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Closed-form approximations for diffusion densities: a path integral approach

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Abstract

In this paper, we investigate the transition probabilities for diffusion processes. In a first part, we show how transition probabilities for rather general diffusion processes can always be expressed by means of a path integral. For several classical models, an exact calculation is possible, leading to analytical expressions for the transition probabilities and for the maximum probability paths. A second part consists of the derivation of an analytical approximation for the transition probability, which is useful in case the path integral is too complex to be calculated. The approximation we present, is based on a convex combination of a new analytical upper and lower bound for the transition probabilities. The fact that the approximation is analytical has some important advantages, e.g., for the investigation of Asian options. Finally, we demonstrate the accuracy of the approximation by means of some graphical illustrations.

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1. Introduction

Dynamic models, and more specifically continuous-time models, are widely used and studied nowadays in pricing and investment theories. Most of the existing one-factor models refer to the

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general diffusion equations, which are stochastic differential equations in the form

$$dY(t) = \mu(Y(t), t)\,dt + \sigma(Y(t), t)\,dW(t). \tag{1}$$

This equation defines a stochastic process $Y = \{Y(s), s \in [0, t]\}$, reflecting e.g., the price process in time. In this equation, $W = \{W(s), s \in [0, t]\}$ is a standard Brownian motion, $\mu(y, t)$ is the drift of the process Y, and $\sigma^2(y, t)$ is the diffusion of Y. Both μ and σ can contain one or more parameters.

In this contribution, we will assume that the drift μ and the diffusion σ^2 do not depend explicitly on time t. Thus, we consider stochastic differential equations of the form

$$dY(t) = A(Y(t))\,dt + B(Y(t))\,dW(t), \tag{2}$$

where, as in the general diffusion model, the functions $A(y)$ and $B(y)$ can contain parameters. Fortunately, this time independence is only a minor restriction, since most of the classical models e.g., for interest rates are members of this class of processes (see also Section 7).

One of the questions in this context is to find a closed-form expression for the probability of the process Y reaching the value y_e at time t_e given the value y_0 at a former point in time $t_0 \leqslant t_e$. We will use the notation

$$p(t_0, y_0; t_e, y_e) = \frac{d}{dy_e}\,\mathrm{Prob}[Y(t_e) \leqslant y_e | Y(t_0) = y_0]$$

for the transition density of the process Y. The knowledge of this density is important, for instance, in the framework of derivative pricing, where the stochastic process Y then reflects the price process.

Contrary to the rather simple form of the diffusion equation (2), such a closed form is only known for a few cases, e.g., the Wiener model, the geometric Wiener model, the Vasicek model, the Cox–Ingersol–Ross model and related models.

In a paper of 1999 (see [1]), Ait-Sahalia presented a method leading to a closed-form approximation for the exact transition density. His approach is based on a Hermite expansion of the density around a normal density function. Due to the closed form, the advantages for derivative pricing remain, be it that the accuracy diminishes. A problem with the method that Ait-Sahalia proposes, is that it converges for $\Delta t = t_e - t_0$ going to zero, but it may lead to bad approximations when the time horizon increases. For financial applications, the author says that Δt is never bigger than 3 or 6 months, so this will not cause any problems. However, in actuarial applications, we may need a much larger horizon.

In the present paper, we want to give an answer towards the solution of the problems sketched above. First, we show how for general types of diffusion processes, whether the time interval is small or big, the transition density $p(t_0, y_0; t_e, y_e)$ can be expressed by means of a Feynman path integral. This is a powerful concept borrowed from quantum mechanics used to describe the amplitude to move between two points if each possible path is given a certain probability. Making use of specific properties and calculation techniques on path integrals, we show that an exact calculation is not only possible for the four classical models mentioned earlier, but also for some other types of processes.

Secondly, starting from the path integral expression for the transition density, we point out how to find a closed-form approximation for the transition density, usable for any diffusion process, with very high accuracy. In contrast with the method developed in [1], this approximation can be used for a short time span as well as for a larger horizon.

The paper is organized as follows. We start with a brief description of the concepts and notations about stochastic differential equations and Feynman path integrals in Section 2. Section 3 contains the first important result, expressing the transition densities for general diffusion processes by means of a path integral. Afterwards in Section 4, we show how the modal path or maximum probability path can be determined. Section 5 is meant to prove how the famous Itô lemma can be translated into the path integral formalism. Section 6 constitutes the “body” of this paper. Here we show, based on the path integral expression, how the transition density for general diffusion processes can be approximated with a closed-form formula. In Section 7 we present examples of the methodology, for common models in the financial theory. We give an expression for the transition probability in each case, together with an explicit calculation if possible. Finally, Section 8 demonstrates the accuracy of our new approximation by means of a few graphical illustrations.

The proofs of the theorems and some explicit computational results about path integrals are brought together in the appendix.

2. Definitions

2.1. Stochastic differential equations

In order to explain the similarities and dissimilarities between Itô integrals and path integrals, we briefly discuss the concept of a general stochastic differential equation. For more details, we refer to [2], [7] and [15].

A θ-stochastic differential equation is defined as

$$\mathrm{d}Y(t) = a(Y(t),t)\,\mathrm{d}t + b(Y(t),t)_\theta\,\mathrm{d}W(t) \tag{3}$$

with solution

$$Y(t) = Y(0) + \int_0^t a(Y(s),s)\,\mathrm{d}s + \int_0^t b(Y(s),s)_\theta\,\mathrm{d}W(s), \tag{4}$$

where $W(t)$ is a standard Brownian motion.

The first integral in (4) is a Riemann-integral, the second one is a θ-stochastic integral.

If $X = \{X(s), s \in [0,t]\}$ is a process adapted to the natural Brownian filtration, the θ-stochastic integral

$$I_t^{(\theta)}(X) = \int_0^t X(s)_\theta\,\mathrm{d}W(s) \tag{5}$$

is defined by

$$I_t^{(\theta)}(X) = \lim_{n\to\infty}\sum_{i=0}^{n-1} X(t_i^\theta)(W(t_{i+1}) - W(t_i)) \tag{6}$$

for any partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ for which $\max(t_{i+1} - t_i) \to 0$, and with t_i^θ equal to $t_i^\theta = t_i + \theta(t_{i+1} - t_i)$.

We want to draw attention to three special choices of θ.

- When θ is equal to zero, the values of X are chosen at the left points, and the θ-stochastic integral coincides with an Itô stochastic integral. We will use the notation

$$I_t^{(0)}(f(W)) = \int_0^t f(W(s)_{\mathrm{L}})\,\mathrm{d}W(s) \tag{7}$$

when we use this type of integration. The most important advantage of this choice is the fact that Itô stochastic integrals satisfy the martingale property. A disadvantage however is that the chain rule of classical calculus is not valid.

- When θ is equal to $\frac{1}{2}$, the values of X are chosen at the midpoints, and the θ-stochastic integral reduces to a Stratonovich stochastic integral. We will use the notation

$$I_t^{(1/2)}(f(W)) = \int_0^t f(W(s))\,\mathrm{d}W(s) \tag{8}$$

(without index) when we use this type of integration. The stochastic integral no longer satisfies the martingale property, but now the classical chain rule is formally satisfied, or

$$\int_0^t f'(W(s))_{\theta=1/2}\,\mathrm{d}W(s) = f(W(t)) - f(W(0)). \tag{9}$$

Since stochastic integrals with $\theta = \frac{1}{2}$ behave like Riemann integrals (to a certain extent), the omittance of an index seems acceptable.

- The situation with θ equal to 1 corresponds to a choice for the right points. We will denote this kind of integration as

$$I_t^{(1)}(f(W)) = \int_0^t f(W(s)_{\mathrm{R}})\,\mathrm{d}W(s). \tag{10}$$

The following relation between general θ-stochastic integrals and Stratonovich stochastic integrals will be very helpful in the development of our methodology:

$$\int_0^t f(W(s)_\theta)\,\mathrm{d}W(s) = \int_0^t f(W(s))\,\mathrm{d}W(s) + \left(\theta - \frac{1}{2}\right)\int_0^t f'(W(s))\,\mathrm{d}s. \tag{11}$$

A proof can be found in an easy way using Taylor expansions.

2.2. Path integrals

Feynman path integrals originate from quantum mechanics, where they are used to describe the amplitude to go from one point to another point, if each possible trajectory is given a certain probability. As such, they provide a very efficient tool in the derivation of transition probabilities. Note that in the original approach, Feynman used an imaginary argument of the exponential function, whereas we use a real argument but keep the formalism.

A Feynman path integral

$$K(t_0, x_0; t_\mathrm{e}, x_\mathrm{e}) = \int_{(t_0,x_0)}^{(t_\mathrm{e},x_\mathrm{e})} Dx(s)\mathrm{e}^{-\int_{t_0}^{t_\mathrm{e}} L(\dot{x}(s),x(s),s)\,\mathrm{d}s}, \tag{12}$$

where $L(\dot{x}(s), x(s), s)$ is called the Lagrangian, is defined by

$$K(t_0, x_0; t_\mathrm{e}, x_\mathrm{e}) = \lim_{n\to\infty} \frac{1}{\left(\sqrt{2\pi\varepsilon}\right)^n} \int \mathrm{d}x_1 \int \mathrm{d}x_2 \ldots \int \mathrm{d}x_{n-1} \times \mathrm{e}^{-\varepsilon\sum_{i=0}^{n-1} L((x_{i+1}-x_i)/\varepsilon,(x_i+x_{i+1})/2,(t_i+t_{i+1})/2)} \tag{13}$$

for a partition $t_0 = t_0 < t_1 < \cdots < t_n = t_\mathrm{e}$ where $\varepsilon = (t_\mathrm{e} - t_0)/n$ and $t_{i+1} = t_i + \varepsilon$ and where we used the notation $x_i = x(t_i)$.

It is important to note that, in fact, this definition makes use of a midpoint choice as it was the case for the partition in a Stratonovich stochastic integral. As a consequence, one has to be very careful when comparing or mixing Itô calculus and Feynman path integrals.

As an example, we consider a Brownian motion, for which the Lagrangian is equal to $L(\dot{x}, x, s) = \dot{x}^2/2$. In that case the multiple integration can be worked out in a straightforward way, resulting in

$$\int_{(t_0,x_0)}^{(t_e,x_e)} Dx(s)\mathrm{e}^{-(1/2)\int_{t_0}^{t_e}\dot{x}(s)^2\,\mathrm{d}s} = \lim_{n\to\infty}\frac{1}{\left(\sqrt{2\pi\varepsilon}\right)^n}\int \mathrm{d}x_1\cdots\int \mathrm{d}x_{n-1}\mathrm{e}^{-(\varepsilon/2)\sum_{i=0}^{n-1}((x_{i+1}-x_i)/\varepsilon)^2}$$

$$= \frac{1}{\sqrt{2\pi(t_e-t_0)}}\,\mathrm{e}^{-(x_e-x_0)^2/2(t_e-t_0)}. \tag{14}$$

This well-known result can be read as the transition probability to go from the starting point x_0 at time t_0 to the final point x_e at time t_e when the underlying process is a standard Brownian motion.

A handsome result about Feynman path integrals can be found in the so-called Kolmogorov property. It shows how to write a path integral as a combination of successive events:

$$K(t_0, x_0; t_e, x_e) = \int_{-\infty}^{+\infty} \mathrm{d}x_s K(t_0, x_0; t_s, x_s)K(t_s, x_s; t_e, x_e), \tag{15}$$

where t_s is any time between t_0 and t_e.

Proofs, applications and more details about this powerful concept can be found e.g., in [3,5,6,11,16,17]. Important computational results are summarized in Appendix A.

3. Transition probabilities

In this section, we show how the transition probability for stochastic processes defined by means of a stochastic differential equation, can be expressed by means of a Feynman path integral. We start with a diffusion equation with unit diffusion, and we generalize the result for equations where the diffusion is a function of the stochastic process. Proofs are provided in Appendix B.

Theorem 3.1. *Consider a θ-stochastic differential equation*

$$\mathrm{d}Y(t) = A(Y(t))\,\mathrm{d}t + \mathrm{d}W(t), \tag{16}$$

where $W(t)$ is a standard Brownian motion.

The transition probability for the stochastic process $Y = \{Y(s), s \in [0, t]\}$ can be written by means of a path integral as

$$p(0, y_0; t, y_t) = \frac{\mathrm{d}}{\mathrm{d}y_t}\mathrm{Prob}[Y(t) \leqslant y_t | Y(0) = y_0]$$

$$= \int_{(0,y_0)}^{(t,y_t)} Dy(s)\mathrm{e}^{-(1/2)\int_0^t \dot{y}^{\,2}\,\mathrm{d}s - 1/2\int_0^t (A(y)^2 + \partial A/\partial y)\,\mathrm{d}s + \int_0^t A(y)\,\mathrm{d}y}. \tag{17}$$

This result is independent of the choice of θ in the definition of the stochastic integral.

The long-term probability for the process Y can be calculated as

$$\bar{p}(\bar{y}) = \lim_{t\to\infty} p(0, y_0; t, \bar{y}) = C(y_0)\mathrm{e}^{2\int_{y_0}^{\bar{y}} A(z)\,\mathrm{d}z}, \tag{18}$$

where the constant $C(y_0)$ is determined by the condition of a total mass equal to one.

Remark 3.1. If the domain of the stochastic process Y is $(0, +\infty)$ instead of $(-\infty, +\infty)$, the differential part $Dy(s)$ has to be replaced by $D_+y(s)$.

Remark 3.2. The last integral in the exponent of (17) behaves as a Stratonovich integral. A transformation into an Itô integral as mentioned in (11), enables us to write the short time transition probability as

$$\begin{aligned} p(0, y_0; \Delta t, y_\Delta) &= \frac{1}{\sqrt{2\pi\Delta t}}\,\mathrm{e}^{-(y_\Delta - y_0)^2/2\Delta t - (\Delta t/2)A^2(y_0) + A(y_0)(y_\Delta - y_0)} \\ &= \frac{1}{\sqrt{2\pi\Delta t}}\,\mathrm{e}^{-(y_\Delta - y_0 - \Delta t A(y_0))^2/2\Delta t}, \end{aligned} \tag{19}$$

which coincides with the classical expression for measures associated with diffusion processes with unit volatility.

Remark 3.3. The constitution of the path integral (17) nicely illustrates Girsanov's theorem (see, e.g., [14]).

Indeed, the process

$$M(t) = \exp\left\{\int_0^t A(W(s)_{\mathrm{L}})\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t A^2(W(s))\,\mathrm{d}s\right\} \tag{20}$$

is the Radon–Nikodym derivative of the measure in (17) to the Wiener measure. As a consequence, the diffusion process defined by (16) is a Brownian motion with respect to the measure defined by the transition probability in (17).

Note that in case the stochastic process Y has domain $(0, +\infty)$ instead of $(-\infty, +\infty)$, the process $M(t)$ of (20) is the Radon–Nikodym like derivative with respect to a Brownian motion killed at zero.

Remark 3.4. The transition probability in (17) also satisfies the forward Fokker–Planck equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y_t^2} - \frac{\partial}{\partial y_t}(A(y_t)p), \tag{21}$$

where p is used as a short hand notation for the probability $p(0, y_0; t, y_t)$, see, e.g., [10].

Theorem 3.2. *Consider a θ-stochastic differential equation*

$$\mathrm{d}Y(t) = A(Y(t))\,\mathrm{d}t + B(Y(t))_\theta\,\mathrm{d}W(t), \tag{22}$$

where $W(t)$ is a standard Brownian motion.

A change of variables $X(t)=\int_0^t \mathrm{d}Y(s)/B(Y(s))_{\theta=1/2}=\psi(Y(t))$ *results in the new stochastic differential equation*

$$\mathrm{d}X(t)=\left(\frac{A(\psi^{-1}(X(t)))}{B(\psi^{-1}(X(t)))}+\left(\theta-\frac{1}{2}\right)\frac{\partial B}{\partial y}(\psi^{-1}(X(t)))\right)\mathrm{d}t+\mathrm{d}W(t). \tag{23}$$

This result is dependent on the choice of θ *in the definition of the stochastic integral.*

Theorem 3.3. *Consider an Itô stochastic differential equation*

$$\mathrm{d}Y(t)=A(Y(t))\,\mathrm{d}t+B(Y(t)_{\mathrm{L}})\,\mathrm{d}W(t), \tag{24}$$

where $W(t)$ *is a standard Brownian motion, and where* $\psi(y)=\int^{y}\mathrm{d}z/B(z)$ *defines a nondecreasing function.*

The transition probability for the stochastic process $Y=\{Y(s),s\in[0,t]\}$ *can be written by means of a path integral as*

$$p(0,y_0;t,y_t)=\frac{\mathrm{d}}{\mathrm{d}y_t}\mathrm{Prob}[Y(t)\leqslant y_t|Y(0)=y_0]$$

$$=\frac{1}{B(y_t)}\int_{(0,\psi(y_0))}^{(t,\psi(y_t))}Dy(s)\mathrm{e}^{-(1/2)\int_0^t\dot{y}^{\,2}\,\mathrm{d}s}\mathrm{e}^{-(1/2)\int_0^t(T(y)^2+\partial T/\partial y)\,\mathrm{d}s+\int_0^t T(y)\,\mathrm{d}y}, \tag{25}$$

where the function T *is defined by*

$$T(z)=\frac{A(\psi^{-1}(z))}{B(\psi^{-1}(z))}-\frac{1}{2}\frac{\partial B}{\partial y}(\psi^{-1}(z)). \tag{26}$$

The long-term probability for the process Y *can be calculated as*

$$\bar{p}(\bar{y})=\lim_{t\to\infty}p(0,y_0;t,\bar{y})=\frac{C(y_0)}{B(\bar{y})}\mathrm{e}^{2\int_{\psi(y_0)}^{\psi(\bar{y})}T(z)\,\mathrm{d}z}, \tag{27}$$

where the constant $C(y_0)$ *is determined by the condition of a total mass equal to one.*

Remark 3.5. If the domain of the stochastic process $\psi(Y)=\{\psi(Y(s)),s\in[0,t]\}$ is $(0,+\infty)$ instead of $(-\infty,+\infty)$, the differential part $Dy(s)$ as before has to be replaced by $D_+y(s)$.

Remark 3.6. For the stochastic process of Theorem 3.3, the short time transition probability is equal to

$$p(0,y_0;\Delta t,y_\Delta)=\frac{1}{B(y_\Delta)}\frac{1}{\sqrt{2\pi\Delta t}}\mathrm{e}^{-[(\psi(y_\Delta)-\psi(y_0)-\Delta tT(\psi(y_0)))^2]/2\Delta t}. \tag{28}$$

Since we are dealing with infinitesimal time periods, we can write

$$\psi(y_\Delta)-\psi(y_0)=(y_\Delta-y_0)\psi'(y_0)=(y_\Delta-y_0)\frac{1}{B(y_0)} \tag{29}$$

and with an explicitation of T, we obtain the classical expression

$$p(0,y_0;\Delta t,y_\Delta)=\frac{1}{\sqrt{2\pi\Delta tB^2(y_0)}}\mathrm{e}^{-(y_\Delta-y_0-\Delta tA(y_0))^2/2\Delta tB^2(y_0)}. \tag{30}$$

Remark 3.7. The transition probability in (25) now satisfies the forward Fokker–Planck equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y_t^2}(B^2(y_t)p) - \frac{\partial}{\partial y_t}(A(y_t)p), \tag{31}$$

where p is used as a short-hand notation for the probability $p(0, y_0; t, y_t)$, see, e.g., [10].

For $\theta = 0$, the previous results were already derived by the same authors in an earlier paper (see [4]); however, in that contribution the path integrals were found in a completely different way, without making use of Itô calculus. The result of Theorem 3.2 for $\theta = 0$ is also mentioned in [1].

4. Maximal probability path

As mentioned before, a Feynman path integral $K(t_0,x_0;t_e,x_e)$ as in (12) describes the amplitude to go from the point x_0 at time t_0 to the point x_e at time t_e, where each trajectory is given a certain probability according to the stochastic process related to the Lagrangian. In fact, in the whole set of trajectories connecting the two points, only paths in the vicinity of the classical or modal path provide important contributions to $K(t_0,x_0;t_e,x_e)$. Indeed, for other paths, there are always neighboring trajectories that cancel out their contribution.

This modal path, or maximum probability path, can be determined (see, e.g., [6]) as the solution of the ordinary second-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{32}$$

subject to the boundary conditions $x(t_0) = x_0$ and $x(t_e) = x_e$.

As an example, if we consider the Brownian motion, the maximum probability path is given by

$$x_{\text{mod}}(s|t_0,x_0;t_e,x_e) = \frac{t_e - s}{t_e - t_0}x_0 + \frac{s - t_0}{t_e - t_0}x_e. \tag{33}$$

Looking for the modal path for stochastic processes defined by stochastic differential equations in a form as in Section 3, the following nice result appears.

Theorem 4.1. *Consider a stochastic process $Y = \{Y(s), s \in [0,t]\}$ defined by a diffusion equation with unit diffusion (16). The maximal probability path $y_{\text{mod}}(s)$ for this process when starting at point y_0 at time 0 and arriving at y_t at time t, can be determined implicitly by*

$$\int_{y_0}^{y_{\text{mod}}(s)} \frac{\mathrm{d}y}{\sqrt{A(y)^2 + \partial A/\partial y + C(y_0, y_t)}} = \pm s, \tag{34}$$

where $C(y_0, y_t)$ is fixed by the condition $y_{\text{mod}}(t) = y_t$. The sign in the right-hand side is equal to the sign of the difference $y_{\text{mod}}(s) - y_0$.

Theorem 4.2. *Consider a stochastic process $Y = \{Y(s), s \in [0,t]\}$ defined by a diffusion equation with non-unit diffusion (24). The maximal probability path $y_{\text{mod}}(s)$ for this process when starting at point y_0 at time 0 and arriving at y_t at time t, can be determined implicitly by*

$$\int_{\psi(y_0)}^{\psi(y_{\text{mod}}(s))} \frac{\mathrm{d}x}{\sqrt{T(x)^2 + \partial T/\partial x + C(y_0, y_t)}} = \pm s, \tag{35}$$

where ψ is defined in Theorem 3.3, *T is defined in* (26), *and where $C(y_0, y_t)$ is fixed by the condition $y_{\text{mod}}(t)=y_t$. The sign in the right-hand side is equal to the sign of the difference $\psi(y_{\text{mod}}(s))-\psi(y_0)$.*

5. The Itô lemma in the path integral formalism

Consider again a stochastic process $Y=\{Y(s), s\in[0,t]\}$ determined by the stochastic differential equation (16), where $W(t)$ is a standard Brownian motion. From Theorem 3.1, we know that the transition probability can be written by means of the path integral

$$p_Y(0,y_0;t,y_t)=\int_{(0,y_0)}^{(t,y_t)} Dy(s)\mathrm{e}^{-(1/2)\int_0^t \dot{y}^{\,2}\,\mathrm{d}s-(1/2)\int_0^t (A(y)^2+\partial A/\partial y)\,\mathrm{d}s+\int_0^t A(y)\,\mathrm{d}y}. \tag{36}$$

Following Itô's lemma, the stochastic differential equation for the process $X=\{X(s), s\in[0,t]\}$ when $Y(t)=f(X(t))$, is given by

$$\begin{aligned}\mathrm{d}X(t)&=\frac{1}{2}\frac{\partial^2}{\partial y^2}[f^{-1}(Y(t)_{\mathrm{L}})]\,\mathrm{d}t+\frac{\partial}{\partial y}[f^{-1}(Y(t)_{\mathrm{L}})]\,\mathrm{d}Y(t)\\&=-\frac{1}{2}\frac{f''(Y(t)_{\mathrm{L}})}{f'(Y(t)_{\mathrm{L}})^3}\,\mathrm{d}t+\frac{1}{f'(Y(t)_{\mathrm{L}})}\,\mathrm{d}Y(t),\end{aligned} \tag{37}$$

or in the other direction

$$\mathrm{d}Y(t)=\frac{1}{2}\frac{f''(X(t)_{\mathrm{L}})}{f'(X(t)_{\mathrm{L}})^2}\,\mathrm{d}t+f'(X(t)_{\mathrm{L}})\,\mathrm{d}X(t). \tag{38}$$

The question that arises is: how can this transformation be extended into the path integral (36)? Note that we have to take into account the difficulty that, contrary to the Itô lemma, the integrations in the path integral are of the Stratonovich type.

Making use of a stochastic time change in the path integral (36), one can prove the following result (see Appendix B):

Theorem 5.1. *Consider the stochastic differential equation*

$$\mathrm{d}Y(t)=A(Y(t))\,\mathrm{d}t+\mathrm{d}W(t), \tag{39}$$

where $W(t)$ is a standard Brownian motion, and a transformation $Y(t)=f(X(t))$, for which the inverse function is well defined.

Starting from the path integral expression for the transition probability for the process $Y=\{Y(s), s\in[0,t]\}$, the transition probability for the process $X=\{X(s), s\in[0,t]\}$ can be found as

$$\begin{aligned}&p_X(0,x_0=f^{-1}(y_0);t,x_t=f^{-1}(y_t))\\&\quad=\frac{1}{\sqrt{f'(f^{-1}(y_0))f'(f^{-1}(y_t))}}\int_{-\infty}^{+\infty}\mathrm{d}\beta\mathrm{e}^{\mathrm{i}\beta t}\int_0^{+\infty}\mathrm{d}t^*\end{aligned}$$

$$\times \int_{(0,f^{-1}(y_0))}^{(t^*,f^{-1}(y_t))} Dx(\sigma)\mathrm{e}^{-(1/2)\int_0^{t^*} \dot{x}^2\,\mathrm{d}\sigma - \mathrm{i}\beta\int_0^{t^*} f'(x)^2\,\mathrm{d}\sigma}$$
$$\times \mathrm{e}^{-(1/2)\int_0^{t^*}(A[f(x)]^2+(\partial A/\partial y)[f(x)])f'(x)^2\,\mathrm{d}\sigma}$$
$$\times \mathrm{e}^{+\int_0^{t^*} A[f(x)]f'(x)\,\mathrm{d}x - (1/8)\int_0^{t^*}[3f''(x)^2/f'(x)^2 - 2f'''(x)/f'(x)]\,\mathrm{d}\sigma}. \tag{40}$$

As an example, consider the transformation $Y(t) = f(X(t)) = g^{-1}(X(t))$ where the function g is chosen in such a way that

$$g'(y) = \mathrm{e}^{-2\int_{y_0}^{y} A(z)\,\mathrm{d}z}. \tag{41}$$

Following Itô's lemma, for this choice we know that

$$\mathrm{d}X(t) = g'(Y(t)_\mathrm{L})\,\mathrm{d}W(t) = \frac{1}{f'(X(t)_\mathrm{L})}\,\mathrm{d}W(t) \tag{42}$$

or

$$\mathrm{d}W(t) = f'(X(t)_\mathrm{L})\,\mathrm{d}X(t). \tag{43}$$

Applying Theorem 5.1, a straightforward calculation leads to the result

$$p_X(0,x_0;t,x_t) = \int_0^{+\infty} \mathrm{d}t^*\delta\left(t - \int_0^{t^*} f'(x)^2\,\mathrm{d}\sigma\right)\cdot\int_{(0,x_0)}^{(t^*,x_t)} Dx(\sigma)\,\mathrm{e}^{-(1/2)\int_0^{t^*}\dot{x}^2\,\mathrm{d}\sigma} \tag{44}$$

which nicely fits with (43).

6. Calculation of the transition probabilities

6.1. Exact results for path integrals

In the previous sections, we showed how to find analytical expressions for the transition probability of diffusion processes by means of path integrals. For the computation of these functional integrations, we can rely on some methods and calculations from quantum mechanics. In Appendix A we summarize some important and useful exact computational results for common classes of path integrals, some of which were derived in the framework of earlier research on annuities with stochastic interest rates.

However, when the Lagrangian appearing in the path integral becomes too complicated, we will have to use approximations instead of exact results. In the following subsections, we will show how to find an approximation based on properties that hold for general path integrals.

In order to make things clear, we will use the notation $K(t_0,x_0;t_\mathrm{e},x_\mathrm{e})$ for Wiener integrals

$$K(t_0,x_0;t_\mathrm{e},x_\mathrm{e}) = \int_{(t_0,x_0)}^{(t_\mathrm{e},x_\mathrm{e})} Dx(s)\,\mathrm{e}^{-(1/2)\int_{t_0}^{t_\mathrm{e}}\dot{x}(s)^2\,\mathrm{d}s} = \frac{1}{\sqrt{2\pi(t_\mathrm{e}-t_0)}}\,\mathrm{e}^{-(x_\mathrm{e}-x_0)^2/2(t_\mathrm{e}-t_0)} \tag{45}$$

and the notation $I(t_0,x_0;t_e,x_e)$ for path integrals which are related but more general than Wiener integrals:

$$I(t_0,x_0;t_e,x_e)=\int_{(t_0,x_0)}^{(t_e,x_e)} Dx(s)\,e^{-(1/2)\int_{t_0}^{t_e}\dot{x}(s)^2\,ds-\int_{t_0}^{t_e}V[x(s)]\,ds}. \tag{46}$$

As a consequence, the transition probabilities for stochastic processes as derived in Section 3 can be expressed as

$$p(t_0,x_0;t_e,x_e)=C(t_0,x_0;t_e,x_e)\,I(t_0,x_0;t_e,x_e), \tag{47}$$

for adequate choices of the function V.

Furthermore, we will make use of the expected value over Wiener paths with known starting point and known final point, which can be written as

$$E_W[e^{-\int_{t_0}^{t_e}V[X(s)]\,ds}]=\frac{I(t_0,x_0;t_e,x_e)}{K(t_0,x_0;t_e,x_e)}. \tag{48}$$

If we are dealing with the absorbed Wiener process due to the restriction of the diffusion process to the domain $(0,+\infty)$, in each of Eqs. (45) and (46), the differential part $Dx(s)$ has to be replaced by $D_+x(s)$. More specifically, Eq. (45) has to be changed into (see [18])

$$\begin{aligned}K(t_0,x_0;t_e,x_e)&=\int_{(t_0,x_0)}^{(t_e,x_e)} D_+x(s)\,e^{-(1/2)\int_{t_0}^{t_e}\dot{x}(s)^2\,ds}\\&=\frac{1}{\sqrt{2\pi(t_e-t_0)}}\,e^{-(x_e-x_0)^2/2(t_e-t_0)}-\frac{1}{\sqrt{2\pi(t_e-t_0)}}\,e^{-(x_e+x_0)^2/2(t_e-t_0)}.\end{aligned} \tag{49}$$

Note that the expectation in (48) can also easily be extended for path integrals different from Wiener path integrals.

Finally, with the same notations, we will write the distribution for Wiener paths with fixed starting and final points as

$$F_s(x)=\mathrm{Prob}[X(s)\leqslant x|X(t_0)=x_0,X(t_e)=x_e] \tag{50}$$

and

$$f_s(x)=\frac{d}{dx}F_s(x)=\frac{K(t_0,x_0;s,x)\cdot K(s,x;t_e,x_e)}{K(t_0,x_0;t_e,x_e)}. \tag{51}$$

A straightforward calculation leads to

$$F_s(x)=\Phi\left(\sqrt{\frac{(t_e-t_0)}{(s-t_0)(t_e-s)}}\left(x-\frac{(t_e-s)x_0+(s-t_0)x_e}{t_e-t_0}\right)\right), \tag{52}$$

where $\Phi(x)$ denotes the standard normal cumulative probability, or $\Phi(x)=\left(1/\sqrt{2\pi}\right)\int_{-\infty}^{x}e^{-t^2/2}\,dt$.

6.2. Approximation of path integrals

In case the path integral cannot be calculated in an exact way with a closed form, as an alternative we can work with an approximation. Making use of techniques from quantum mechanics on the one hand and of the properties of convex ordered risks on the other hand, we can find a lower and upper bound for the path integrals. A combination of both expressions together with a correct scaling will enable us to find an approximation that seems to be very accurate.

The most important advantage of our methodology has to be found in the fact that it results in an analytical expression for the approximation, whereas most techniques that are presented in the literature lead to exclusively numerical approximations. Consequently, this new methodology is very interesting in the framework of the pricing of Asian options.

6.2.1. Upperbound

The method we propose to find accurate upperbounds for the transition probabilities, makes use of convex order. We briefly recall the most important concepts and necessary results; we refer to [9,12] for proofs and more details.

A variable A is said to be smaller than B in convex ordering, $A \leqslant_{\mathrm{cx}} B$, if for each convex function $u : \mathfrak{R} \to \mathfrak{R} : x \mapsto u(x)$ the expected values (provided they exist) are ordered as $E[u(A)] \leqslant E[u(B)]$.

As a consequence, $E[A] = E[B]$ and $E[(A-k)_+] \leqslant E[(B-k)_+]$ for all k, with $(x)_+ = \max(0, x)$.

Once an expression is known for the stop-loss premium $E[(B-k)_+]$, the distribution of the variable B can be easily found. Indeed, there is a well-known link between stop-loss premiums and the distribution, stating that the right-hand derivative of a stop-loss premium $E[(B-k)_+]$ with respect to k equals $\mathrm{Prob}[B \leqslant k] - 1$.

The notion of convex ordering can be extended from two single variables to two sums of variables, discrete or continuous. The results are summarized in the following two propositions (for a proof see [9,12]). For the distributions, we make use of the notation

$$F_X(x) = \mathrm{Prob}[X \leqslant x], \tag{53}$$

the inverse distributions are defined in the classical way as

$$F_X^{-1}(p) = \inf\{x \in \mathfrak{R} : F_X(x) \geqslant p\}. \tag{54}$$

Proposition 6.1. *Consider a sum of functions of random variables*

$$A = g_1(X_1) + g_2(X_2) + \cdots + g_n(X_n) \tag{55}$$

and for U an arbitrary random variable that is uniformly distributed on $[0, 1]$, *define the related stochastic quantity*

$$B = F^{-1}_{g_1(X_1)}(U) + F^{-1}_{g_2(X_2)}(U) + \cdots + F^{-1}_{g_n(X_n)}(U). \tag{56}$$

Then $A \leqslant_{\mathrm{cx}} B$.

Remark 6.1. The corresponding terms in the sums A and B are all mutually identically distributed, or $g_j(X_j) \stackrel{\mathrm{d}}{=} F^{-1}_{g_j(X_j)}(U)$.

Proposition 6.2. *Consider a functional integration*

$$A = \int_{t_0}^{t_e} g(X(s))\,\mathrm{d}s \tag{57}$$

and for U an arbitrary random variable that is uniformly distributed on $[0,1]$, *define the related stochastic quantity*

$$B = \int_{t_0}^{t_e} F^{-1}_{g(X(s))}(U)\,\mathrm{d}s. \tag{58}$$

Then $A \leqslant_{\mathrm{cx}} B$.

An application of the method of convex upperbounds to the transition probabilities of diffusion processes, brings us to the following result:

Theorem 6.1. *For a path integral with a structure as mentioned in Eq.* (46), *an upperbound can be found as*

$$I(t_0,x_0;t_e,x_e) \leqslant I^{\mathrm{upp}}(t_0,x_0;t_e,x_e), \tag{59}$$

where

$$I^{\mathrm{upp}}(t_0,x_0;t_e,x_e) = K(t_0,x_0;t_e,x_e)E_{\mathrm{U}}\left[\mathrm{e}^{-\int_{t_0}^{t_e} F^{-1}_{V(X(\tau))}(U)\,\mathrm{d}\tau}\right], \tag{60}$$

with U uniformly distributed on $[0,1]$.

The expectation can also be written as

$$I^{\mathrm{upp}}(t_0,x_0;t_e,x_e) = K(t_0,x_0;t_e,x_e)\int_{-\infty}^{+\infty} \mathrm{e}^{-k}\frac{\partial^2}{\partial k^2}G(k|t_0,x_0;t_e,x_e)\,\mathrm{d}k \tag{61}$$

with

$$G(k|t_0,x_0;t_e,x_e) = E_{\mathrm{U}}\left[\left(\int_{t_0}^{t_e} F^{-1}_{V(X(\tau))}(U)\,\mathrm{d}\tau - k\right)_+\right]. \tag{62}$$

6.2.2. Lowerbound

In order to improve the upperbound, we also need to derive a lowerbound for the transition probability of diffusion processes. The present result mainly originates from an application of the well-known inequality of Jensen.

Theorem 6.2. *For a path integral with a structure as mentioned in Eq.* (46), *a lower bound can be found as*

$$I(t_0,x_0;t_e,x_e) \geqslant I^{\mathrm{low}}(t_0,x_0;t_e,x_e), \tag{63}$$

where

$$I^{\mathrm{low}}(t_0,x_0;t_e,x_e) = K(t_0,x_0;t_e,x_e)E_{X(t_s)}\left[\mathrm{e}^{-\int_{t_0}^{t_s} E_{\mathrm{W}}[V(X(\tau))]\,\mathrm{d}\tau}\,\mathrm{e}^{-\int_{t_s}^{t_e} E_{\mathrm{W}}[V(X(\tau))]\,\mathrm{d}\tau}\right] \tag{64}$$

with t_s any point in time between t_0 and t_e.

Remark 6.2. The notation $E_{\mathrm{W}}[\dots]$ means an expectation over Wiener paths (or absorbed Wiener paths if the domain of the stochastic process is $(0,+\infty)$) between the two boundary time points.

6.2.3. *Approximation*

Consider a stochastic process $Y=\{Y(s), s\in[0,t]\}$ for which the transition probability can be expressed as in (47).

From Theorems 6.1 and 6.2, we know that

$$p(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}}) \leqslant C(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}})I^{\mathrm{upp}}(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}}),$$

$$p(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}}) \geqslant C(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}})I^{\mathrm{low}}(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}}), \tag{65}$$

for specific choices of the function V.

A problem with these bounds is the fact that we are no longer dealing with density functions. Therefore, we suggest to use a convex combination

$$\tilde{p}(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}}) = C(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}})\{z(t_0,x_0;t_{\mathrm{e}})I^{\mathrm{low}}(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}}) + (1-z(t_0,x_0;t_{\mathrm{e}}))I^{\mathrm{upp}}(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}})\} \tag{66}$$

resulting in an analytical approximation (which is a density) for the transition probability.

The factor $z(t_0,x_0;t_{\mathrm{e}})$ can be determined by the condition of a total mass equal to one, or

$$z(t_0,x_0;t_{\mathrm{e}}) = \frac{\int_{-\infty}^{+\infty} C(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}})\,I^{\mathrm{upp}}(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}})\,\mathrm{d}x - 1}{\int_{-\infty}^{+\infty} C(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}})\,[I^{\mathrm{upp}}(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}}) - I^{\mathrm{low}}(t_0,x_0;t_{\mathrm{e}},x_{\mathrm{e}})]\,\mathrm{d}x_{\mathrm{e}}}. \tag{67}$$

There is an extra advantage when working with this convex combination, due to the factor $z(t_0,x_0;t_{\mathrm{e}})$. Indeed, in the situation where one of the bounds turns out to be not that accurate, the contribution of that bound will have a less important impact on the approximation. Either of the bounds has an influence on the final approximation, but the closer the bound to the exact transition density, the larger the impact of that bound.

7. Examples

In the present section, we show how the transition probability for some common classes of diffusion processes can be expressed by means of a Feynman path integral, as explained in the previous section. Table 1 contains the results for the maximal probability path and for the long term probability; in Table 2, the results for the transition probability are summarized.

Table 1
Results for maximal probability path and long-term probability for special classes of diffusion processes

Model	Diffusion equation	Max. prob. path $y_{\mathrm{mod}}(s\|0,Y_0;t,y_t)$	Long-term probability $\bar{p}(\bar{y})$
Wiener model	$\mathrm{d}Y(t)=\mu\,\mathrm{d}t+\sigma\,\mathrm{d}W(t)$ Domain: $(-\infty,+\infty)$	$\frac{(t-s)y_0+sy_t}{t}$	$C\mathrm{e}^{(2\mu/\sigma^2)\bar{y}}$ (instable)
Geometric Wiener model	$\mathrm{d}Y(t)=\left(\mu+\frac{\sigma^2}{2}\right)Y(t)\,\mathrm{d}t$ $+\sigma Y(t)\,\mathrm{d}W(t)$ Domain: $(0,+\infty)$	$y_0^{(t-s)/t}\,y_t^{s/t}$	$C\bar{y}^{\,2\mu/\sigma^2-1}$ (instable)
Vasicek model	$\mathrm{d}Y(t)=\kappa(\alpha-Y(t))\,\mathrm{d}t+\sigma\,\mathrm{d}W(t)$ Domain: $(-\infty,+\infty)$	$\alpha+\frac{(y_t-\alpha)\sinh(\kappa s)}{\sinh(\kappa t)}+\frac{(y_0-\alpha)\sinh(\kappa(t-s))}{\sinh(\kappa t)}$	$C\mathrm{e}^{-(\kappa/\sigma^2)(\bar{y}-\alpha)^2}$ with $C=\sqrt{\frac{\kappa}{\pi\sigma^2}}$
Cox Ingersoll Ross model	$\mathrm{d}Y(t)=\kappa(\alpha-Y(t))\,\mathrm{d}t+\sigma\sqrt{Y(t)}\,\mathrm{d}W(t)$ where it is assumed that $2\kappa\alpha/\sigma^2\geqslant 1$ Domain: $(0,+\infty)$	$(y_0+C)\cosh(\kappa s)-C+\sinh(\kappa s)$ $\times\sqrt{(y_0+C)^2-C^2+\frac{\sigma^4}{4\kappa^2}\left(\frac{2\kappa\alpha}{\sigma^2}-\frac{1}{2}\right)\left(\frac{2\kappa\alpha}{\sigma^2}-\frac{3}{2}\right)}$ with $C=\frac{1}{\cosh(\kappa t)-1}(y_0+y_t)+\frac{\sinh(\kappa t)}{\cosh(\kappa t)-1}$ $\times\sqrt{\frac{\sigma^4}{4\kappa^2}\left(\frac{2\kappa\alpha}{\sigma^2}-\frac{1}{2}\right)\left(\frac{2\kappa\alpha}{\sigma^2}-\frac{3}{2}\right)+y_0y_t\frac{2}{\cosh(\kappa t)-1}}$	$C\bar{y}^{\,2\kappa\alpha/\sigma^2-1}\,\mathrm{e}^{-(2\kappa/\sigma^2)\bar{y}}$ with $C=(2\kappa/\sigma^2)^{2\kappa\alpha/\sigma^2}/\Gamma(2\kappa\alpha/\sigma^2)$
Adapted geometric Wiener model	$\mathrm{d}Y(t)=\left(\left(\delta+\frac{\sigma^2}{2}\right)Y(t)-1\right)\,\mathrm{d}t$ $+\sigma Y(t)\,\mathrm{d}W(t)$ Domain: $(0,+\infty)$	$\frac{\mathrm{e}^{-As}}{4A^2B}((2\delta-\sigma^2)^2-4A^2$ $+\,2B(2\delta-\sigma^2)\mathrm{e}^{As}+B^2\mathrm{e}^{2As})$ where the constants A and B have to be determined numerically by the constraints $y_{\mathrm{mod}}(0)=y_0$ and $y_{\mathrm{mod}}(t)=y_t$	$C\bar{y}^{\,2\delta/\sigma^2-1}\mathrm{e}^{2/(\sigma^2\bar{y})}$ (instable)
Bessel model with drift	$\mathrm{d}Y(t)=\left(\frac{1}{Y(t)}-2\right)\,\mathrm{d}t+\mathrm{d}W(t)$ Domain: $(0,+\infty)$	Solution of implicit equation $\frac{2}{C^3}\ln\left(\frac{C+\sqrt{C^2-4/y}}{C-\sqrt{C^2-4/y}}\right)+\frac{y}{C^2}\sqrt{C^2-4/y}=s$ where the constant C follows from the boundary condition $y_{\mathrm{mod}}(t)=y_t$	$C\bar{y}^{\,2}\mathrm{e}^{-4\bar{y}}$ with $C=32$
Inverse of Feller's square root model	$\mathrm{d}Y(t)=Y(t)(\kappa-(\sigma^2-\kappa\alpha)Y(t))\,\mathrm{d}t$ $+\sigma Y(t)^{3/2}\,\mathrm{d}W(t)$	$\cosh(\kappa s)\left(\frac{1}{y_0}+C\right)-C+\sinh(\kappa s)$ $\times\sqrt{\left(\frac{1}{y_0}+C\right)^2-C^2+\frac{\sigma^4}{4\kappa^2}\left(\frac{7}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)\left(\frac{5}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)}$	$C\,\bar{y}^{\,-\left(5-\frac{2\kappa\alpha}{\sigma^2}\right)}\,\mathrm{e}^{-\frac{2\kappa}{\sigma^2}\frac{1}{\bar{y}}}$ with $C=\left(\frac{2\kappa}{\sigma^2}\right)^{4-\frac{2\kappa\alpha}{\sigma^2}}/\Gamma\left(4-\frac{2\kappa\alpha}{\sigma^2}\right)$

Table 1 (Continued)

Model	Diffusion equation	Max. prob. path $y_{\mathrm{mod}}(s\mid 0, Y_0; t, y_t)$	Long-term probability $\bar{p}(\bar{y})$
	where it is assumed that $2\kappa\alpha/\sigma^2 \geqslant 1$ Domain: $(0,+\infty)$	with $C = \frac{1}{\cosh(\kappa t)-1}\left(\frac{1}{y_0}+\frac{1}{y_t}\right)+\frac{\sinh(\kappa t)}{\cosh(\kappa t)-1}$ $\times\sqrt{\frac{\sigma^4}{4\kappa^2}\left(\frac{7}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)\left(\frac{5}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)+\frac{1}{y_0 y_t}\frac{2}{\cosh(\kappa t)-1}}$	
Linear drift CEV model	$\mathrm{d}Y(t)=\kappa(\alpha-Y(t))\,\mathrm{d}t$ $+\sigma Y(t)^{3/2}\,\mathrm{d}W(t)$ Domain: $(0,+\infty)$	Solution of implicit equation $\int_{y_0}^{y_{\mathrm{mod}}(s)}\left(\frac{3\sigma^4}{16}z^4+C\sigma^2 z^3+\kappa^2\left(1-\frac{3\alpha\sigma^2}{\kappa}\right)z^2\right.$ $\left.-2\kappa^2\alpha z+\kappa^2\alpha^2\right)^{-1/2}\,\mathrm{d}z=s$ where the constant C follows from the boundary condition $y_{\mathrm{mod}}(t)=y_t$	$C\frac{1}{\bar{y}^3}\exp\left\{\frac{2\kappa}{\sigma^2}\frac{1}{\bar{y}}-\frac{\kappa\alpha}{\sigma^2}\frac{1}{\bar{y}^2}\right\}$ with $C=\frac{\sigma^2}{2\kappa\alpha}\left[1+\sqrt{\frac{\pi\kappa}{\sigma^2\alpha}}\,\mathrm{e}^{\frac{\kappa}{\sigma^2\alpha}}\right.$ $\left.\times\left(1+\mathrm{erf}\left(\sqrt{\frac{\kappa}{\sigma^2\alpha}}\right)\right)\right]$
Nonlinear mean reversion model	$\mathrm{d}Y(t)=(\alpha_{-1}Y(t)^{-1}-\alpha_0+\alpha_1 Y(t)$ $+\alpha_2 Y(t)^2)\,\mathrm{d}t+\sigma Y(t)^{3/2}\,\mathrm{d}W(t)$ Domain: $(0,+\infty)$	Solution of implicit equation $\int_{y_0}^{y_{\mathrm{mod}}(s)}(Az^6+Gz^5+Bz^4+Cz^3$ $+Dz^2+Ez+F)^{-1/2}z\,\mathrm{d}z=s$ where $A=\frac{\sigma^4}{4}\left(\frac{3}{2}-\frac{2\alpha_2}{\sigma^2}\right)\left(\frac{1}{2}-\frac{2\alpha_2}{\sigma^2}\right)$ $B=\alpha_1^2-\alpha_0\sigma^2\left(3-\frac{2\alpha_2}{\sigma^2}\right)$ $C=2\alpha_1\alpha_0-\alpha_{-1}\sigma^2\left(4-\frac{2\alpha_2}{\sigma^2}\right)$ $D=\alpha_0^2+2\alpha_1\alpha_{-1}$ $E=2\alpha_0\alpha_{-1}$ $F=\alpha_{-1}^2$ and where G follows from the boundary condition $y_{\mathrm{mod}}(t)=y_t$	$C\bar{y}^{-\left(3-\frac{2\alpha_2}{\sigma^2}\right)}\mathrm{e}^{-\frac{2\alpha_1}{\sigma^2}\frac{1}{\bar{y}}-\frac{\alpha_0}{\sigma^2}\frac{1}{\bar{y}^2}-\frac{2\alpha_{-1}}{3\sigma^2}\frac{1}{\bar{y}^3}}$ where C follows from the condition of a total mass equal to one
Double well potential model	$\mathrm{d}Y(t)=(Y(t)-Y(t)^3)\,\mathrm{d}t+\mathrm{d}W(t)$ Domain: $(-\infty,+\infty)$	Solution of implicit equation $\int_{y_0}^{y_{\mathrm{mod}}(s)}\frac{\mathrm{d}z}{\sqrt{z^6-2z^4-2z^2+C}}=s$ where the constant C follows from the boundary condition $y_{\mathrm{mod}}(t)=y_t$	$C\mathrm{e}^{\bar{y}^2-\frac{\bar{y}^4}{2}}$ with $C=\frac{2}{\pi}\frac{\mathrm{e}^{-1/4}}{I_{1/4}\left(\frac{1}{4}\right)+I_{-1/4}\left(\frac{1}{4}\right)}$

Table 2
Closed-form expressions for transition densities for special classes of diffusion processes

Model	Path integral expression for $p(0, y_0; t, y_t)$	Transition probability $p(0, y_0; t, y_t)$
Wiener model	$\frac{1}{\sigma}\mathrm{e}^{-\frac{\mu^2 t}{2\sigma^2}+\frac{\mu}{\sigma^2}(y_t-y_0)} \int_{\left(0,\frac{1}{\sigma}y_0\right)}^{\left(t,\frac{1}{\sigma}y_t\right)} Dx(s)\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{x}^2\,\mathrm{d}s}$	$\frac{1}{\sqrt{2\pi\sigma^2 t}}\mathrm{e}^{-\frac{(y_t-y_0-\mu t)^2}{2\sigma^2 t}}$
Geometric Wiener model	$\frac{1}{\sigma y_t}\left(\frac{y_t}{y_0}\right)^{\frac{\mu}{\sigma^2}} \mathrm{e}^{-\frac{\mu^2 t}{2\sigma^2}} \int_{\left(0,\frac{1}{\sigma}\ln y_0\right)}^{\left(t,\frac{1}{\sigma}\ln y_t\right)} Dx(s)\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{x}^2\,\mathrm{d}s}$	$\frac{1}{\sqrt{2\pi\sigma^2 t}\,y_t}\,\mathrm{e}^{-\frac{1}{2\sigma^2 t}\left(\ln\frac{y_t}{y_0}-\mu t\right)^2}$
Vasicek model	$\frac{1}{\sigma}\mathrm{e}^{-\frac{\kappa^2\alpha^2 t}{2\sigma^2}+\frac{\kappa t}{2}}\,\mathrm{e}^{\frac{\kappa\alpha}{\sigma^2}(y_t-y_0)-\frac{\kappa}{2\sigma^2}(y_t^2-y_0^2)}$	$\sqrt{\frac{\kappa}{\pi\sigma^2(1-\mathrm{e}^{-2\kappa t})}}$
	$\int_{\left(0,\frac{1}{\sigma}y_0\right)}^{\left(t,\frac{1}{\sigma}y_t\right)} Dx(s)\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{x}^2\,\mathrm{d}s}\,\mathrm{e}^{-\frac{\kappa^2}{2}\int_0^t x^2\,\mathrm{d}s+\frac{\kappa^2\alpha}{\sigma}\int_0^t x\,\mathrm{d}s}$	$\exp\left\{-\frac{\kappa}{\sigma^2(1-\mathrm{e}^{-2\kappa t})}\left((y_t-\alpha)-(y_0-\alpha)\mathrm{e}^{-\kappa t}\right)^2\right\}$
Cox Ingersoll Ross model	$\frac{1}{\sigma\sqrt{y_t}}\left(\frac{y_t}{y_0}\right)^{\frac{\kappa\alpha}{\sigma^2}-\frac{1}{4}}\,\mathrm{e}^{\frac{\kappa^2\alpha t}{\sigma^2}}\,\mathrm{e}^{-\frac{\kappa}{\sigma^2}(y_t-y_0)}$	$\frac{2\kappa\mathrm{e}^{-\kappa t/2}}{\sigma^2(1-\mathrm{e}^{-\kappa t})}\,\mathrm{e}^{\kappa^2\alpha t/\sigma^2}\left(\frac{y_t}{y_0}\right)^{\frac{\kappa\alpha}{\sigma^2}-\frac{1}{2}}$
	$\int_{\left(0,\frac{2}{\sigma}\sqrt{y_0}\right)}^{\left(t,\frac{2}{\sigma}\sqrt{y_t}\right)} D_+x(s)\,\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{x}^2\,\mathrm{d}s-\frac{\kappa^2}{8}\int_0^t x^2\,\mathrm{d}s}$	$\exp\left\{-\frac{2\kappa}{\sigma^2(1-\mathrm{e}^{-\kappa t})}\left(y_0\mathrm{e}^{-\kappa t}+y_t\right)\right\}$
	$\times\mathrm{e}^{\left(\left(\frac{\sqrt{2}\kappa\alpha}{\sigma^2}-\frac{1}{\sqrt{2}}\right)^2-\frac{1}{8}\right)\int_0^t\frac{1}{x^2}\,\mathrm{d}s}$	$I_{2\kappa\alpha/\sigma^2-1}\left(\frac{4\kappa\mathrm{e}^{-\kappa t/2}}{\sigma^2(1-\mathrm{e}^{-\kappa t})}\sqrt{y_0 y_t}\right)$
Adapted geometric Wiener model	$\frac{1}{\sigma y_t}\left(\frac{y_t}{y_0}\right)^{\frac{\delta}{\sigma^2}}\,\mathrm{e}^{-\frac{\delta^2 t}{2\sigma^2}}\,\mathrm{e}^{\frac{1}{\sigma^2}\left(\frac{1}{y_t}-\frac{1}{y_0}\right)}$	$\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\frac{1}{\sigma^5}\frac{1}{\sqrt{t}}\frac{1}{y_0^2}\sqrt{\frac{y_0}{y_t}}\exp\left\{-\frac{\delta^2 t}{2\sigma^2}-\frac{\sigma^2}{2}\left(1-\frac{\delta}{\sigma}\right)^2 t+\frac{2\pi^2}{\sigma^2 t}\right\}$
	$\int_{\left(0,\frac{1}{\sigma}\ln y_0\right)}^{\left(t,\frac{1}{\sigma}\ln y_t\right)} Dx(s)\,\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{x}^2\,\mathrm{d}s}$	$\int_0^{\infty}\mathrm{d}s\,\frac{\mathrm{e}^{-\frac{4}{\sigma^2}\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)}}{\sinh^2\left(\frac{2s}{\sigma^2}\right)}\exp\left\{-\frac{1}{\sigma^2}\frac{1}{\tanh\left(\frac{2s}{\sigma^2}\right)}\left(\frac{1}{y_t}+\frac{1}{y_0}\right)\right\}$
	$\times\mathrm{e}^{-\frac{1}{2}\left(1-\frac{2\delta}{\sigma^2}\right)\int_0^t\mathrm{e}^{-\sigma x}\,\mathrm{d}s-\frac{1}{2\sigma^2}\int_0^t\mathrm{e}^{-2\sigma x}\,\mathrm{d}s}$	$\int_0^{\infty}\mathrm{d}z\,\mathrm{e}^{-\frac{2z^2}{\sigma^2 t}}\sinh(z)\sin\left(\frac{4\pi z}{\sigma^2 t}\right)\exp\left\{-\frac{2}{\sigma^2}\frac{1}{\sqrt{y_0 y_t}}\frac{\cosh(z)}{\sinh\left(\frac{2s}{\sigma^2}\right)}\right\}$
Bessel model with drift	$\left(\frac{y_t}{y_0}\right)\mathrm{e}^{-2t}\,\mathrm{e}^{-2(y_t-y_0)}\int_{(0,y_0)}^{(t,y_t)} D+y(s)\,\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{y}^2\,\mathrm{d}s+2\int_0^t\frac{1}{y}\,\mathrm{d}s}$	$\left(\frac{y_t}{y_0}\right)\mathrm{e}^{-2t}\,\mathrm{e}^{-2(y_t-y_0)}\int_{-\infty}^{+\infty}\mathrm{d}\beta\,\mathrm{e}^{\mathrm{i}\beta t}\int_0^{+\infty}\mathrm{d}\vartheta\,\mathrm{e}^{8\vartheta}$ $\frac{\sqrt{2\mathrm{i}\beta}}{\sinh\left(2\sqrt{2\mathrm{i}\beta\vartheta}\right)}I_1\left(\frac{2\sqrt{2\mathrm{i}\beta}\sqrt{y_0 y_t}}{\sinh\left(2\sqrt{2\mathrm{i}\beta\vartheta}\right)}\right)\exp\left(-\frac{2\sqrt{2\mathrm{i}\beta}(y_0+y_t)}{\tanh\left(2\sqrt{2\mathrm{i}\beta\vartheta}\right)}\right)$
Inverse of Feller's	$\frac{1}{\sigma y_t^{3/2}}\left(\frac{y_0}{y_t}\right)^{\frac{1}{2}\left(\frac{7}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)}\mathrm{e}^{\frac{\kappa}{2}\left(4-\frac{2\kappa\alpha}{\sigma}\right)t}\,\mathrm{e}^{-\frac{\kappa}{\sigma^2}\left(\frac{1}{y_t}-\frac{1}{y_0}\right)}$	$\frac{2\kappa}{\sigma^2 y_t^2}\frac{1}{(1-\mathrm{e}^{-\kappa t})}\left(\frac{y_0}{y_t}\right)^{\frac{1}{2}\left(3-\frac{2\kappa\alpha}{\sigma^2}\right)}\mathrm{e}^{\frac{\kappa}{2}\left(3-\frac{2\kappa\alpha}{\sigma^2}\right)t}$
square root model	$\int_{\left(0,-\frac{2}{\sigma\sqrt{y_0}}\right)}^{\left(t,-\frac{2}{\sigma\sqrt{y_t}}\right)} D_-x(s)\,\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{x}^2\,\mathrm{d}s-\frac{\kappa^2}{8}\int_0^t x^2\,\mathrm{d}s}$	$\exp\left\{-\frac{\kappa}{\sigma^2}\left(\frac{1}{y_t}-\frac{1}{y_0}\right)-\frac{4\kappa}{\sigma^2}\left(\frac{1}{y_t}+\frac{1}{y_0}\right)\frac{(1+\mathrm{e}^{-\kappa t})}{(1-\mathrm{e}^{-\kappa t})}\right\}$

Table 2 (Continued)

Model	Path integral expression for $p(0, y_0; t, y_t)$	Transition probability $p(0, y_0; t, y_t)$
	$\times \mathrm{e}^{-\frac{1}{2}\left(\frac{7}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)\left(\frac{5}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)\int_0^t \frac{1}{x^2}\,\mathrm{d}s}$	$I_{3-\frac{2\kappa\alpha}{\sigma^2}}\left(s\frac{4\kappa}{\sigma^2}\frac{\mathrm{e}^{-\kappa t/2}}{\sqrt{y_0 y_t}}\frac{1}{(1-\mathrm{e}^{-\kappa t})}\right)$
Linear drift CEV	$\frac{\mathrm{e}^{-\kappa t}}{\sigma y_t^{3/2}}\left(\frac{y_0}{y_t}\right)^{\frac{3}{4}} \mathrm{e}^{\frac{\kappa}{\sigma^2}\left(\frac{1}{y_t}-\frac{1}{y_0}\right)} \mathrm{e}^{-\frac{\kappa\alpha}{2\sigma^2}\left(\frac{1}{y_t^2}-\frac{1}{y_0^2}\right)}$	Cannot be calculated exactly.
model	$\int_{\left(0,-\frac{2}{\sigma\sqrt{y_0}}\right)}^{\left(t,-\frac{2}{\sigma\sqrt{y_t}}\right)} D_{-}x(s)\,\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{x}^2\,\mathrm{d}s}$	A convex combination as in (66) with
	$\times \mathrm{e}^{-\frac{1}{8}\int_0^t\left[\frac{3}{x^2}+\kappa^2\left(1-\frac{3\alpha\sigma^2}{\kappa}\right)x^2-\frac{\kappa^2\alpha\sigma^2}{2}x^4+\frac{\kappa^2\alpha^2\sigma^4}{16}x^6\right]\mathrm{d}s}$	$V[z]=\frac{1}{8}\left[\frac{3}{z^2}+\kappa^2\left(1-\frac{3\alpha\sigma^2}{\kappa}\right)z^2-\frac{\kappa^2\alpha\sigma^2}{2}z^4+\frac{\kappa^2\alpha^2\sigma^4}{16}z^6\right]$ results in a closed-form approximation
Nonlinear mean	$\frac{1}{\sigma y_t^{3/2}}\left(\frac{y_0}{y_t}\right)^{\frac{1}{2}\left(\frac{3}{2}-\frac{2\alpha_2}{\sigma^2}\right)} \mathrm{e}^{\frac{\alpha_1}{2}\left(2-\frac{2\alpha_2}{\sigma^2}\right)t}$	Cannot be calculated exactly.
reversion model	$\mathrm{e}^{-\frac{\alpha_1}{\sigma^2}\left(\frac{1}{y_t}-\frac{1}{y_0}\right)} \mathrm{e}^{-\frac{\alpha_0}{2\sigma^2}\left(\frac{1}{y_t^2}-\frac{1}{y_0^2}\right)} \mathrm{e}^{-\frac{\alpha_{-1}}{3\sigma^2}\left(\frac{1}{y_t^3}-\frac{1}{y_0^3}\right)}$	A convex combination as in (66) with
	$\int_{\left(0,-\frac{2}{\sigma\sqrt{y_0}}\right)}^{\left(t,-\frac{2}{\sigma\sqrt{y_t}}\right)} D_{-}x(s)\,\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{x}^2\,\mathrm{d}s}$	$V[z]=Az^{-2}+Bz^2+Cz^4+Dz^6+Ez^8+Fz^{10}$
	$\times \mathrm{e}^{-\int_0^t [Ax^{-2}+Bx^2+Cx^4+Dx^6+Ex^8+Fx^{10}]\,\mathrm{d}s}$ where $A=\frac{1}{2}\left(\frac{3}{2}-\frac{2\alpha_2}{\sigma^2}\right)\left(\frac{1}{2}-\frac{2\alpha_2}{\sigma^2}\right)$ $B=\frac{1}{8}\left[\alpha_1^2-\alpha_0\sigma^2\left(3-\frac{2\alpha_2}{\sigma^2}\right)\right]$ $C=\frac{\sigma^2}{32}\left[2\alpha_1\alpha_0\sigma^2-\alpha_{-1}\sigma^2\left(4-\frac{2\alpha_2}{\sigma^2}\right)\right]$ $D=\frac{\sigma^4}{128}(\alpha_0^2+2\alpha_1\alpha_{-1})$ $E=\frac{\sigma^6}{256}\alpha_0\alpha_{-1}$ $F=\frac{\sigma^8}{2048}\alpha_{-1}^2$	results in a closed-form approximation
Double Well potential	$\mathrm{e}^{\frac{1}{2}(y_t^2-y_0^2)-\frac{1}{4}(y_t^4-y_0^4)-\frac{1}{2}t}$	Cannot be calculated exactly.
model	$\int_{(0,y_0)}^{(t,y_t)} Dy(s)\,\mathrm{e}^{-\frac{1}{2}\int_0^t \dot{y}^2\,\mathrm{d}s-\frac{1}{2}\int_0^t (y^6-2y^4-2y^2)\,\mathrm{d}s}$	A convex combination as in (66) with $V[z]=\frac{1}{2}(z^6-2z^4-2z^2)$ results in a closed-form approximation

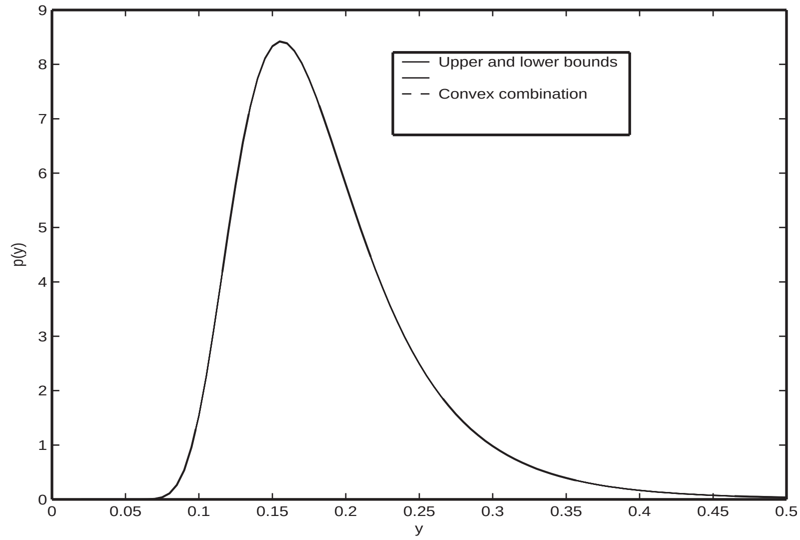

Fig. 1. Approximation of the density function for the CEV diffusion model, with $t = 1$ and starting point $y_0 = 0.2$.

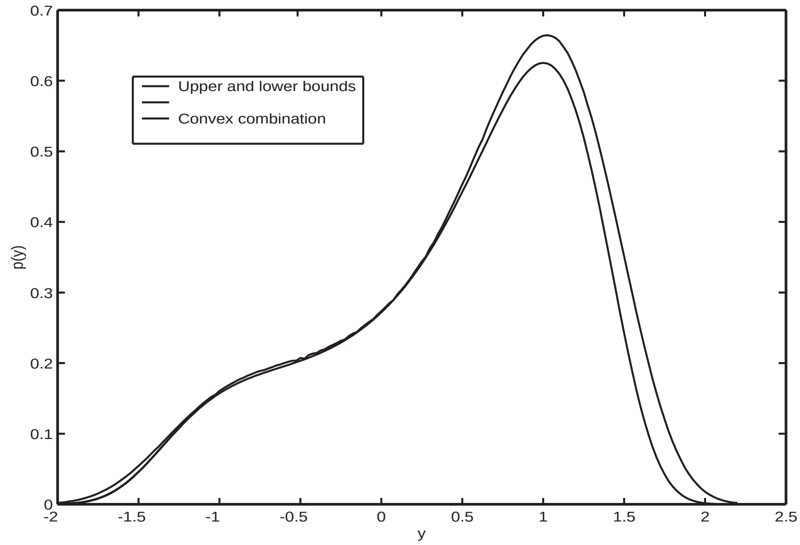

Fig. 2. Approximation of the density function for the double well potential model, with $t = 1$ and starting point $y_0 = 0.5$.

8. Numerical illustration

In this last section, we want to show the high accuracy of our approximations. We present graphs for the CEV Diffusion Model (Fig. 1), for the double well potential (Figs. 2 and 3) and for the nonlinear mean reversion model (Fig. 4), three models for which the exact transition probability is not known in a closed form. For the parameters appearing in the models, use has been made of the

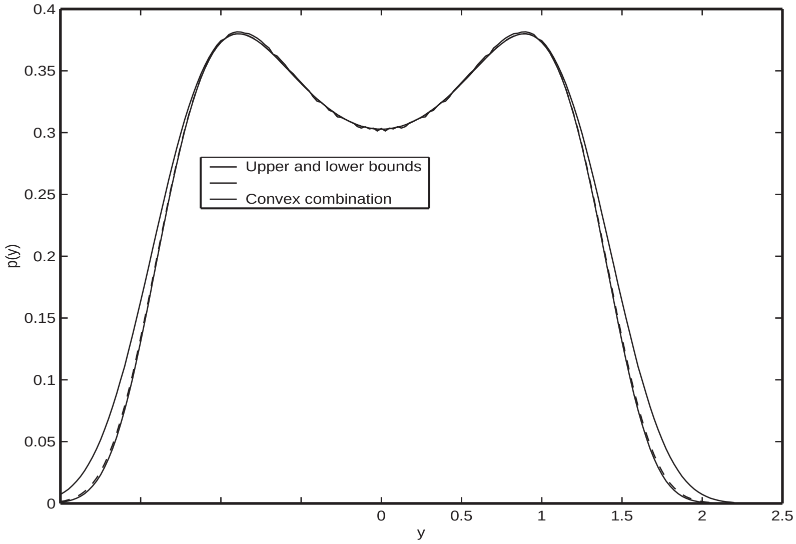

Fig. 3. Approximation of the density function for the double well potential model, with $t = 1$ and starting point $y_0 = 0$.

same values as mentioned in the paper of Ait-Sahalia (see [1]):

CEV diffusion model	$\alpha = 0.0808$
	$\kappa = 0.0972$
	$\sigma = 0.7224$
Nonlinear mean reversion model	$\alpha_{-1} = 0.00107$
	$\alpha_0 = -0.0517$
	$\alpha_1 = 0.877$
	$\alpha_2 = -4.604$
	$\sigma = 0.8047.$

Each figure contains our upper and lower bound, and the final new approximation which is based on a convex combination of the two bounds.

For the CEV diffusion model, upper and lower bounds are almost equal, such that the convex combination provides a very efficient approximation of the exact transition probability density. As can be seen in the graphs for the double well potential, the accuracy of both bounds is still very high, be it that the lower bound performs slightly better. For the nonlinear mean reversion model, the upper bound is less accurate, but fortunately the lower bound does better. Thanks to the fact that the total mass of the lower bound is only little lower than 1, it seems that also in this case, the final approximation performs very well.

Acknowledgements

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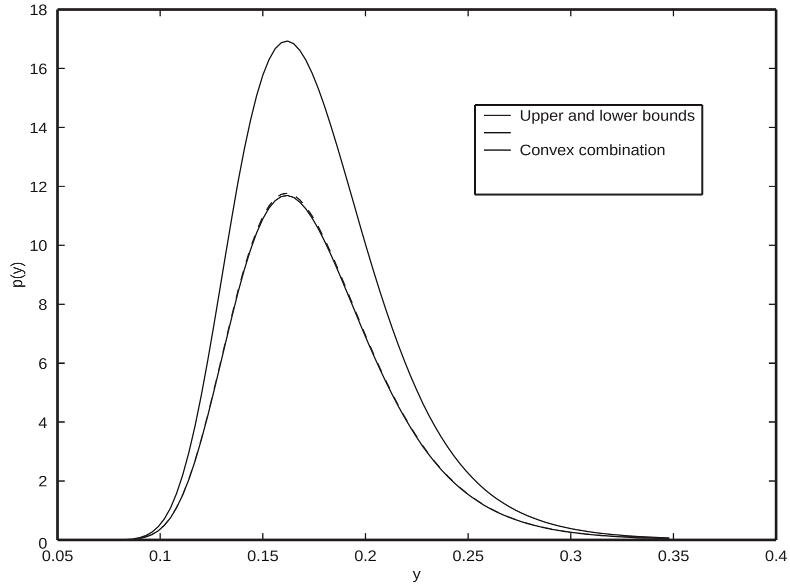

Fig. 4. Approximation of the density function for the nonlinear mean reversion model, with $t=1$ and starting point $y_0=1$.

Appendix A. Computational results for path integrals

In this first appendix we mention some useful results about path integrals for which an explicit expression is known. For the calculation of the Wiener and Gaussian integrals, use has been made of the methods of Feynman and Hibbs [6]. The results for the Calogero integrals are based on a result of Goovaerts [8] and Vanneste et al. [18]. For the calculation of the exponential integral, we used a special coordinate transformation as was done in a similar proof in [4]; this enables us to rewrite the result of the Calogero integral into a result for the exponential path integral.

A.1. Wiener integrals

Ordinary Wiener process

$$\int_{(t_0,x_0)}^{(t_e,x_e)} Dx(s)\mathrm{e}^{-(1/2)\int_{t_0}^{t_e}\dot{x}(s)^2\,\mathrm{d}s} = \frac{1}{\sqrt{2\pi(t_e-t_0)}}\,\mathrm{e}^{-(x_e-x_0)^2/2(t_e-t_0)}. \tag{A.1}$$

Absorbed Wiener process

$$\int_{(t_0,x_0)}^{(t_e,x_e)} D_+x(s)\mathrm{e}^{-(1/2)\int_{t_0}^{t_e}\dot{x}(s)^2\,\mathrm{d}s}$$

$$=\frac{1}{\sqrt{2\pi(t_e-t_0)}}\,\mathrm{e}^{-(x_e-x_0)^2/2(t_e-t_0)} - \frac{1}{\sqrt{2\pi(t_e-t_0)}}\,\mathrm{e}^{-(x_e+x_0)^2/2(t_e-t_0)}. \tag{A.2}$$

A.2. Gaussian integrals

$$\int_{(t_0,x_0)}^{(t_e,x_e)} Dx(s)\mathrm{e}^{-(1/2)\int_{t_0}^{t_e}\dot{x}(s)^2\,\mathrm{d}s-\frac{a^2}{2}\int_{t_0}^{t_e}x(s)^2\,\mathrm{d}s+b\int_{t_0}^{t_e}x(s)\,\mathrm{d}s}$$

$$=\sqrt{\frac{a}{\pi(1-\mathrm{e}^{-2a(t_e-t_0)})}}\exp\left\{-\frac{a}{2}(t_e-t_0)+\frac{b^2}{2a^2}(t_e-t_0)\right\}$$

$$\times\exp\left\{\frac{(1+\mathrm{e}^{-2a(t_e-t_0)})}{a^3(1-\mathrm{e}^{-2a(t_e-t_0)})}(2b^2-2a^2b(x_0+x_e)+a^4(x_0^2+x_e^2))\right\}$$

$$\times\exp\left\{-\frac{2\mathrm{e}^{-a(t_e-t_0)}}{a^3(1-\mathrm{e}^{-2a(t_e-t_0)})}(b-a^2x_0)(b-a^2x_e)\right\}. \tag{A.3}$$

A.3. Calogero integrals

$$\int_{(t_0,x_0)}^{(t_e,x_e)} Dx(s)\mathrm{e}^{-(1/2)\int_{t_0}^{t_e}\dot{x}(s)^2\,\mathrm{d}s-\frac{a^2}{2}\int_{t_0}^{t_e}x(s)^2\,\mathrm{d}s-b\int_{t_0}^{t_e}\frac{\mathrm{d}s}{x(s)^2}}$$

$$=\frac{a\sqrt{x_0x_e}}{\sinh(a(t_e-t_0))}I_{\sqrt{2b+\frac{1}{4}}}\left[\frac{ax_0x_e}{\sinh(a(t_e-t_0))}\right]\exp\left\{-\frac{a(x_0^2+x_e^2)}{\tanh(a(t_e-t_0))}\right\}, \tag{A.4}$$

where $I_g[z]$ denotes a modified Bessel function, which can be expressed in a series as

$$I_g[z]=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(g+k+1)}\left(\frac{z}{2}\right)^{g+2k}, \tag{A.5}$$

and arises as a solution of the differential equation

$$\psi''(z)+\frac{1}{z}\psi'(z)-\left(1+\frac{g^2}{z^2}\right)\psi(z)=0. \tag{A.6}$$

A.4. Exponential path integrals

$$\int_{(t_0,x_0)}^{(t_e,x_e)} Dx(s)\mathrm{e}^{-(1/2)\int_{t_0}^{t_e}\dot{x}(s)^2\,\mathrm{d}s-\frac{a^2}{2}\int_{t_0}^{t_e}\mathrm{e}^{-2\sigma x}\,\mathrm{d}s-b\int_{t_0}^{t_e}\mathrm{e}^{-\sigma x}\,\mathrm{d}s}$$

$$=\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\frac{a^2}{\sigma^2}\frac{1}{\sqrt{t_e-t_0}}\mathrm{e}^{2\pi^2/\sigma^2(t_e-t_0)-\frac{\sigma}{2}(x_0+x_e)}$$

$$\times\int_0^{\infty}\mathrm{d}s\,\mathrm{e}^{-(4b/\sigma^2)s}\frac{1}{\sinh^2(2as/\sigma)}\exp\left\{-\frac{a}{\sigma}\frac{1}{\tanh(2as/\sigma)}(\mathrm{e}^{-\sigma x_0}+\mathrm{e}^{-\sigma x_e})\right\}$$

$$\times\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{-2y^2/\sigma^2(t_e-t_0)}\sinh(y)\sin\left(\frac{4\pi y}{\sigma^2(t_e-t_0)}\right)$$

$$\times\exp\left\{-\frac{2a}{\sigma}\frac{\cosh(y)}{\sinh(2as/\sigma)}\mathrm{e}^{-(\sigma/2)(x_0+x_e)}\right\}. \tag{A.7}$$

Appendix B. Proofs of the theorems

Proof of Theorem 3.1. From probability theory, we know that the solution of the stochastic differential equation (16) is unique. Hence, a solution that is found in another way, automatically leads to the same transition probability.

We start with a discretisation of equation (16),

$$y_{i+1} - y_i = (A(y_i) + \theta(A(y_{i+1}) - A(y_i)))\varepsilon + w_{i+1} - w_i \tag{B.1}$$

for $\varepsilon = t/n$, $t_i = i\varepsilon$ and $y_i = Y(t_i)$. This can be rewritten as

$$y_{i+1} - y_i = A(y_i)\varepsilon + \theta \frac{\partial A(y_i)}{\partial y}(y_{i+1} - y_i)\varepsilon + w_{i+1} - w_i. \tag{B.2}$$

In order to find a path integral expression for the transition probability of the process $Y = \{Y(s), s \in [0,t]\}$, we will perform a change of variables from w to y in the Brownian path integral

$$\int_{(0,w_0)}^{(t,w_t)} Dw(s)\mathrm{e}^{-(1/2)\int_0^t \dot{w}(s)^2\,\mathrm{d}s} \tag{B.3}$$

when written as the limit of an $(n-1)$-fold integration. Due to the nature of the variables, the Jacobian matrix $J = \partial(w_1, w_2, \ldots, w_{n-1})/\partial(y_1, y_2, \ldots, y_{n-1})$ is an uppertriangular matrix, and therefore

$$|J| = \prod_{i=1}^{n-1}\left(1 - \theta\varepsilon\frac{\partial A(y_{i-1})}{\partial y}\right) \approx \mathrm{e}^{-\theta\varepsilon\sum_{i=1}^{n-1}\partial A(y_{i-1})/\partial y}. \tag{B.4}$$

This brings us for the path integral to

$$\begin{aligned}\int_{(0,w_0)}^{(t,w_t)} Dw(s)\,\mathrm{e}^{-(1/2)\int_0^t \dot{w}(s)^2\,\mathrm{d}s} = {} & \lim_{n\to\infty}\frac{1}{\sqrt{2\pi\varepsilon}^n}\int \mathrm{d}y_1 \ldots \int \mathrm{d}y_{n-1} \\ & \times \mathrm{e}^{-(\varepsilon/2)\sum_{i=0}^{n-1}\left(\frac{y_{i+1}-y_i}{\varepsilon}\right)^2}\mathrm{e}^{-\theta\varepsilon\sum_{i=1}^{n-1}(\partial A/\partial y)(y_{i-1})} \\ & \times \mathrm{e}^{-(\varepsilon/2)\sum_{i=0}^{n-1}(A^2(y_i)+2\theta A(y_i)(\partial A/\partial y)(y_i)(y_{i+1}-y_i))} \\ & \times \mathrm{e}^{+\varepsilon\sum_{i=0}^{n-1}(A(y_i)+\theta(\partial A/\partial y)(y_i)(y_{i+1}-y_i))\left(\frac{y_{i+1}-y_i}{\varepsilon}\right)}.\end{aligned} \tag{B.5}$$

Now, we can return to a Feynman path integral expression by eliminating the limit of the $(n-1)$-fold integration. This results in

$$\begin{aligned}&\int_{(0,w_0)}^{(t,w_t)} Dw(s)\mathrm{e}^{-(1/2)\int_0^t \dot{w}(s)^2\,\mathrm{d}s} \\ &\quad = \int_{(0,y_0)}^{(t,y_t)} Dy(s)\mathrm{e}^{-(1/2)\int_0^t \dot{y}(s)^2\,\mathrm{d}s}\mathrm{e}^{-\theta\int_0^t \partial A/\partial y\,\mathrm{d}s}\,\mathrm{e}^{-(1/2)\int_0^t A^2(y(s))_\theta\,\mathrm{d}s}\,\mathrm{e}^{+\int_0^t A(y(s))_\theta\,\mathrm{d}y(s)}.\end{aligned} \tag{B.6}$$

Since the first integral in the last line of Eq. (B.6) is a Riemann integral, the θ has no influence. The second integral in the last line of (B.6) behaves as a general stochastic integral in the sense of (5); θ can be eliminated when making use of (11)—which completes the proof for the first part. Note that this last step is necessary due to the fact that Feynman integrations make use of a midpoint definition.

The result for the long term probability immediately follows from the forward Fokker–Planck equations when choosing $\partial p/\partial t = 0$. □

Proof of Theorem 3.2. In a first step, we rewrite the θ-stochastic differential equation into a Stratonovich equation. This is necessary in order to justify the use of the classical chain rule.

Assume that the process $Y=\{Y(s), s\in[0,t]\}$ satisfies Eq. (22), and define the notations $\varepsilon=t/n$, $t_i=i\varepsilon$, $y_i = Y(t_i)$ and $y_i^\theta = Y(t_i + \theta(t_{i+1}-t_i))$. We then have

$$\int_0^t \mathrm{d}Y(s) = \int_0^t \{A(Y(s))\,\mathrm{d}s + B(Y(s))_\theta\,\mathrm{d}W(s)\}$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1}\{\varepsilon A(y_i) + B(y_i^\theta)(w_{i+1}-w_i)\}$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left\{\varepsilon A(y_i) + \left[B(y_i^{1/2}) + \left(\theta-\frac{1}{2}\right)\frac{\partial B}{\partial y}(y_i)(y_{i+1}-y_i)\right](w_{i+1}-w_i)\right\}$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left\{\varepsilon A(y_i) + B(y_i^{1/2})(w_{i+1}-w_i) + \left(\theta-\frac{1}{2}\right)\varepsilon\frac{\partial B}{\partial y}(y_i)B(y_i)\right\}$$

$$= \int_0^t \left\{\left[A(Y(s)) + \left(\theta-\frac{1}{2}\right)\frac{\partial B}{\partial y}(Y(s))B(Y(s))\right]\mathrm{d}s + B(Y(s))\,\mathrm{d}W(s)\right\}. \tag{B.7}$$

Now, if we use a change of variables as suggested in Theorem 3.2, the previous reasoning brings us to

$$\int_0^t \mathrm{d}X(s) = \int_0^t \frac{\mathrm{d}Y(s)}{B(Y(s))}$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left\{\frac{1}{B(y_i^{1/2})}[y_{i+1}-y_i]\right\}$$

$$= \lim_{n\to\infty}\sum_{i=0}^{n-1}\left\{\frac{\varepsilon A(y_i) + B(y_i^{1/2})(w_{i+1}-w_i) + \left(\theta-\frac{1}{2}\right)\varepsilon\frac{\partial B}{\partial y}(y_i)B(y_i)}{B(y_i^{1/2})}\right\}$$

$$= \int_0^t \left\{\left[\frac{A(Y(s))}{B(Y(s))} + \left(\theta-\frac{1}{2}\right)\frac{\partial B}{\partial y}(Y(s))\right]\mathrm{d}s + \mathrm{d}W(s)\right\}, \tag{B.8}$$

which completes the proof. □

Proof of Theorem 3.3. This immediately follows from the change of variables mentioned in Theorem 3.2. Indeed,

$$p(0,y_0;t,y_t) = \frac{\mathrm{d}}{\mathrm{d}y_t}\,\mathrm{Prob}[Y(t) \leqslant y_t | Y(0) = y_0]$$
$$= \frac{1}{B(y_t)}\frac{\mathrm{d}}{\mathrm{d}x_t}\,\mathrm{Prob}[X(t) \leqslant \psi(y_t) | X(0) = \psi(y_0)]. \tag{B.9}$$

Applying Theorem 3.1, the desired result follows. □

Proof of Theorem 4.1. The Lagrangian for the process $Y(t)$ equals

$$L(\dot{y},y,s) = \frac{1}{2}\dot{y}^{\,2} + \frac{1}{2}A(y)^2 + \frac{1}{2}\frac{\partial A(y)}{\partial y} - A(y)\dot{y}. \tag{B.10}$$

Therefore, applying (32), the maximal probability path is determined by

$$\ddot{y} = A(y)\frac{\partial A(y)}{\partial y} + \frac{1}{2}\frac{\partial^2 A(y)}{\partial y^2}. \tag{B.11}$$

After multiplying both sides by $\dot{y}$, two integrations lead to the desired result. □

Proof of Theorem 5.1. In order to prove (41), we start from the path integral expression for the transition probability for the process Y, as stated in Theorem 3.1:

$$p(0,y_0;t,y_t) = \int_{(0,y_0)}^{(t,y_t)} Dy(s)\mathrm{e}^{-(1/2)\int_0^t \dot{y}^{\,2}\,\mathrm{d}s - (1/2)\int_0^t (A(y)^2 + \partial A/\partial y)\,\mathrm{d}s + \int_0^t A(y)\,\mathrm{d}y}. \tag{B.12}$$

As suggested by the stochastic differential equation (38), we make use of a coordinate transformation as developed in [13], in discretized version

$$y_{i+1} - y_i = \frac{1}{2}\frac{f''(x_i)}{f'(x_i)^2}(t_{i+1} - t_i) + f'(x_i)(x_{i+1} - x_i), \tag{B.13}$$

where $\varepsilon = t/n$, $t_i = i\varepsilon$, $y_i = Y(t_i)$, $x_i = X(t_i)$, for $i = 0,\ldots,n$.

This transformation results in

$$p(0,y_0;t,y_t) = \int_{(0,y_0)}^{(t,y_t)} [f'(x(s)_{\mathrm{L}})Dx(s)]\mathrm{e}^{-(1/2)\int_0^t f'(x(s))^2\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2\mathrm{d}s}$$
$$\times \mathrm{e}^{-(1/2)\int_0^t \frac{f''(x(s))^2}{f'(x(s))^4}\,\mathrm{d}s + \frac{1}{4}\int_0^t \frac{f'''(x(s))}{f'(x(s))^3}\,\mathrm{d}s}\;\mathrm{e}^{-(1/2)\int_0^t (A[f(x(s))]^2 + \frac{\partial A}{\partial y}[f(x(s))])\,\mathrm{d}s}$$
$$\times \mathrm{e}^{+\int_0^t A[f(x(s))]\,f'(x(s))\,\mathrm{d}x(s)}. \tag{B.14}$$

Examining this path integral expression, it is clear that we still need a stochastic time change from t and $\mathrm{d}s$ to t^* and $\mathrm{d}\sigma$ in order to get both a kinetic term and a differential measure that are independent of the transformation function. This can be done when choosing

$$t_{i+1} - t_i = f'(x_i)\,f'(x_{i+1})(\sigma_{i+1} - \sigma_i) \tag{B.15}$$

or

$$\mathrm{d}s = f'(x(\sigma)_{\mathrm{L}})\,f'(x(\sigma)_{\mathrm{R}})\,\mathrm{d}\sigma. \tag{B.16}$$

The transformation is obvious for all integrations except for the kinetic term in the exponent of (B.14). For this kinetic term, we can rely on the fact that

$$f'(x(s))\,\mathrm{d}x = f'(x(s)_{\mathrm{L}})\,\mathrm{d}x + \frac{1}{2}\frac{f''(x(s))}{f'(x(s))^2}\,\mathrm{d}s \tag{B.17}$$

$$= f'(x(s)_{\mathrm{R}})\,\mathrm{d}x - \frac{1}{2}\frac{f''(x(s))}{f'(x(s))^2}\,\mathrm{d}s, \tag{B.18}$$

resulting in

$$-\frac{1}{2}\int_0^t f'(x(s))^2\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2\mathrm{d}s = -\frac{1}{2}\int_0^{t^*}\left(\frac{\mathrm{d}x}{\mathrm{d}\sigma}\right)^2\mathrm{d}\sigma + \frac{1}{8}\int_0^{t^*}\frac{f''(x(\sigma))^2}{f'(x(\sigma))^2}\,\mathrm{d}\sigma. \tag{B.19}$$

For the path differential measure, the stochastic time change leads to

$$[f'(x(s)_{\mathrm{L}})Dx(s)] = \lim_{n\to\infty}\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi f'(x_{i-1})f'(x_i)(\sigma_i-\sigma_{i-1})}}\prod_{i=1}^{n-1}f'(x_i)\,\mathrm{d}x_i$$

$$= \frac{1}{\sqrt{f'(x_0)f'(x_t)}}\,Dx(\sigma). \tag{B.20}$$

Altogether, this results in the final path integral

$$px(0, x_0 = f^{-1}(y_0); t, x_t = f^{-1}(y_t))$$

$$= \frac{1}{\sqrt{f'(f^{-1}(y_0))f'(f^{-1}(y_t))}}\int_{(0,f^{-1}(y_0))}^{(t^*,f^{-1}(y_t))} Dx(\sigma)\mathrm{e}^{-(1/2)\int_0^{t^*}\dot{x}^2\,\mathrm{d}\sigma}$$

$$\times\mathrm{e}^{-(1/2)\int_0^{t^*}\left(A[f(x)]^2+\frac{\partial A}{\partial y}[f(x)]\right)f'(x)^2\,\mathrm{d}\sigma}$$

$$\times\mathrm{e}^{+\int_0^{t^*}A[f(x)]f'(x)\,\mathrm{d}x-\frac{1}{8}\int_0^t\left[3\frac{f''(x)^2}{f'(x)^2}-2\frac{f'''(x)}{f'(x)}\right]\mathrm{d}\sigma}, \tag{B.21}$$

where we still have to impose the condition

$$t = \int_0^{t^*} f'(x)^2\,\mathrm{d}\sigma. \tag{B.22}$$

This can be done by adding an integration with a Dirac function,

$$\int_0^{+\infty}\mathrm{d}t^*\,\delta\left(t - \int_0^{t^*} f'(x)^2\,\mathrm{d}\sigma\right) \tag{B.23}$$

or

$$\int_{-\infty}^{+\infty}\mathrm{d}\beta\int_0^{+\infty}\mathrm{d}t^*\,\mathrm{e}^{\mathrm{i}\beta\left(t-\int_0^{t^*}f'(x)^2\,\mathrm{d}\sigma\right)}, \tag{B.24}$$

which completes the proof. □

Proof of Theorem 6.1. The path integral can be written as

$$I(t_0,x_0;t_e,x_e)=K(t_0,x_0;t_e,x_e)E_W[e^{-\int_{t_0}^{t_e}V[X(s)]\,ds}]. \tag{B.25}$$

Applying Proposition 6.2, we know that the variable $A=\int_{t_0}^{t_e}V[X(s)]\,ds$ is smaller than $B=\int_{t_0}^{t_e}F_{V(X(s))}^{-1}(U)\,ds$ in convex ordering. Since the exponential function is convex, it follows immediately from the definition of convex ordering that

$$I(t_0,x_0;t_e,x_e)\leqslant I^{upp}(t_0,x_0;t_e,x_e) \tag{B.26}$$

with

$$I^{upp}(t_0,x_0;t_e,x_e)=K(t_0,x_0;t_e,x_e)E_U[e^{-\int_{t_0}^{t_e}F_{V(X(s))}^{-1}(U)\,ds}]. \tag{B.27}$$

If we rewrite the expectation in Eq. (B.27) as an expectation over $B=\int_{t_0}^{t_e}F_{V(X(s))}^{-1}(U)\,ds$ instead of over U, a derivation with respect to k leads to the second result, which completes the proof. □

Proof of Theorem 6.2. We start by writing the path integral as

$$I(t_0,x_0;t_e,x_e)=K(t_0,x_0;t_e,x_e)\cdot E_\Lambda[E_W[e^{-\int_{t_0}^{t_e}V[X(\tau)]\,d\tau}|\Lambda]], \tag{B.28}$$

for an arbitrary stochastic variable Λ.

Applying the inequality of Jensen, it follows that

$$I(t_0,x_0;t_e,x_e)\geqslant I^{low}(t_0,x_0;t_e,x_e) \tag{B.29}$$

with

$$I^{low}(t_0,x_0;t_e,x_e)=K(t_0,x_0;t_e,x_e)\cdot E_\Lambda[e^{-\int_{t_0}^{t_e}E_W[V[X(\tau)]|\Lambda]\,d\tau}]. \tag{B.30}$$

If we choose $\Lambda=X(t_s)$, with t_s such that $t_0\leqslant t_s\leqslant t_e$, the final result immediately follows. □

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