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SINGULARITIES OF THE REPRESENTATION VARIETY OF THE BRAID GROUP ON 3 STRANDS

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ABSTRACT. The singularities of $\text{rep}_n B_3$ are studied, where B_3 is the knot group on 3 strands. Specifically, we determine which semisimple representations are smooth points of $\text{rep}_n B_3$.

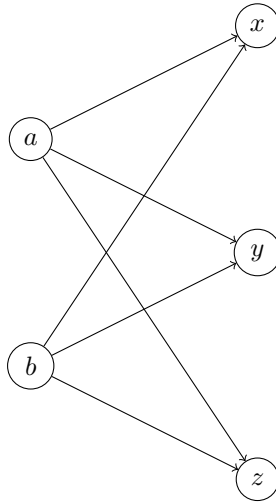


FIGURE 1. The quiver Q for Γ

representation variety, braid group ¹

1. INTRODUCTION

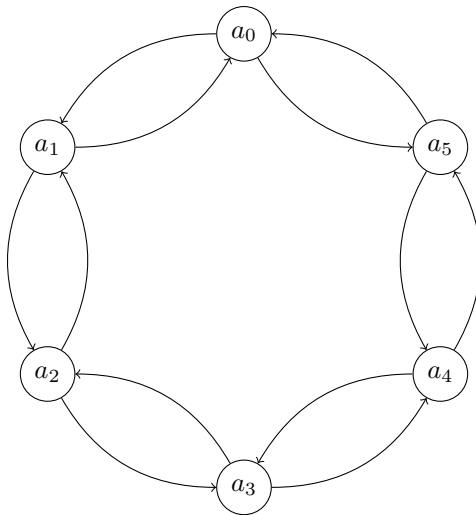
The braid group $B_3 = \langle \sigma_1, \sigma_2 | \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle = \langle X, Y | X^2 = Y^3 \rangle$ on 3 strands plays an important part in knot theory, for example its representations played an important role in [3] for determining knot-vertibility. In order to study the representation variety

$$\text{rep}_n B_3 = \{(A, B) \in \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) | A^2 = B^3\},$$

one uses the fact that $B_3/(X^2) \cong \mathbb{Z}_2 * \mathbb{Z}_3 = \Gamma$, whose representation theory is decoded by the quiver Q of Figure 1, see for example [7] or [1]. In particular, we have

$$\text{rep}_n \Gamma = \{(A, B) \in \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) | A^2 = B^3 = 1\} = \sqcup_{\alpha} \text{rep}_{\alpha} \Gamma$$

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FIGURE 2. The one quiver Q' for Γ

with $\alpha = (a, b; x, y, z)$ a dimension vector for Q fulfilling the requirement $a + b = x + y + z = n$. Consequently, $\mathbf{rep}_n \Gamma$ is smooth, such that one can compute the local quiver Q' in the semisimple representation

$$S_{1,1}^{\oplus a_0} \oplus S_{-1,\omega}^{\oplus a_1} \oplus S_{1,\omega^2}^{\oplus a_2} \oplus S_{-1,1}^{\oplus a_3} \oplus S_{1,\omega}^{\oplus a_4} \oplus S_{-1,\omega^2}^{\oplus a_5}$$

where $S_{\rho,\tau}(\overline{X}) = \rho$, $S_{\rho,\tau}(\overline{Y}) = \tau$ and ω is a primitive 3rd root unity, which can be found in Figure 2. Questions like ‘which dimension vectors determine components for which the generic member is simple’ and ‘what is the dimension of $\mathbf{rep}_\alpha \Gamma // \mathbf{PGL}_n(\mathbb{C})$ ’ can then be easily solved using the quiver technology from [5].

However, $\mathbf{rep}_n B_3$ is far from smooth. Although $\mathbf{rep}_n B_3$ is smooth in any simple representation, in the semisimple but not simple representations singularities can occur. The main result, Theorem 8, will give a necessary and sufficient conditions for a semisimple representation to be a smooth point of $\mathbf{rep}_n B_3$. In addition, we show that almost all of these singularities come from the fact that they are intersection points of irreducible components of $\mathbf{rep}_n B_3$, except in the case that there is a 2-dimensional simple representation with multiplicity ≥ 2 in the decomposition of the semisimple module.

Remark 1. *Determining the singularities in semisimple points of $\mathbf{rep}_n B_3$ is not a full classification of all singularities of $\mathbf{rep}_n B_3$, as a sum of indecomposables V may be smooth or singular if its semisimplification V^{ss} is singular. However, if V is singular, then its semisimplification V^{ss} is necessarily singular, as $V^{ss} \in \mathcal{O}(V)$ and the singular locus is a closed $\mathbf{PGL}_n(\mathbb{C})$ -subvariety of $\mathbf{rep}_n B_3$.*

2. THE CONNECTION BETWEEN B_3 AND Γ

As B_3 is a central extension of Γ with \mathbb{Z}

$$1 \longrightarrow \mathbb{Z} \longrightarrow B_3 \longrightarrow \Gamma \longrightarrow 1$$

any simple representation S of B_3 has the property that $X^2 = Y^3$ acts by scalar multiplication λI_n on S for some $\lambda \in \mathbb{C}^*$.

In addition, for each n , we have an action of \mathbb{C}^* on $\mathbf{rep}_n B_3$ defined by

$$t \cdot (A, B) = (t^3 A, t^2 B), \text{ for } t \in \mathbb{C}^*, (A, B) \in \mathbf{rep}_n B_3,$$

which is of course coming from the fact that $\mathbf{rep}_1 B_3 = \mathbf{V}(x_1^2 - x_2^3) \setminus \{(0, 0)\} \cong \mathbb{C}^*$, so this action is just the same as taking the tensor product with a 1-dimensional representation of B_3 . Using this action, we find that any simple representation of B_3 can be found in the \mathbb{C}^* -orbit of a simple representation of Γ . Equivalently, the corresponding map

$$\mathbb{C}^* \times \mathbf{irrep}_n \Gamma \xrightarrow{f_n} \mathbf{irrep}_n B_3$$

is surjective and 6-to-1, as for $t \in \mu_6 = \{\rho \in \mathbb{C}^* | \rho^6 = 1\}$ and $(A, B) \in \mathbf{irrep}_n \Gamma$, $t \cdot (A, B)$ is also a simple Γ -representation. For $S \in \mathbf{irrep}_n \Gamma$, $\lambda \in \mathbb{C}^*$, we denote λS for $f_n(\lambda, S)$.

3. GEOMETRICAL INTERPRETATION OF EXTENSIONS

In here we review parts of [5, Chapter 3 and 4] which we will need. Given an algebra \mathcal{A} and a n -dimensional representation V , we define the normal space to the $\mathrm{PGL}_n(\mathbb{C})$ -orbit of V in $\mathbf{rep}_n \mathcal{A}$ to be the vector space

$$N_V \mathbf{rep}_n \mathcal{A} = \frac{T_V \mathbf{rep}_n \mathcal{A}}{T_V \mathcal{O}(V)},$$

where $\mathcal{O}(V)$ is the $\mathrm{PGL}_n(\mathbb{C})$ -orbit of V . Recall that $\mathbf{rep}_n \mathcal{A}$ is a $\mathrm{PGL}_n(\mathbb{C})$ -variety by base change, that is, if V is determined by the algebra morphism

$$\mathcal{A} \xrightarrow{\rho} \mathbf{M}_n(\mathbb{C}),$$

then $g \cdot \rho$ is the composition

$$\mathcal{A} \xrightarrow{\rho} \mathbf{M}_n(\mathbb{C}) \xrightarrow{g(-)g^{-1}} \mathbf{M}_n(\mathbb{C}).$$

In [5, Example 3.13] it is shown that $N_V \mathbf{rep}_n \mathcal{A} = \mathrm{Ext}_{\mathcal{A}}^1(V, V)$, with for N, M 2 finite dimensional \mathcal{A} -modules with corresponding algebra morphisms σ and ρ we define

$$\mathrm{Ext}_{\mathcal{A}}^1(N, M) = \frac{\{\delta \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathrm{Hom}_{\mathbb{C}}(N, M)) | \forall a, a' \in \mathcal{A} : \delta(aa') = \rho(a)\delta(a') + \delta(a)\sigma(a')\}}{\{\delta \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathrm{Hom}_{\mathbb{C}}(N, M)) | \exists \beta \in \mathrm{Hom}_{\mathbb{C}}(N, M) : \forall a \in \mathcal{A} : \delta(a) = \rho(a)\beta - \beta\sigma(a)\}}$$

The top vector space is called the space of cycles and is denoted by $Z(N, M)$, while the lower vector space is the space of boundaries and is denoted by $B(N, M)$.

Equivalently, $\mathrm{Ext}_{\mathcal{A}}^1(N, M)$ is the set of equivalence classes of short exact sequences of \mathcal{A} -modules

$$e : 0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

where 2 exact sequences (P, e) and (P', e') are equivalent if there is an \mathcal{A} -isomorphism $\phi \in \mathrm{Hom}_{\mathcal{A}}(P, P')$ such that the following diagram is commutative

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow id_M & & \downarrow \phi & & \downarrow id_N \\ 0 & \longrightarrow & M & \longrightarrow & P' & \longrightarrow & N \longrightarrow 0 \end{array}$$

These 2 type of definitions of $\text{Ext}_{\mathcal{A}}^1(N, M)$ will both occur in this paper.

4. CALCULATING EXTENSIONS

Following the previous section it is obvious that extensions of finite dimensional B_3 -modules will play an important role in determining the singular points of $\text{rep}_n B_3$.

Theorem 2. *Let S, T be irreducible representations of Γ and let $\lambda, \mu \in \mathbb{C}^*$. We have*

- (1) $\text{Ext}_{B_3}^1(\lambda S, \mu T) = 0$ except if $\frac{\lambda}{\mu} \in \boldsymbol{\mu}_6$,
- (2) $\text{Ext}_{B_3}^1(\lambda S, \mu T) = \text{Ext}_{\Gamma}^1(S, \frac{\mu}{\lambda} T)$ if $\frac{\mu}{\lambda} \in \boldsymbol{\mu}_6$ and $S \not\cong \frac{\mu}{\lambda} T$, and lastly
- (3) $\text{Ext}_{B_3}^1(\lambda S, \lambda T) = \text{Ext}_{\Gamma}^1(S, T) \oplus \mathbb{C}$ if $S \cong T$.

Proof. For the first case, we may assume by the action of \mathbb{C}^* that $\lambda = 1$ and $\mu \notin \boldsymbol{\mu}_6$. Let σ , respectively τ be the algebra morphisms from $\mathbb{C}B_3$ to $\text{End}_{\mathbb{C}}(S)$, respectively $\text{End}_{\mathbb{C}}(T)$ coming from the representations S and T . Let δ be a cycle, that is, a linear map

$$\mathbb{C}B_3 \xrightarrow{\delta} \text{Hom}_{\mathbb{C}}(S, \mu T)$$

such that

$$\forall a, b \in \mathbb{C}B_3 : \delta(ab) = \delta(a)\sigma(b) + (\mu\tau)(a)\delta(b).$$

We need to prove that there exists a $\beta \in \text{Hom}_{\mathbb{C}}(S, \mu T)$ such that

$$\forall a \in \mathbb{C}B_3 : \delta(a) = (\mu\tau)(a)\beta - \beta\sigma(a).$$

Using the defining relation of B_3 , we see that

$$\begin{aligned} c &= \delta(X^2) = \delta(X)\sigma(X) + \mu^3\tau(X)\delta(X) \\ &= \delta(Y^3) = \delta(Y)\sigma(Y)^2 + \mu^2\tau(Y)\delta(Y)\sigma(Y) + \mu^4\tau(Y)^2\delta(Y). \end{aligned}$$

Assuming that $\mu \notin \boldsymbol{\mu}_6$, we find for $\beta = \frac{c}{\mu^6-1} \in \text{Hom}_{\mathbb{C}}(S, \mu T)$

$$\begin{aligned} &(\mu\tau)(X)\beta - \beta\sigma(X) \\ &= \frac{1}{\mu^6-1}(\mu^3\tau(X)\delta(X)\sigma(X) + \mu^6\tau(X)^2\delta(X) \\ &\quad - \delta(X)\sigma(X)^2 - \mu^3\tau(X)\delta(X)\sigma(X)) \\ &= \delta(X) \end{aligned}$$

and

$$\begin{aligned} &(\mu\tau)(Y)\beta - \beta\sigma(Y) \\ &= \frac{1}{\mu^6-1}(\mu^2\tau(Y)\delta(Y)\sigma(Y)^2 + \mu^4\tau(Y)^2\delta(Y)\sigma(Y) + \mu^6\tau(Y)^3\delta(Y) \\ &\quad - \delta(Y)\sigma(Y)^3 - \mu^2\tau(Y)\delta(Y)\sigma(Y)^2 - \mu^4\tau(Y)^2\delta(Y)\sigma(Y)) \\ &= \delta(Y). \end{aligned}$$

So $\delta \subset B(S, \mu T) \subset \text{Hom}_{\mathbb{C}}(S, \mu T)$ and consequently, $\text{Ext}_{B_3}^1(S, \mu T) = 0$.

For the second case, we may again assume that $\lambda = 1$ and in addition that $\mu = 1$, for μT is just a Γ -representation in this case. In order to prove that $\text{Ext}_{B_3}^1(S, T) = \text{Ext}_{\Gamma}^1(S, T)$ for $S \not\cong T$, we have to prove that each extension of S

by T as B_3 -module is actually a Γ -module. Equivalently, we have to prove that for any cycle

$$\mathbb{C}B_3 \xrightarrow{\delta} \text{Hom}_{\mathbb{C}}(S, T),$$

we have $\delta(X^2) = 0$, as we then get $\delta(X^2 - 1) = \delta(Y^3 - 1) = 0$ which makes sure that this extension is a Γ -module. We have as before

$$\begin{aligned} c &= \delta(X^2) = \delta(X)\sigma(X) + \tau(X)\delta(X) \\ &= \delta(Y^3) = \delta(Y)\sigma(Y)^2 + \tau(Y)\delta(Y)\sigma(Y) + \tau(Y)^2\delta(Y). \end{aligned}$$

Now we get

$$\begin{aligned} &c\sigma(X) - \tau(X)c \\ &= \delta(X)\sigma(X)^2 + \tau(X)\delta(X)\sigma(X) \\ &- \tau(X)\delta(X)\sigma(X) - \tau(X)^2\delta(X) = 0 \end{aligned}$$

and

$$\begin{aligned} &c\sigma(Y) - \tau(Y)c \\ &= \delta(Y)\sigma(Y)^3 + \tau(Y)\delta(Y)\sigma(Y)^2 + \tau(Y)^2\delta(Y)\sigma(Y) \\ &- \tau(Y)\delta(Y)\sigma(Y)^2 - \tau(Y)^2\delta(Y)\sigma(Y) - \tau(Y)^3\delta(Y) = 0. \end{aligned}$$

Consequently, $c \in \text{Hom}_{\Gamma}(S, T)$, which is 0 if $S \not\cong T$, so δ defines a Γ -module, that is, $\delta \in \text{Ext}_{\Gamma}^1(S, T)$. The inclusion $\text{Ext}_{\Gamma}^1(S, T) \subset \text{Ext}_{B_3}^1(S, T)$ is obvious.

For the last part, if $S \cong T$, we have for c as in the second case that $c \in \text{End}_{\Gamma}(S) \cong \mathbb{C}$. In this case, the following matrices define an element of $\text{Ext}_{B_3}^1(S, S)$

$$X \mapsto \begin{bmatrix} \sigma(X) & \frac{c}{2}\sigma(X) \\ 0 & \sigma(X) \end{bmatrix}, Y \mapsto \begin{bmatrix} \sigma(Y) & \frac{c}{3}\sigma(Y) \\ 0 & \sigma(Y) \end{bmatrix}.$$

So the function

$$\mathbb{C}B_3 \xrightarrow{\delta_{\sigma}} \text{End}_{\mathbb{C}}(S)$$

defined by $\delta_{\sigma}(X) = \frac{c}{2}\sigma(X)$, $\delta_{\sigma}(Y) = \frac{c}{3}\sigma(Y)$ defines a derivation on $\mathbb{C}B_3$ with $\delta_{\sigma}(X^2) = c$. But then $\delta - \delta_{\sigma} \in \text{Ext}_{\Gamma}^1(S, S)$, which shows that

$$\dim_{\mathbb{C}} \text{Ext}_{B_3}^1(S, S) \leq \dim_{\mathbb{C}} \text{Ext}_{\Gamma}^1(S, S) + 1.$$

Let $S \in \text{rep}_{\alpha}\Gamma$ for a dimension vector α for Γ , then the map

$$\mathbb{C}^* \times \text{irrep}_{\alpha}\Gamma \xrightarrow{f_{\alpha}} \text{irrep}_n B_3$$

has finite fibers and is surjective on $\text{PGL}_n(\mathbb{C})$ -orbits, from which it follows that

$$\dim_{\mathbb{C}} \text{Ext}_{B_3}^1(S, S) \geq \dim_{\mathbb{C}} \text{Ext}_{\Gamma}^1(S, S) + 1,$$

leading to the other inequality. \square

Corollary 3. *For all simple B_3 -modules S and T , we have*

$$\dim \text{Ext}_{B_3}^1(S, T) = \dim \text{Ext}_{B_3}^1(T, S)$$

Proof. If $S \cong T$ the statement is trivial, so assume $S \not\cong T$. Either there exists a $\lambda \in \mathbb{C}^*$ such that λS and λT are Γ -modules or not. In the first case, if $S \not\cong T$ it is true because

$$\mathrm{Ext}_{B_3}^1(S, T) = \mathrm{Ext}_{\Gamma}^1(\lambda S, \lambda T) = \mathrm{Ext}_{\Gamma}^1(\lambda T, \lambda S) = \mathrm{Ext}_{B_3}^1(T, S),$$

which follows from the symmetry of the one quiver for Γ . In the second case both $\mathrm{Ext}_{B_3}^1(S, T)$ and $\mathrm{Ext}_{B_3}^1(T, S)$ are 0. \square

5. COMPONENTS OF $\mathrm{rep}_n B_3$

Theorem 4. *Let $\alpha_i = (a_i, b_i; x_i, y_i, z_i)$ for $1 \leq i \leq k$ be simple dimension vectors for Γ , let $n_i = a_i + b_i$ and let $n = \sum_{i=1}^k n_i$. Then the closure of the image of the map*

$$\mathrm{PGL}_n(\mathbb{C}) \times (\prod_{i=1}^k \mathrm{GL}_{n_i}(\mathbb{C}))/\mathbb{C}^* \left((\mathbb{C}^*)^k \times \prod_{i=1}^k \mathrm{rep}_{\alpha_i} \Gamma \right) \xrightarrow{f_{(\alpha_1, \dots, \alpha_k)}} \mathrm{rep}_n(B_3)$$

is an irreducible component of $\mathrm{rep}_n(B_3)$.

Proof. Let $X = \mathrm{PGL}_n(\mathbb{C}) \times (\prod_{i=1}^k \mathrm{GL}_{n_i}(\mathbb{C}))/\mathbb{C}^* \left((\mathbb{C}^*)^k \times \prod_{i=1}^k \mathrm{rep}_{\alpha_i} \Gamma \right)$ and put $Y = \mathrm{rep}_n(B_3)$. On the open subset of X defined by $\forall 1 \leq i \leq k : V_i \in \mathrm{rep}_{\alpha_i} \Gamma$ simple, the map $f_{(\alpha_1, \dots, \alpha_k)}$ has finite fibers, so we have $\dim \overline{f_{(\alpha_1, \dots, \alpha_k)}(X)} = \dim X$. In order to prove that $\overline{f_{(\alpha_1, \dots, \alpha_k)}(X)}$ is indeed a component of $\mathrm{rep}_n(B_3)$, it is then enough to prove that there exists an element of $x \in X$ such that $\dim T_x X = \dim T_{f_{(\alpha_1, \dots, \alpha_k)}(x)} Y$. Let $V = \bigoplus_{i=1}^k t_i V_i$ be a semisimple B_3 -representation such that

- $V_i \in \mathrm{rep}_{\alpha_i} \Gamma$ is a simple Γ -module and
- $\frac{t_i}{t_j} \notin \mu_6$ for all $1 \leq i < j \leq k$.

Then by Theorem 2 we have

$$\dim \mathrm{Ext}_{B_3}^1(V, V) = k + \sum_{i=1}^k \dim \mathrm{Ext}_{\Gamma}^1(V_i, V_i).$$

As both $((t_i)_{i=1}^k, (V_i)_{i=1}^k)$ and V have the same $\mathrm{PGL}_n(\mathbb{C})$ -stabilizer, we are done, as we have

$$\begin{aligned} \dim T_{(1, (t_i)_{i=1}^k, (V_i)_{i=1}^k)} X &= \dim \mathcal{O}(1, (t_i)_{i=1}^k, (V_i)_{i=1}^k) + \dim (\mathbb{C}^*)^k + \sum_{i=1}^k \dim \mathrm{Ext}_{\Gamma}^1(V_i, V_i) \\ &= n^2 - k + k + \sum_{i=1}^k \dim \mathrm{Ext}_{\Gamma}^1(V_i, V_i) = n^2 + \sum_{i=1}^k \dim \mathrm{Ext}_{\Gamma}^1(V_i, V_i), \\ \dim T_V Y &= \dim \mathcal{O}(V) + \mathrm{Ext}_{B_3}^1(V, V) \\ &= n^2 - k + k + \sum_{i=1}^k \dim \mathrm{Ext}_{\Gamma}^1(V_i, V_i) = n^2 + \sum_{i=1}^k \dim \mathrm{Ext}_{\Gamma}^1(V_i, V_i). \end{aligned}$$

\square

In fact, this proof shows that the image of elements fulfilling $\frac{t_i}{t_j} \notin \mu_6$ belong to the smooth locus of a component of $\mathrm{rep}_n B_3$.

Corollary 5. *Let $V = \bigoplus_{i=1}^k S_i^{e_i}$ be a n -dimensional semi-simple B_3 -module and let $S_i = \lambda_i S'_i$ with S'_i a simple Γ -module, $S'_i \in \mathbf{rep}_{\alpha_i} \Gamma$. Then V belongs to a component of dimension*

$$n^2 + \sum_{i=1}^k e_i \dim \mathrm{Ext}_{\Gamma}^1(S'_i, S'_i).$$

Proof. Let $S'_i \in \mathbf{rep}_{\alpha_i} \Gamma$, then V belongs to the image of the map $f(\underbrace{\alpha_1, \dots, \alpha_1}_{e_1}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{e_k})$.

But then the torus $(\mathbb{C}^*)^{\sum_{i=1}^k e_i}$ acts on $\mathrm{Im}(f(\underbrace{\alpha_1, \dots, \alpha_1}_{e_1}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{e_k}))$ such that for

a generic element of $\mathfrak{t} \in (\mathbb{C}^*)^{\sum_{i=1}^k e_i}$, there are no extensions between any simple subrepresentations of $\mathfrak{t}V$. It follows that the dimension of this component is

$$\begin{aligned} \dim \overline{\mathrm{Im}(f(\underbrace{\alpha_1, \dots, \alpha_1}_{e_1}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{e_k}))} &= n^2 - \sum_{i=1}^k e_i + \sum_{i=1}^k e_i (\dim \mathrm{Ext}_{B_3}^1(S_i, S_i)) \\ &= n^2 + \sum_{i=1}^k e_i \dim \mathrm{Ext}_{\Gamma}^1(S'_i, S'_i). \end{aligned}$$

□

Proposition 6. *One of the components of $\mathbf{rep}_n B_3$ is isomorphic to $\mathrm{GL}_n(\mathbb{C})$ as $\mathrm{PGL}_n(\mathbb{C})$ -variety.*

Proof. Let X be the irreducible component of $\mathbf{rep}_n B_3$ with generic element being a direct sum of 1-dimensional representations. This component can be described as

$$X = \{(A, B) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \mid A^2 = B^3, AB = BA\}.$$

Define the maps

$$\mathrm{GL}_n(\mathbb{C}) \xrightarrow{f_1} X, \quad X \xrightarrow{f_2} \mathrm{GL}_n(\mathbb{C})$$

as $f_1(G) = (G^3, G^2)$ and $f_2(A, B) = AB^{-1}$. Then f_1 and f_2 are clearly $\mathrm{PGL}_n(\mathbb{C})$ -equivariant maps and $f_1 \circ f_2 = \mathrm{Id}_X$ and $f_2 \circ f_1 = \mathrm{Id}_{\mathrm{GL}_n(\mathbb{C})}$. □

Lemma 7. *The only components of $\mathbf{rep}_n \Gamma$ equal to the $\mathrm{PGL}_n(\mathbb{C})$ -orbit of a single simple representation occur in dimension 1 and are equal to the 6 1-dimensional representations of Γ .*

Proof. This amounts to proving that $\mathbf{rep}_{\alpha} \Gamma // \mathrm{PGL}_n(\mathbb{C})$ has dimension ≥ 1 if $n > 1$ and α is a simple dimension vector. If $\alpha = (a, b; x, y, z)$ and $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$ is a dimension vector for Q' such that

$$\begin{cases} a = a_0 + a_2 + a_4, \\ b = a_1 + a_3 + a_5, \\ x = a_0 + a_3 \\ y = a_1 + a_4, \\ z = a_2 + a_5, \end{cases}$$

then the dimension of the moduli space $\mathbf{rep}_\alpha \Gamma / \mathrm{PGL}_n(\mathbb{C})$ is equal to $1 - \chi_{Q'}(\mathbf{a}, \mathbf{a})$, with Q' the Euler form on $\mathbb{Z}^6 \times \mathbb{Z}^6$ associated to Q' , which in this case is determined by the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

see for example [5, Chapter 4]. In addition, $\mathbf{rep}_\alpha Q'$ contains simple representations of Γ if and only if either \mathbf{a} lies in the \mathbb{Z}_6 -orbit of $e_1 = (1, 0, 0, 0, 0, 0)$ or \mathbf{a} fulfils the condition

$$(5.1) \quad \forall i \in \mathbb{Z}_6 : a_i \leq a_{i-1} + a_{i+1},$$

indices taken in \mathbb{Z}_6 . In the case that $\mathbf{supp}(\mathbf{a})$ is the union of 2 consecutive vertices that fulfil condition 5.1, the dimension vector has to lie in the \mathbb{Z}_6 -orbit of $(1, 1, 0, 0, 0, 0)$. For α itself this boils down to

$$(5.2) \quad \max\{x, y, z\} \leq \min\{a, b\}$$

except if either x, y or z is equal to 0, in which case α has to lie in the \mathbb{Z}_6 -orbit of either $(1, 0; 1, 0, 0, 0)$ or $(1, 1; 1, 1, 0)$. A quick calculation shows that $\chi_{Q'}(e_1, e_1) = 1$. So assume that \mathbf{a} fulfils condition 5.1 and that $\chi_{Q'}(\mathbf{a}, \mathbf{a}) = 1$. We have

$$\begin{aligned} \chi_{Q'}(\mathbf{a}, \mathbf{a}) &= \sum_{i=0}^5 a_i^2 - \sum_{i=0}^2 2a_{2i}(a_{2i-1} + a_{2i+1}) \leq \sum_{i=0}^2 a_{2i}^2 - \sum_{i=0}^2 a_{2i+1}^2 \\ &= \sum_{i=0}^5 a_i^2 - \sum_{i=0}^2 2a_{2i+1}(a_{2i} + a_{2(i+1)}) \leq \sum_{i=0}^2 a_{2i+1}^2 - \sum_{i=0}^2 a_{2i}^2 \end{aligned}$$

which is impossible because this would imply that $1 \leq 0$. \square

In addition, we also have for representations $V \in \mathbf{rep}_\alpha Q', W \in \mathbf{rep}_\beta Q'$ that

$$\dim \mathrm{Hom}_{\mathbb{C}Q'}(V, W) - \dim \mathrm{Ext}_{\mathbb{C}Q'}^1(V, W) = \chi_{Q'}(\mathbf{a}, \mathbf{b}),$$

see for example [5, Chapter 4]. In particular, if V is simple, then $\dim \mathrm{Ext}_{\mathbb{C}Q'}^1(V, V)$ is 0 if and only if $\chi_{Q'}(\mathbf{a}, \mathbf{a}) = 1$, which we have just shown is only true if V is 1-dimensional. Now we can prove the main theorem.

Theorem 8. *Let $V = \bigoplus_{i=1}^k S_i^{e_i}$ be a semisimple n -dimensional representation of V with $S_i = \lambda_i S'_i$ with $S'_i \in \mathbf{rep}_{\alpha_i} \Gamma$ simple. Then $\mathbf{rep}_n B_3$ is smooth in V if and only if*

- $\forall 1 \leq i < j \leq k : \mathrm{Ext}_{B_3}^1(S_i, S_j) = 0$ and
- $\forall 1 \leq i \leq k : \dim S_i = 1$ or $e_i = 1$.

Proof. We have shown that V belongs to a component of dimension

$$n^2 + \sum_{i=1}^k e_i \dim \mathrm{Ext}_{\Gamma}^1(S'_i, S'_i).$$

So $\text{rep}_n B_3$ is smooth in V if and only if the dimension of the tangent space in V to $\text{rep}_n B_3$ is equal to this number. Calculating the dimension of the tangent space, we find

$$\begin{aligned} \dim T_V \text{rep}_n B_3 &= n^2 - \sum_{i=1}^k e_i^2 + \sum_{i=1}^k e_i^2 \dim \text{Ext}_{B_3}^1(S_i, S_i) + \sum_{1 \leq i < j \leq k} 2e_i e_j \dim \text{Ext}_{B_3}^1(S_i, S_j) \\ &= n^2 + \sum_{i=1}^k e_i^2 \dim \text{Ext}_{\Gamma}^1(S'_i, S'_i) + \sum_{1 \leq i < j \leq k} 2e_i e_j \dim \text{Ext}_{B_3}^1(S_i, S_j). \end{aligned}$$

So $\text{rep}_n B_3$ is smooth in V if and only if

$$\sum_{i=1}^k e_i(e_i - 1) \dim \text{Ext}_{\Gamma}^1(S'_i, S'_i) + \sum_{1 \leq i < j \leq k} 2e_i e_j \dim \text{Ext}_{B_3}^1(S_i, S_j) = 0.$$

As this is a sum of positive numbers, this can only be 0 if each term is 0, which already leads to the first condition of the theorem. It additionally follows that

$$\forall 1 \leq i \leq k : e_i = 1 \text{ or } \dim \text{Ext}_{\Gamma}^1(S'_i, S'_i) = 0.$$

So if $e_i \neq 0$, then $\dim S'_i = 1$ in light of the previous lemma. \square

Most of the singularities are the consequence of the following theorem.

Theorem 9. *If $V = \bigoplus_{i=1}^k S_i^{e_i}$ is a singular point of $\text{rep}_n B_3$ such that*

- *either $\exists 1 \leq i < j \leq k : \text{Ext}_{B_3}^1(S_i, S_j) \neq 0$ or*
- *$\exists 1 \leq i \leq k : \dim S_i \geq 3$ and $e_i \geq 2$.*

Then V lies on the intersection of 2 components.

Proof. Let $S_i = \lambda_i S'_i$ with $S'_i \in \text{rep}_{\alpha_i} \Gamma$ as before. In the first case, we may assume that $(i, j) = (1, 2)$ and that $\lambda_1 = \lambda_2 = 1$. Then V belongs to the image of

$$f(\underbrace{\alpha_1, \dots, \alpha_1}_{e_1}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{e_k})$$

and

$$f(\alpha_1 + \alpha_2, \underbrace{\alpha_1, \dots, \alpha_1}_{e_1 - 1}, \underbrace{\alpha_2, \dots, \alpha_2}_{e_2 - 1}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{e_k}).$$

$\alpha_1 + \alpha_2$ is also a simple dimension vector of Γ , so the claim follows.

In the second case, we may again assume that $i = 1$ and that S_1 is a Γ -module. If $\dim S_1 \geq 3$, then $2\alpha_1$ also fulfils requirement 5.2 and V lies on the intersection of the image of

$$f(\underbrace{\alpha_1, \dots, \alpha_1}_{e_1}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{e_k})$$

and

$$f(2\alpha_1, \underbrace{\alpha_1, \dots, \alpha_1}_{e_1 - 2}, \underbrace{\alpha_2, \dots, \alpha_2}_{e_2 - 1}, \dots, \underbrace{\alpha_k, \dots, \alpha_k}_{e_k}).$$

\square

However, if there exists an S of dimension 2 in the decomposition of V with multiplicity ≥ 2 , then V is still a singular point of $\text{rep}_n B_3$, but V is not necessarily the intersection of 2 components with generically semisimple elements.

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