right ideals $VQ = \operatorname{rann}_Q L$. However, Lemma 7.2 entails $1 \in QI$. Hence $LT \subset LQIT \subset LV = 0$, that is, $TQ \subset \operatorname{rann}_Q L$. We thus arrive at a contradiction.

We conclude that $QT \subset TQ$, that is, TQ is an ideal of Q for any ideal T of A. Now $QT = \operatorname{lann}_Q K$ where we put $K = \operatorname{rann}_Q QT$. As $TAK \subset TK = 0$, we have $AK \subset K$; hence $K \cap A$ is an ideal of A. By Proposition 6.5(iv) $K = (K \cap A)Q$. Then K is an ideal of Q by the first part of the proof, and so is its left annihilator QT. Hence $TQ \subset QT$ as well. □

Lemma 7.4. *Let N be the prime radical of A and J the Jacobson radical of Q. Then $J = NQ$ and* $\operatorname{lann}_Q N = \operatorname{rann}_Q N$.

Proof. The prime radical of a right Goldie ring is its largest nilpotent ideal [18]. Since Q is right Artinian, J is the largest nilpotent ideal of Q. It is clear therefore that $J \cap A \subset N$. Now Proposition 6.5(iv) yields $J = (J \cap A)Q \subset NQ$. By Lemma 7.3 $NQ = QN$. Hence NQ is a nilpotent ideal of Q, which implies that $NQ \subset J$.

Since $J = NQ$, we have $\operatorname{lann}_Q N = \operatorname{lann}_Q J$. Similarly, the equality $J = QN$ implies $\operatorname{rann}_Q N = \operatorname{rann}_Q J$. The proof will be completed once we establish that $\operatorname{lann}_Q J = \operatorname{rann}_Q J$. However, the left-(respectively right-)hand side of the last equality is equal to the right (respectively left) socle of Q. The two socles coincide because Q is quasi-Frobenius [38, Chapter XIV, Exercise 9]. □

Lemma 7.5. *Each $I \in \mathcal{E}_H(A)$ contains a regular element of A.*

Proof. Let N be as in Lemma 7.4, and let $X = \{x \in A \mid \operatorname{rann}_A x = \operatorname{rann}_A x^2\}$. Clearly X consists precisely of those $x \in A$ for which $xA \cap \operatorname{rann}_A x = 0$. Now pick $x \in I \cap X$ for which the right ideal $\operatorname{rann}_A x$ has minimal uniform dimension. We claim that $K = I \cap \operatorname{rann}_A x$ is a nil right ideal of A.

Suppose that $u \in K$ is not nilpotent. For a sufficiently large integer $n > 0$ the element $v = u^n$ lies in $K \cap X$ by ACC on right annihilators. Put $y = x + v$. Then $y^2 = x^2 + vx + v^2$ since $xv = 0$. Note that $vx + v^2 \in K$ and $xA \cap K = 0$ since $K \subset \operatorname{rann}_A x$. If $y^2 a = 0$ for some $a \in A$, we must have therefore $x^2 a = 0$, whence $xa = 0$, and then $v^2 a = 0$, whence $va = 0$. This shows that

$$\operatorname{rann}_A y^2 = \operatorname{rann}_A y = \operatorname{rann}_A x \cap \operatorname{rann}_A v.$$

Thus $y \in X$. Furthermore, $vA + \operatorname{rann}_A y \subset \operatorname{rann}_A x$, and the sum here is direct because $\operatorname{rann}_A y \subset \operatorname{rann}_A v$. It follows that $\operatorname{rann}_A y$ has smaller uniform dimension than $\operatorname{rann}_A x$, which contradicts the minimality condition in the choice of x.

We conclude that K is nil. By [18] K is nilpotent; hence $K \subset N$. Suppose that the left ideal $L = \operatorname{lann}_Q x$ of Q is nonzero. Since N is nilpotent, there exists $0 \neq t \in L$ such that $Nt = 0$. By Lemma 7.4 $tN = 0$ as well, and so $tJ = 0$ where we put $J = xA + K$. By Lemma 5.5

$xA + \operatorname{rann}_A x \in \mathcal{E}_H(A)$. Intersecting with I, we deduce that $J \in \mathcal{E}_H(A)$ too. This shows that t lies in the $\mathcal{E}_H(A)$-torsion submodule of Q_A. But then Lemma 6.1(i) yields $t = 0$, a contradiction.

Thus $L = 0$, that is, x is left regular in Q. By [22, Proposition 3.1.1] left regular elements of a left Artinian ring are invertible. We conclude that x is invertible in Q and so regular in A. □

Proof of Proposition 7.1. If $u \in A$ is a regular element then $uA \in \mathcal{E}_H(A)$ by Lemma 5.5 and the assignment $ua \mapsto a$ for $a \in A$ defines an $\mathcal{M}_A$-morphism $\varphi : uA \to A$. There exists $v \in Q$ such that $\varphi(b) = vb$ for all $b \in uA$ (see Lemma 6.1(ii)). Then $vu = 1$. Since Q is right Artinian, u is invertible in Q.

Let $x \in Q$ be an arbitrary element. Then $xI \subset A$ for some $I \in \mathcal{E}_H(A)$. By Lemma 7.5 I contains a regular element u. We get $a = xu \in A$ and $x = au^{-1}$. These two properties are exactly what is needed in the definition of classical quotient rings. □

Corollary 7.6. *Under the hypotheses of Proposition* 7.1 $E_H(A)$ *is* H*-simple if and only if* A *is* H*-prime, that is, the product of any two nonzero* H*-stable ideals of* A *is nonzero.*

Proof. If J is any H-stable ideal of Q, then so also is $L = \operatorname{lann}_Q J$. In this case $J \cap A$ and $L \cap A$ are H-stable ideals of A with zero product. If A is H-prime, we must have either $J \cap A = 0$ or $L \cap A = 0$. Since Q is an essential extension of A in $\mathcal{M}_A$, this gives $J = 0$ or $L = 0$. If $L = 0$ then $J = \operatorname{rann}_Q L = Q$.

Conversely, suppose that Q is H-simple and $IK = 0$ for two H-stable ideals I, K of A. By Lemma 7.3 QI and KQ are ideals of Q. At least one of them is proper since $QI \cdot KQ = 0$. Then either $I = 0$ or $K = 0$. □

Remarks. If R is a ring with prime radical N such that $Q_{\max}(R)$ is quasi-Frobenius with Jacobson radical J, then for $Q_{\max}(R)$ to be a classical right quotient ring of R it is necessary and sufficient that $N = J \cap R$ [41].

In the hypotheses of Theorem 0.1 it suffices to assume that H is only right admissible and has a bijective antipode. Under these weaker assumptions it still follows from Lemmas 6.4 and 3.4 that $Q = E_H(A)$ is right Goldie and right selfinjective. Hence Q is quasi-Frobenius by [10], and Proposition 7.1 still applies.

8. Applications and related results

In the next result based on [36] no assumptions about H are needed.

Proposition 8.1. *Suppose that* A *is semilocal* H*-semisimple and* $M \in {}_H\mathcal{M}_A$*. If either* M *is finitely generated in* $\mathcal{M}_A$ *or* H *is finitely generated in* $\mathcal{M}_k$ *then* M *is projective in* $\mathcal{M}_A$*.*

Proof. Since A is a direct product of finitely many H-simple algebras, the proof reduces to the case where A is itself H-simple. Under this assumption, λA must equal either A or 0 for each $\lambda \in k$. It follows that $\mathfrak{p} = \{\lambda \in k \mid \lambda A = 0\}$ is a prime ideal of k, and A is an algebra over the field of fractions $\kappa(\mathfrak{p})$ of $k/\mathfrak{p}$. Replacing H with the Hopf algebra $H \otimes \kappa(\mathfrak{p})$ over $\kappa(\mathfrak{p})$, we may assume that k is a field. In this case the conclusion follows from [36, Theorem 7.6]. □

If H is finitely projective then $H^* = [H, k]$ has a Hopf algebra structure and H^* is also finitely projective. By Proposition 4.4 H and H^* are admissible. Denote by $\int \subset H$ and $\int^* \subset H^*$ the

k-submodules of either left or right integrals (it does not matter which exactly). Applying the counity maps $\varepsilon : H \to k$ and $H^* \to k$, the evaluation at $1 \in H$, we get two ideals $\varepsilon(\int)$ and $\int^*(1)$ of k.

Theorem 8.2. *Let H be finitely projective and A semilocal H-semisimple. Then*:

(i) *A and $A \# H$ are quasi-Frobenius.*
(ii) *If $\varepsilon(\int)A = A$ then $A \# H$ is semisimple Artinian.*
(iii) *If $\int^*(1)A = A$ then A is semisimple Artinian.*

Proof. We may view $B = A \# H$ as a right H-comodule algebra with respect to the comodule structure $a \# h \mapsto \sum_{(h)} (a \# h_{(1)}) \otimes h_{(2)}$ where $a \in A$ and $h \in H$. Then B is also a left H^*-module algebra. By [26, Lemma 1.3] the assignment $I \mapsto I \# H$ defines a bijection between the H-stable ideals of A and the H^*-stable ideals of B. In particular, B is H^*-simple whenever A is H-simple. In general $A \cong \prod_{I \in \Omega} A/I$ where Ω is the finite set of maximal H-stable ideals of A. Then each $(A/I) \# H$ is an H^*-simple algebra and $B \cong \prod_{I \in \Omega} (A/I) \# H$. This shows that B is H^*-semisimple.

There is a category equivalence ${}_B\mathcal{M} \approx {}_{H^{\mathrm{cop}}}\mathcal{M}_{A^{\mathrm{op}}}$. Applying Proposition 8.1 with A^{op}, H^{cop} in place of A, H, we deduce that all left B-modules are projective in ${}_A\mathcal{M}$. In particular, L/K is projective in ${}_A\mathcal{M}$, so that $L \cong K \oplus L/K$ in ${}_A\mathcal{M}$, for any pair of left ideals $K \subset L$ of B.

If $V \in {}_A\mathcal{M}$ is finitely generated then the factor module $V/\operatorname{Jac}(A)V$ has finite length, say j_V, since the ring $A/\operatorname{Jac}(A)$ is semisimple Artinian. By Nakayama's Lemma $V = 0$ if and only if $j_V = 0$. Furthermore, $j_{V \oplus W} = j_V + j_W$ for any two finitely generated left A-modules. Clearly $B \cong A \otimes H$ is finitely generated in ${}_A\mathcal{M}$, and so are all left ideals of B since they are ${}_A\mathcal{M}$-direct summands of B. Given any set $\mathcal{X}$ of left ideals of B, we can find $L \in \mathcal{X}$ with minimal j_L. If $K \subset L$ where $K \in \mathcal{X}$ then $j_K = j_L$ by the minimality assumption, and the equality $j_L = j_K + j_{L/K}$ yields $j_{L/K} = 0$, showing that $K = L$. Thus L is a minimal element of $\mathcal{X}$.

We can conclude that B is left Artinian. Then B is semiprimary, and Theorem 0.3 shows that B is quasi-Frobenius. In particular, B is right Artinian. Note that $B = (A \# 1) \oplus (A \# H^+)$, a direct sum of left A-submodules, where $H^+ = \operatorname{Ker} \varepsilon$. This implies that $IB \cap A = I$ for each right ideal I of A. The lattice of right ideals of A is therefore embedded into the lattice of right ideals of B; hence A is right Artinian. Another application of Theorem 0.3 shows that A is quasi-Frobenius.

The proof of (ii) and (iii) is reduced to the case where A is H-simple. We may assume moreover that k is a field (see the proof of Proposition 8.1). Note that $\varepsilon(\int) \neq 0$ implies that H is semisimple and $\int^*(1) \neq 0$ implies that H^* is semisimple [40]. In the first case B is a separable extension of A, that is, every short exact sequence in ${}_B\mathcal{M}$ splits provided it splits in ${}_A\mathcal{M}$ [8, Theorem 4]. Since left B-modules are projective in ${}_A\mathcal{M}$, all short exact sequences in ${}_B\mathcal{M}$ split. This property characterizes semisimple Artinian rings. Under the hypothesis of (iii) A is a right H^*-comodule algebra, and the desired conclusion follows from [36, Theorem 5.2]. □

Theorem 8.3. *Let H be finitely projective and A either* (a) *semiprime right Goldie or* (b) *H-semiprime right Noetherian. Then*:

(i) *A and $A \# H$ have quasi-Frobenius classical right quotient rings.*
(ii) *If $\operatorname{lann}_A \varepsilon(\int) = 0$ then $A \# H$ is semiprime.*
(iii) *If $\operatorname{lann}_A \int^*(1) = 0$ then A is semiprime.*

Proof. (i) Denote by Q the classical right quotient ring of A. Under hypothesis (a) it exists by classical Goldie's Theorem. Under hypothesis (b) we apply Theorem 0.1. In both cases Q is quasi-Frobenius. By Theorem 2.2 Q is a left H-module algebra. By Theorem 0.3 Q is H-semisimple. By Theorem 8.2 $Q \# H$ is quasi-Frobenius. We will check that $Q \# H$ is a classical right quotient ring of $A \# H$.

One has $q \# h = \sum_{(h)}(1 \# h_{(2)})(S^{-1}(h_{(1)})q \# 1)$ for all $q \in Q$ and $h \in H$. Thus $Q \# H = (1 \# H)(Q \# 1)$. Given any $x \in Q \# H$, we can find, using common denominators, a regular element u of A such that $y = xu \in (1 \# H)(A \# 1) \subset A \# H$ (we identify u with $u \# 1$). Clearly u is invertible in $Q \# H$ (in particular, u is regular in $A \# H$) and $x = yu^{-1}$. Suppose that t is any regular element of $A \# H$. Then t is right regular in $Q \# H$ since the equality $tyu^{-1} = 0$ with y, u as above implies $ty = 0$. The ring $Q \# H$ is right Artinian since so is Q and $Q \# H$ is finitely generated in $\mathcal{M}_Q$. It follows that t is invertible in $Q \# H$. This completes the verification.

(ii) The ideal $\operatorname{lann}_Q \varepsilon(\int)$ of Q has to be zero since it has zero intersection with A. It follows that $\varepsilon(\int)Q = Q$ since all right ideals of Q are right annihilators. By Theorem 8.2 $Q \# H$ is semisimple Artinian. Since $Q \# H$ is also a classical right quotient ring of $A \# H$, the conclusion in (ii) is a standard fact [22, Proposition 2.3.1].

(iii) This is proved similarly using Theorem 8.2(iii). □

Remarks. There is an algebra isomorphism $(A \# H)^{\mathrm{op}} \cong A^{\mathrm{op}} \# H^{\mathrm{cop}}$ (under which $a \# h$ in the first algebra corresponds to $(1 \# S^{-1}h)(a \# 1)$ in the second). It follows that the right conditions in (a), (b), (i) of Theorem 8.3 can be replaced with their left versions.

A close relationship between (ii) and (iii) of Theorem 8.3 was discovered in [26, Theorems 8.10, 8.11] and further exploited in [19]. As the H^*-stable nilpotent ideals of $B = A \# H$ are of the form $I \# H$ where I is an H-stable nilpotent ideal of A, the H-semiprimeness of A implies the H^*-semiprimeness of B. Hence (ii) follows from (iii) applied to B. Conversely, $B \# H^* \cong A \otimes \operatorname{End}_k H$ is Morita equivalent to A by duality principle [2,44]. If $B \# H^*$ is semiprime then so too is A, that is, (iii) follows from (ii) applied to B. In the framework of Montgomery and Schneider's paper [26] it is possible to reformulate Theorem 8.3 in terms of Galois extensions rather than module algebras.

Denote by $\mathcal{I}$ the set of ideals of A with zero left and right annihilators. Clearly $\mathcal{I}$ satisfies conditions (T2), (T3) from Section 5. The left, right and symmetric Martindale quotient rings $Q^l_M(A)$, $Q^r_M(A)$, $Q_M(A)$ are defined with respect to $\mathcal{I}$ similarly to the construction of localizations with respect to Gabriel topologies (see, e.g., [23, §6.4]).

Theorem 8.4. *All elements of H operate continuously on A with respect to the $\mathcal{I}$-topology if either* (a) *A is semiprime right Goldie or* (b) *A is H-semiprime right Noetherian and H is admissible. In these cases the H-module structure extends to $Q^l_M(A)$, $Q^r_M(A)$, $Q_M(A)$.*

Proof. Denote by Q the classical right quotient ring of A. We claim that for an ideal I of A the three conditions:

(i) I contains a regular element,
(ii) $I \in \mathcal{E}_H(A)$,
(iii) $\operatorname{lann}_A I = 0$

are equivalent to each other. The implication (i) ⇒ (ii) follows from Lemma 5.5, and (ii) ⇒ (iii) follows from Lemma 5.3. Suppose that $\mathrm{lann}_A I = 0$. Since $\mathrm{lann}_Q I$ is a submodule of Q_A which has zero intersection with A, we must have $\mathrm{lann}_Q I = 0$. Then $IQ = Q$ since Q is quasi-Frobenius (either by classical Goldie's Theorem or by Theorem 0.1). All elements of IQ are of the form au^{-1} where $a \in I$ and $u \in A$ is a regular element. In particular, $1 = au^{-1}$ for some a, u as above, which shows that $u \in I$. Hence (iii) ⇒ (i). Clearly (i) implies that $\mathrm{rann}_A I = 0$ as well. We see that each of the conditions (i), (ii), (iii) is equivalent to $I \in \mathcal{I}$.

Suppose that $h \in H$ and $I \in \mathcal{I}$. We can find $C \in \mathcal{F}$ such that $h \in C$. By Lemma 5.1 $I_C \in \mathcal{E}_H$. Since $[C, I]$ is an ideal of $[C, A]$, it follows from the definition of I_C in Section 5 that I_C is an ideal of A. Hence $I_C \in \mathcal{I}$, and we have $hI_C \subset I$. This shows that h operates on A as a continuous transformation. The final assertion follows from [24, Lemma 3.3, Theorem 3.4]. □

Remark. If H is finitely projective and A as in Theorem 8.4 then for each $I \in \mathcal{I}$ the H-stable ideal I_H also lies in $\mathcal{I}$. In this case $Q^l_M(A)$, $Q^r_M(A)$, $Q_M(A)$ coincide with the H-analogs of Martindale quotient rings introduced by Cohen [9].

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