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Pointwise bornological spaces

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ABSTRACT

With each metric space (X, d) we can associate a bornological space $(X, \mathcal{B}_d)$ where $\mathcal{B}_d$ is the set of all subsets of X with finite diameter. Equivalently, $\mathcal{B}_d$ is the set of all subsets of X that are contained in a ball with finite radius. If the metric d can attain the value infinite, then the set of all subsets with finite diameter is no longer a bornology. Moreover, if d is no longer symmetric, then the set of subsets with finite diameter does not coincide with the set of subsets that are contained in a ball with finite radius. In this text we will introduce two structures that capture the concept of boundedness in both symmetric and non-symmetric extended metric spaces.

1. Introduction

Throughout the text we will call a function $d : X \times X \to [0, \infty]$ that is zero on the diagonal of $X \times X$ and satisfies the triangular inequality a $pq\infty$-metric. If d is symmetric or only attains finite values, then we will respectively drop the suffix q or ∞. For more information about the categorical notions that are used in the text we refer the reader to [1].

A bornology on a set X is an ideal in 2^X that contains all singletons. It generalizes the concept of a set of bounded subsets of X. The set of relatively compact subsets of a topological space and the set of totally bounded subsets of a uniform space are examples of bornologies. The morphisms between bornological spaces—maps that preserve boundedness—are called bounded maps. Vector spaces endowed with a bornology for which the addition and scalar multiplication are bounded maps are called bornological vector spaces and play an important role in functional analysis. See for example [2] and [3].

With each p-metric space (X, d) one can associate the bornology $\mathcal{B}_d$ of bounded sets. A set $A \subseteq X$ is an element of this bornology iff

$$\exists x \in X\ \exists R > 0: \quad A \subseteq B(x, R).$$

Because the distance between two points is always finite this last definition is equivalent with

$$\forall x \in X\ \exists R > 0: \quad A \subseteq B(x, R).$$

It is easy to see that this is in fact a bornology.

If we would now start from a $p\infty$-metric and define $\mathcal{B}_d$ in the same way as we did before, then we would still obtain a downset that contains all singletons, but we would in general not obtain a set that is closed under finite unions. This set would however satisfy the following weaker condition:

$$A, B \in \mathcal{B}, \quad A \cap B \neq \phi \quad \Rightarrow \quad A \cup B \in \mathcal{B}.$$

In both cases a set A is an element of $\mathcal{B}_d$ iff it has a finite diameter, but in the $p\infty$-metric case we can no longer say that a set is bounded iff for all $x \in X$ we can find an $R > 0$ such that $A \subseteq B(x, R)$.

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We can try to do the same thing for a $pq\infty$-metric, but then $A \subseteq B(x, R)$ no longer implies that A has a finite diameter. Therefore boundedness is no longer a quality from the set A itself, but is a notion relative to a point. We say that a set A is bounded according to x iff A is contained in a ball with finite radius and center x. Clearly, the set of all subsets that are bounded according to x is an ideal that contains $\{x\}$ and if $\{y\}$ is bounded according to x and A is bounded according to y, then A is bounded according to x.

All of the above motivates the claim that bornological spaces are no longer the right objects to examine boundedness to deal with $p\infty$-metric or $pq\infty$-metric spaces. In Sections 2 and 3 we will therefore introduce two new structures that capture the idea of bounded sets in these spaces and in Section 4 the categorical connections between both structures will be examined. In the last section we will look into the relations between these new structures and quasi-uniform spaces.

2. Extended bornological spaces

Let (X, d) be an extended pseudo-metric space. As we saw in the introduction, the set of all subsets of X that are contained in a ball with finite radius is not a bornology. It is a downset that contains all singletons and if two of its members have a non-empty intersection, then their union is also an element of that set. In this section we will call a set with such a structure an extended bornological space. We will introduce a category the objects of which are precisely these spaces and we will see that this category is a topological construct that contains the category of bornological spaces and bounded maps as a reflective, finally dense subcategory.

Definition 1. A downset $\mathcal{B}$ in 2^X is a called an *extended bornology* on X if it has the following properties:

(E1) $\forall x \in X$: $\{x\} \in \mathcal{B}$,
(E2) $(A, B \in \mathcal{B}, A \cap B \neq \phi) \Rightarrow A \cup B \in \mathcal{B}$.

Its elements are called *bounded* sets. A pair $(X, \mathcal{B})$, where $\mathcal{B}$ is an extended bornology on X, is an extended bornological space and a map f from X to Y is called a *bounded* map from $(X, \mathcal{B})$ to $(Y, \mathcal{B}')$ if it maps bounded sets to bounded sets.

Proposition 2. *The category of extended bornological spaces and bounded maps* $\mathbf{Bor}^\infty$ *is a topological construct.*

Proof. Let $(f_i : X \to (X_i, \mathcal{B}_i))_{i\in I}$ be a structured source where all $(X_i, \mathcal{B}_i)$ are objects in $\mathbf{Bor}^\infty$. Define $\mathcal{B}$ as the set $\{B \subseteq X \mid \forall i \in I\colon\ f_i(B) \text{ is bounded}\}$. It is clear that this is a downset that contains all singletons in X. Now let A and B be two elements of $\mathcal{B}$ that have a non-empty intersection. If $x \in A \cap B$ then $f_i(x) \in f_i(A \cap B) \subseteq f_i(A) \cap f_i(B)$ and thus $f_i(A \cup B)$, being equal to $f_i(A) \cup f_i(B)$, is bounded. Because this is true for all $i \in I$ we obtain that $A \cup B \in \mathcal{B}$, so $(X, \mathcal{B})$ is an extended bornological space and by definition each f_i is a bounded map. Suppose $f_i \circ g : (Y, \mathcal{B}') \to (X_i, \mathcal{B}_i)$ is a bounded map for each $i \in I$. Take $B' \in \mathcal{B}'$. By definition we get that $f_i(g(B'))$ is bounded for each $i \in I$ and so $g(B')$ is an element of $\mathcal{B}$. Hence g is a bounded map and we have proved that $(X, \mathcal{B})$ is the initial object for the given source. □

Remark 3. Let $(X, \mathcal{B})$ be an extended bornological space and $A \subseteq X$. From the previous proposition we obtain that the subspace structure on A, denoted as $\mathcal{B}|_A$, is given by $\{B \subseteq A \mid B \in \mathcal{B}\}$.

If $((X_i, \mathcal{B}_i))_{i\in I}$ is a family of extended bornological spaces, then the product structure on $\prod_{i\in I} X_i$, denoted as $\prod_{i\in I} \mathcal{B}_i$, is given by the set of subsets of elements of $\{\prod_{i\in I} B_i \mid \forall i \in I\colon\ B_i \in \mathcal{B}_i\}$.

Proposition 4. *The full subcategory* $\mathbf{Bor}$ *of bornological spaces is initially closed in* $\mathbf{Bor}^\infty$.

Proof. Let $(f_i : (X, \mathcal{B}) \to (X_i, \mathcal{B}_i))_{i\in I}$ be an initial source where each $(X_i, \mathcal{B}_i)$ is a bornological space. If A and B are bounded in X then we know that $f_i(A \cup B)$, being equal to $f_i(A) \cup f_i(B)$, is an element of $\mathcal{B}_i$ for each $i \in I$ because each $\mathcal{B}_i$ is a bornology. From the construction of initial structures, above, we obtain that $A \cup B$ is an element of $\mathcal{B}$. □

Proposition 5. *The bornology of the bornological reflection of an extended bornological space* $(X, \mathcal{B})$ *is given by the set off all finite unions of elements in* $\mathcal{B}$.

Proof. It is clear that $\mathcal{B}_b$ is a bornology and that $1_X : (X, \mathcal{B}) \to (X, \mathcal{B}_b)$ is a bounded map. Suppose $f : (X, \mathcal{B}) \to (Y, \mathcal{B}')$ is a bounded map and $(Y, \mathcal{B}')$ a bornological space. We want to prove that $f : (X, \mathcal{B}_b) \to (Y, \mathcal{B}')$ is a bounded map. An element B of $\mathcal{B}_b$ is a finite union of elements in $\mathcal{B}$, say $\bigcup_{k=1}^n B_k$. Each B_k is mapped to an element in $\mathcal{B}'$, but because the latter is a bornology the union of these elements, which is equal to the image of B, is again a bounded set. □

Definition 6. Let X be a set and $\mathcal{A} \subseteq 2^X$. We will call $\mathcal{A}$ *linked* if for all elements $A, A' \in \mathcal{A}$ there is a finite sequence $A = A_0, \ldots, A_n = A'$ in $\mathcal{A}$ such that $A_{k-1} \cap A_k \neq \phi$ for each $k \in \{1, \ldots, n\}$. We will call such a sequence a connecting sequence.

Lemma 7. *If* $\mathcal{A}, \mathcal{A}' \subseteq 2^X$ *are linked and* $\bigcup \mathcal{A}$ *and* $\bigcup \mathcal{A}'$ *have a non-empty intersection, then* $\mathcal{A} \cup \mathcal{A}'$ *is linked.*

Proof. Take $A, A' \in \mathcal{A} \cup \mathcal{A}'$. If both A and A' are elements of either $\mathcal{A}$ or $\mathcal{A}'$, then there is nothing left to prove, so let us suppose that this is not the case and that $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$. Since $(\bigcup \mathcal{A}) \cap (\bigcup \mathcal{A}')$ is not empty there are $B \in \mathcal{A}$ and $B' \in \mathcal{A}'$ such that $B \cap B' \neq \phi$. Let $A_0, \ldots, A_n$ and $A_{n+1}, \ldots, A_{n+m}$ be the sequences that respectively connect A to B and B' to A'. Clearly the sequence $A_0, \ldots, A_{n+m}$ connects A to A'. □

Proposition 8. *If $(X, \mathcal{B})$ is an extended bornological space and $\mathcal{A} \subseteq \mathcal{B}$ is finite and linked, then $\bigcup \mathcal{A}$ is an element of $\mathcal{B}$.*

Proof. We will prove this by induction. If $\mathcal{A}$ consists of only one element, then $\bigcup \mathcal{A} \in \mathcal{B}$ is clearly true. Suppose this proposition is true if $\mathcal{A}$ has a number of elements that is less than or equal to n and let $\mathcal{A}$ contain $n+1$ elements. Take an arbitrary $A_1 \in \mathcal{A}$ and an $A_2 \in \mathcal{A}$ that satisfy $A_1 \cap A_2 \neq \phi$ and denote $A_1 \cup A_2$ as A. The set $\mathcal{A}'$, defined as $(\mathcal{A} \setminus \{A_1, A_2\}) \cup \{A\}$, is a linked set with n elements in $\mathcal{B}$ and its union therefore is an element of $\mathcal{B}$. Since $\bigcup \mathcal{A}$ equals $\bigcup \mathcal{A}'$ this implies that $\bigcup \mathcal{A}$ is in $\mathcal{B}$. □

Proposition 9. *Let X be a set and $\mathcal{C} \subseteq 2^X$ with $\bigcup \mathcal{C} = X$. If we define $\mathcal{B}$ as the set of all subsets of unions of finite, linked subsets of $\mathcal{C}$ then $\mathcal{B}$ is an extended bornology that contains $\mathcal{C}$. We say that $\mathcal{C}$ generates $\mathcal{B}$.*

Proof. By definition $\mathcal{B}$ is a downset that contains all singletons of X. Now take elements A_1 and A_2 in $\mathcal{B}$ with non-empty intersection. There are finite, linked subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{C}$ such that $A_k \subseteq \bigcup \mathcal{A}_k$ for $k \in \{1, 2\}$. Because A_1 and A_2 have a non-empty intersection we obtain that $\mathcal{A}_1 \cup \mathcal{A}_2$ is a finite, linked set and $A_1 \cup A_2 \subseteq \bigcup(\mathcal{A}_1 \cup \mathcal{A}_2)$. This yields that $A_1 \cup A_2$ is an element of $\mathcal{B}$ and that the latter is an extended bornology. □

Proposition 10. *Let $(f_i : (X_i, \mathcal{B}_i) \to X)_{i \in I}$ be a structured sink where $(X_i, \mathcal{B}_i)$ are objects in $\mathbf{Bor}^\infty$, then the final structure $\mathcal{B}$ on X for this sink is generated by the union of the set of all singletons in X and*

$$\bigcup_{i \in I} f_i(\mathcal{B}_i).$$

Proof. Suppose that $g \circ f_i : (X_i, \mathcal{B}_i) \to (Y, \mathcal{B}')$ is a bounded map for each $i \in I$. The image of a singleton in X and the image of a set $f_i(B)$, where B is an element of $\mathcal{B}$, is always an element of $\mathcal{B}'$. So if $\mathcal{A}$ is a finite, linked set consisting of singletons and elements of $\bigcup_{i \in I} f_i(\mathcal{B}_i)$, then the image of $\bigcup \mathcal{A}$ under g is the union of a finite, linked subset of $\mathcal{B}'$ and thus itself an element of $\mathcal{B}'$ which implies that g is a bounded map. □

Remark 11. Let $(X, \mathcal{B})$ be an extended bornological space and $\sim$ an equivalence relation on X. We will denote the equivalence class of an element x as $\bar{x}$ and $\overline{B}$ is defined as the set $\{\bar{x} \mid x \in B\}$. The quotient structure on $X/\sim$ is generated by $\{\overline{B} \subseteq \overline{X} \mid B \in \mathcal{B}\}$.

For a family of extended bornological spaces $((X_i, \mathcal{B}_i))_{i \in I}$ the direct sum structure on $\coprod_{i \in I} X_i$ is given by $\bigcup_{i \in I} \mathcal{B}_i$. A set $B \subseteq \coprod_{i \in I} X_i$ is bounded iff it is a subset of $\bigcup \mathcal{A}$ for some finite, linked subset of $\bigcup_{i \in I} \mathcal{B}_i$. Because all X_i's are disjoint, $\mathcal{A}$ being linked implies that it is a subset of $\mathcal{B}_i$ for some $i \in I$. Hence $\bigcup \mathcal{A}$, and thus B, are either elements of $\mathcal{B}_i$ for some $i \in I$.

Definition 12. If $(X, \mathcal{B})$ is an extended bornological space then the relation $x \overset{b}{\sim} y$, defined as $\{x, y\} \in \mathcal{B}$, is an equivalence relation. The equivalence classes of this relation will be called the components of $(X, \mathcal{B})$.

Proposition 13. *Each component of $(X, \mathcal{B})$ endowed with the subspace structure is a bornological space.*

Proof. Let A be a component of $(X, \mathcal{B})$ and take B_1 and B_2 in $\mathcal{B}|_A$. If either B_1 or B_2 is empty then it is obvious that $B_1 \cup B_2 \in \mathcal{B}|_A$, so let us suppose both are non-empty. Choose $b_1 \in B_1$ and $b_2 \in B_2$. Both are in the same component, so $\{b_1, b_2\} \in \mathcal{B}$. Because the sets B_1, $\{b_1, b_2\}$ and B_2 are linked, this yields that $B_1 \cup B_2$ is an element of $\mathcal{B}$. Hence $B_1 \cup B_2 \in \mathcal{B}|_A$. □

Proposition 14. *Each extended bornological space is a direct sum of bornological spaces.*

Proof. We will prove that each extended bornological space $(X, \mathcal{B})$ is the direct sum of its components. The components of an equivalence relation form a partition, so the set X is indeed the direct sum of these components. We still need to prove that $\mathcal{B}$ is the union of all $\mathcal{B}|_A$, where A is a component of $(X, \mathcal{B})$. Take $B \in \mathcal{B}$. By definition $\{b_1, b_2\} \in \mathcal{B}$ holds for all $b_1, b_2 \in B$, so all elements of $\mathcal{B}$ are in the same component. Hence B is a subset of an equivalence class. □

Corollary 15. *The category* **Bor** *is finally dense in* $\mathbf{Bor}^\infty$.

3. Pointwise bornological spaces

In a $pq\infty$-metric space a set can be contained in a ball with finite radius and have diameter infinity at the same time. Take for example the set of reals with the metric d, where $d(x,y)$ is defined as $y-x$ if $y \geqslant x$ and ∞ otherwise. The set $[0,1]$ is contained in the ball with center 0 and radius 1, but has diameter infinity since $d(1,0)=\infty$. We do know, however, that the set of subsets of X that are contained in a ball with center x and finite radius is an ideal that contains $\{x\}$ and if $\{y\}$ is in a ball with center x and A is contained in a ball with center y, then A also contained in a ball with center x. We will generalize this concept in the notion of a pointwise bornological space and define a category with these spaces as objects. This category will turn out to be topological construct that contains the category of extended bornological spaces as a full subconstruct that is both reflectively and coreflectively embedded.

Definition 16. A map β from a set X into the set of ideals in 2^X will be called a *pointwise bornology* iff it satisfies:

(P1) $\{x\} \in \beta(x)$,
(P2) $(\{y\} \in \beta(x), A \in \beta(y)) \Rightarrow A \in \beta(x)$.

If β is a pointwise bornology, then the pair (X,β) will be called a *pointwise bornological space*. An element of $\beta(x)$ will be called *bounded according to x*. A map f from X to Y is called a *bounded* map from (X,β) to (Y,β') if $f(A) \in \beta'(f(x))$ for all $A \in \beta(x)$.

Proposition 17. *The category* **pBor** *of pointwise bornological spaces and bounded maps is a topological construct.*

Proof. Let $(f_i : X \to (X_i,\beta_i))_{i\in I}$ be a structured source where all (X_i,β_i) are objects in **pBor**. Define $\beta(x)$ for an $x \in X$ as the set $\{A \subseteq X \mid \forall i \in I\colon\ f_i(A) \in \beta_i(f_i(x))\}$. It is clear that this is an ideal that contains the singleton $\{x\}$. Now suppose $\{y\} \in \beta(x)$ and $A \in \beta(y)$. By definition this means that $\{f_i(y)\} \in \beta_i(f_i(x))$ and $f_i(A) \in \beta_i(f_i(y))$, and therefore $f_i(A) \in \beta_i(f_i(x))$ for each $i \in I$. So we obtain that $A \in \beta(x)$. This proves that β is a local bornology.

In order to prove that this structure is initial suppose that $f_i \circ g : (Y,\beta') \to (X_i,\beta_i)$ is a bounded map for each $i \in I$. Take a $y \in Y$. If $A \in \beta'(y)$ then $f_i(g(A)) \in \beta_i(f_i(g(y)))$ and thus, by definition, $g(A) \in \beta(g(y))$. We obtain that $g : (Y,\beta') \to (X,\beta)$ is a bounded map, which completes our proof. □

Remark 18. Let (X,β) be a pointwise bornological space and A a subset of X. The subspace structure on A, denoted as $\beta|_A$ is given by

$$\beta|_A(x) = \{B \subseteq A \mid B \in \beta(x)\}$$

for each $x \in X$.

Suppose $((X_i,\beta_i))_{i\in I}$ is a family of objects in **pBor**. The product structure on $\prod_{i\in I} X_i$, denoted as $\prod_{i\in I}\beta_i$, is given by

$$\left(\prod_{i\in I}\beta_i\right)(x) = \left\{B \subseteq \prod_{i\in I} X_i \;\middle|\; \forall i \in I\ \exists B_i \in \beta_i(\pi_i(x))\colon\ B \subseteq \prod_{i\in I} B_i\right\}.$$

A subset $C \subseteq \prod_{i\in I} X_i$ is an element of $\prod_{i\in I}\beta_i(c)$ iff $\pi_i(C) \in \beta_i(\pi_i(c))$, where π_i is the projection map to X_i, for all $i \in I$. Since $C \subseteq \prod_{i\in I}\pi_i(C)$ this is equivalent to $C \subseteq \prod_{i\in I} A_i$ with $A_i \in \beta_i(\pi_i(c))$.

Definition 19. If X is a set and $\alpha(x)$ is a downset in 2^X for each $x \in X$ such that $\{x\} \in \alpha(x)$ then we say that α links A to x iff there is a finite sequence $x_0, x_1, \ldots, x_n$ where $x_0 = x$, $\{x_k\} \in \alpha(x_{k-1})$ for all $1 \leqslant k \leqslant n$ and $A \in \alpha(x_n)$.

Proposition 20. *If X is a set and $\alpha(x)$ is a downset in 2^X for each $x \in X$ such that $\{x\} \in \alpha(x)$ and we define $\beta(x)$ as the set of all finite unions of sets that are linked to x then β is a pointwise bornology. We say that β is generated by α.*

Proof. First of all it is easy to see that $\{x\} \in \beta(x)$. Now take a $\{y\} \in \beta(x)$ and an $A \in \beta(y)$. The set A is a finite union of sets that are linked to y, say $A = \bigcup_{k\in K} A_k$. If the sequence $x_0, \ldots, x_n$ links $\{y\}$ to x and $x_{n+1}, \ldots, x_{n+m}$ links A_k to y, then $x_0, \ldots, x_{n+m}$ links A_k to x. So we obtain that α links each A_k to x and that A is a finite union of sets that are linked to x and therefore is an element of $\beta(x)$. □

Lemma 21. *If (X,β) is a pointwise bornological space and β links A to x, then A is an element of $\beta(x)$.*

Proof. We will prove this by induction. Suppose that the sequence that links A to x consists of only one element. By definition we obtain $A \in \beta(x)$. Let us now assume that the statement we want to prove is true if the connecting sequence has n elements or less and let $x_0, \ldots, x_n$ be a sequence that links A to x. From our induction hypothesis we obtain that $\{x_n\}$ is an element of $\beta(x)$ and because $A \in \beta(x_n)$ this yields $A \in \beta(x)$. □

Proposition 22. *Let $(f_i : (X_i, \beta_i)_i \to X)_{i\in I}$ be a structured sink with $(X_i, \beta_i)_{i\in I}$ a family of objects in* **locBor**. *If we define $\alpha(x) \subseteq 2^X$ as $\{\{x\}\}$ if $x \notin \bigcup_{i\in I} f_i(X_i)$ and as $\{f_i(A) \mid A \in \beta_i(y),\ f_i(y) = x\}$ otherwise, then the pointwise bornology β generated by α is final for the given sink.*

Proof. Suppose that $g \circ f_i : (X_i, \beta_i) \to (Y, \beta')$ is a bounded map for all $i \in I$. Take $x \in X$ and $B \in \beta(x)$. There is a finite sequence $x_0, \ldots, x_n$ that links B to x through α. Clearly each element of $\alpha(x)$ is mapped to an element of $\beta'(g(x))$ and thus $g(x_0), \ldots, g(x_n)$ is a sequence that links $g(B)$ to $g(x)$ through β'. Because β' is a pointwise bornology this yields that $g(B) \in \beta'(g(x))$. So g is a bounded map and β is the final structure for the given structured sink. □

Remark 23. From the previous proposition we obtain that if (X, β) is pointwise bornological space and $\sim$ an equivalence relation on X, then the quotient structure on $X/\sim$ is generated by α where—with the same notations as in Remark 11—$\alpha(\bar{x})$ is defined as

$$\{\bar{A} \mid \exists y \in \bar{x}:\ A \in \beta(y)\}.$$

For a family of objects in **pBor** $((X_i, \beta_i))_{i\in I}$ the direct sum structure β on $\coprod_{i\in I} X_i$ is given by

$$\beta(x) = \beta_i(x)$$

for an $x \in X_i$. The direct sum structure is generated by α with $\alpha(x)$ defined as $\beta_i(x)$ if $x \in X_i$. Take $x \in X_j$ and an $A \subseteq \coprod_{i\in I} X_i$ that is linked to x by a sequence $x_0, \ldots, x_n$. By definition of α we get that all x_k are elements of X_j and therefore A an element of $\beta_j(x)$.

Proposition 24. *Let $(X, \mathcal{B})$ be an extended bornological space. If we define $\beta_{\mathcal{B}}(x)$ as the ideal $\{A \subseteq X \mid A \cup \{x\} \in \mathcal{B}\}$, then $\beta_{\mathcal{B}}$ is a pointwise bornology on X.*

Proof. It is clear that $\{x\} \in \beta_{\mathcal{B}}$ for each $x \in X$. Now suppose that $\{y\} \in \beta_{\mathcal{B}}$ and $A \in \beta_{\mathcal{B}}$. By definition this means that both $\{x, y\}$ and $A \cup \{y\}$ are elements of $\mathcal{B}$. Because they have a non-empty intersection their union and the subset $A \cup \{x\}$ are in $\mathcal{B}$ and we obtain $A \in \beta_{\mathcal{B}}(x)$. □

Proposition 25. *If $f : (X, \mathcal{B}) \to (Y, \mathcal{B}')$ is a morphism in* $\mathbf{Bor}^{\infty}$*, then $f : (X, \beta_{\mathcal{B}}) \to (Y, \beta_{\mathcal{B}'})$ is a morphism in* **pBor**.

Proof. If $A \in \beta_{\mathcal{B}}(x)$, then $A \cup \{x\} \in \mathcal{B}$ and thus $f(A) \cup \{f(x)\} \in \mathcal{B}'$. By definition the latter means that $f(A) \in \beta_{\mathcal{B}'}$ and we obtain that f is a bounded map. □

Corollary 26. *If we define $P(X, \mathcal{B})$ as $(X, \beta_{\mathcal{B}})$ for each extended bornological space $(X, \mathcal{B})$, then P is concrete functor from* $\mathbf{Bor}^{\infty}$ *to* **pBor**.

Proposition 27. *The functor P is a full embedding.*

Proof. In order to prove that P is full suppose that $f : (X, \beta_{\mathcal{B}}) \to (Y, \beta_{\mathcal{B}'})$ is a bounded map. Take a $B \in \mathcal{B}$. If B is the empty set then $f(B)$ automatically is an element of $\mathcal{B}'$. In the case that B is non-empty, B is an element of $\beta_{\mathcal{B}}(b)$ for an arbitrary $b \in B$. Hence $f(B) \in \beta_{\mathcal{B}'}(f(b))$ and by definition this yields that $f(B) \in \mathcal{B}'$ and that $f : (X, \mathcal{B}) \to (Y, \mathcal{B}')$ is a bounded map.

To prove that P is an embedding it is sufficient to show that $\mathcal{B} = \mathcal{B}'$ if $\beta_{\mathcal{B}} = \beta_{\mathcal{B}'}$. Now suppose the latter is true and take $A \in \mathcal{B}$. Again if A is empty there is nothing left to prove. So let A be non-empty. This implies that, as we noticed above, that $A \in \beta_{\mathcal{B}}(a)$ for an arbitrary $a \in A$. Hence $A \in \beta_{\mathcal{B}'}(a)$ and $A \in \mathcal{B}'$. Of course this also works the other way around so we conclude $\mathcal{B} = \mathcal{B}'$. □

Corollary 28. *Both* $\mathbf{Bor}^{\infty}$ *and* **Bor** *are isomorphic to full subcategories of* **pBor**.

Proposition 29. *A pointwise bornological space (X, β) is extended bornological iff it satisfies*:

(E) $\forall x, y \in X\colon \{y\} \in \beta(x) \Rightarrow \{x\} \in \beta(y)$.

We will denote the extended bornology associated with a local bornology β that satisfies E as $\mathcal{B}_{\beta}$.

Proof. If there is some extended bornology $\mathcal{B}$ on X such that $\beta = \beta_{\mathcal{B}}$ and $\{y\} \in \beta(x)$, then by definition $\{x, y\} \in \mathcal{B}$ which yields $\{x\} \in \beta(y)$.

On the other hand if condition E holds then we can define $\mathcal{B}$ as

$$\bigcup_{x\in X} \beta(x).$$

It is clear that this is a downset that contains each singleton of X, but it is also an extended bornology. Take two sets $A, B \in \mathcal{B}$ with non-empty intersection, say $x \in A \cap B$. We can find $a, b \in X$ such that $A \in \beta(a)$ and $B \in \beta(b)$. From condition E we obtain $\{a\}$ and $\{b\}$ are in $\beta(x)$ so $A, B \in \beta(x)$, which yields $A \cup B \in \beta(x)$ and $A \cup B \in \mathcal{B}$. Now to prove that $\beta = \beta_{\mathcal{B}}$ take $A \in \beta_{\mathcal{B}}(x)$. We have $A \cup \{x\} \in \mathcal{B}$ and by definition there is a $z \in X$ such that $A \cup \{x\} \in \beta(z)$. From this we obtain $\{x\} \in \beta(z)$, and from condition E we get $\{z\} \in \beta(x)$ and $A \in \beta(z)$. This implies $A \in \beta(x)$. Conversely if $A \in \beta(x)$ then $A \cup \{x\} \in \beta(x) \subseteq \mathcal{B}$ and $A \in \beta_{\mathcal{B}}(x)$. □

Definition 30. Let (X, β) be a pointwise bornological space and take $x \in X$. We define $C(x)$ as $\bigcup_{A \in \beta(x)} A$.

Proposition 31. *Let (X, β) be a pointwise bornological space, then $C(x)$ is given by $\{y \in X \mid \{y\} \in \beta(x)\}$.*

Proof. That $\{y \in X \mid \{y\} \in \beta(x)\}$ is subset of the component of x is a direct consequence of the definition. On the other hand if $A \in \beta(x)$ then $\{a\} \in \beta(x)$ for each $a \in A$ which implies that A is a subset of $\{y \in X \mid \{y\} \in \beta(x)\}$. □

Corollary 32. *If (X, β) is extended bornological, then $C(x)$ is equal to the component of the extended bornological space that contains x.*

Proposition 33. *A local bornological space (X, β) is extended bornological iff the set $\{C(x) \mid x \in X\}$ is a partition of X.*

Proof. If (X, β) is extended bornological then $\{C(x) \mid x \in X\}$ is a partition of X. This is a consequence of the previous corollary and Definition 12. Conversely suppose that $\{C(x) \mid x \in X\}$ is a partition and that $\{y\} \in \beta(x)$. Now $\{y\}$ is an element of $C(x)$ and of $C(y)$, so they have to be equal. This implies that $\{x\} \in \beta(y)$. □

Proposition 34. *A pointwise bornological space (X, β) is bornological iff it satisfies one of the following equivalent properties*:

(B1) $\forall x, y \in X$: $\{y\} \in \beta(x)$,
(B2) $\forall x, y \in X$: $\beta(x) = \beta(y)$.

Proof. Suppose there is a bornology $\mathcal{B}$ such that $\beta = \beta_{\mathcal{B}}$. We know that each finite set is an element of $\mathcal{B}$ so for each $x, y \in X$ holds $\{y\} \in \beta(x)$. Let us now prove that (B1) implies (B2). Of course it is sufficient to show that for each $x, y \in X$ holds $\beta(y) \subseteq \beta(x)$. Take $A \in \beta(y)$. Because $\{y\} \in \beta(x)$ we obtain that $A \in \beta(x)$ and that $\beta(y) \subseteq \beta(x)$. Finally, suppose that (B2) holds. Define $\mathcal{B}$ as $\beta(x)$, where x is of course arbitrary. The set $\mathcal{B}$ is a bornology. We already know that it is an ideal and because $\beta(y) = \mathcal{B}$ we obtain that $\{y\} \in \mathcal{B}$ for each $y \in X$. By definition $A \in \beta_{\mathcal{B}}(x)$ iff $A \cup \{x\} \in \mathcal{B}$ which, because $\mathcal{B}$ is a bornology, is equivalent to $A \in \mathcal{B} = \beta(x)$. □

Corollary 35. *A local bornological space (X, β) is bornological iff for all $x \in X$ we have $C(x) = X$.*

Proposition 36. **Bor**$^{\infty}$ *is a concretely reflective subcategory of* **pBor**.

Proof. Let (X, β) be a pointwise bornological space. We write $x \overset{s}{\sim} y$ iff there exists a finite sequence $x_0, x_1, \dots x_n$ such that $x = x_0$, $y = x_n$, and for each $k \in \{1, \dots, n\}$ either $\{x_k\} \in \beta(x_{k-1})$ or $\{x_{k-1}\} \in \beta(x_k)$. Clearly this is an equivalence relation. Now define $\beta_r(x)$ as the ideal generated by

$$\bigcup_{x \overset{s}{\sim} y} \beta(y).$$

Because $x \overset{s}{\sim} x$ and $\{x\} \in \beta(x)$ we get $\{x\} \in \beta_r(x)$. Furthermore, if $\{y\} \in \beta_r(x)$ and $A \in \beta_r(y)$ then $y \overset{s}{\sim} x$ and there is a $z \in X$ such that $z \overset{s}{\sim} y$ and $A \in \beta(z)$. All this implies that $z \overset{s}{\sim} x$ and thus by definition $A \in \beta_r(x)$. So far we have proved that β_r is a pointwise bornology. Now we want to show that it satisfies E. If $\{y\} \in \beta_r(x)$ then $y \overset{s}{\sim} x$ and because $\{x\} \in \beta(x)$ this implies $\{x\} \in \beta_r(y)$.

Clearly $1_X : (X, \beta) \to (X, \beta_r)$ is a bounded map. Suppose now that

$$f : (X, \beta) \to (Y, \beta')$$

is a morphism in **pBor** where (Y, β') is an extended bornological space. Take $A \in \beta_r(x)$ and let $x_0, x_1, \dots, x_n$ be a finite sequence such that $x = x_0$, $A \in \beta(x_n)$ and for each $k \in \{1, \dots, n\}$ either $x_k \in \beta(x_{k-1})$ or $x_{k-1} \in \beta(x_k)$. Because f is bounded and (Y, β') is extended bornological we get $f(x_k) \in \beta'(f(x_{k-1}))$ for each $k \in \{1, \dots, n\}$ and $f(A) \in \beta'(x_n)$. Hence $f(A) \in \beta'(f(x))$ and $f : (X, \beta_r) \to (Y, \beta')$ is bounded. □

Proposition 37. *A set $B \subseteq X$ is bounded in the extended bornological reflection of a pointwise bornological space (X, β) iff it is an element of the extended bornology $\mathcal{B}$ generated by*

$$\bigcup_{x \in X} \beta(x).$$

Proof. An element of $\beta_r(x)$ is a finite union $\bigcup_{l \in L} B_l$ of sets that are in some $\beta(y)$ with $x \overset{s}{\sim} y$. If there exists a finite sequence $x_0, x_1, \dots x_n$ such that $x = x_0$, $y = x_n$, for each $k \in \{1, \dots, n\}$ either $\{x_k\} \in \beta(x_{k-1})$ and $B_l \in \beta(y)$, then the set that contains all $\{x_{k-1}, x_k\}$ and $\{y\} \cup B$ is a finite, linked subset of $\bigcup_{x \in X} \beta(x)$. Therefore, all subsets of its union, and thus $B_l \cup \{x\}$, are elemenst of $\mathcal{B}$. This implies that $\bigcup_{l \in L} B_l$, as a subset of the union of all $B_l \cup \{x\}$, is in $\mathcal{B}$.

Conversely, suppose B is the subset of a union of a finite, linked subset $\mathcal{A}$ of $\bigcup_{x \in X} \beta(x)$. Take an arbitrary $b_0 \in B$, let A_0 be the element of $\mathcal{A}$ that contains b_0 and take an $A \in \mathcal{A}$. There is connecting sequence $A_0, \dots A_n$ from A_0 to A. Let x_k be an element that is in $A_{k-1} \cap A_k$. Each A_k is an element of some $\beta(y_k)$. Now the sequence $x, x_1, y_1, x_2, y_2, \dots, x_n, y_n$ has the property that if a and b are successive elements, then either $\{a\} \in \beta(b)$ or $\{b\} \in \beta(a)$ and thus $b_0 \overset{s}{\sim} y_n$. This implies that $A \in \beta_r(x)$. Since this is true for all A_k we obtain that B is an element of $\beta_r(x)$. □

Corollary 38. Bor *is a concretely reflective subcategory of* **pBor**.

Proposition 39. *The bornology of the bornological reflection of a pointwise bornological space (X, β) is equal to the ideal generated by*

$$\bigcup_{x \in X} \beta(x).$$

Proof. The bornological reflection of a pointwise bornological space is equal to the bornological reflection of its extended bornological reflection. This implies that the bornology $\mathcal{B}$ of the bornological reflection is the ideal generated by the set $\mathcal{B}_{\beta_r}$. Since all elements of $\mathcal{B}_{\beta_r}$ are in the ideal generated by

$$\bigcup_{x \in X} \beta(x)$$

this yields that $\mathcal{B}$ itself is equal to this ideal. □

Proposition 40. $\mathbf{Bor}^\infty$ *is a concretely coreflective subcategory of* **pBor**.

Proof. Let (X, β) be a pointwise bornological space. Define for each $x \in X$ the set $S(x) \subseteq X$ as the set that contains all $y \in X$ that satisfy $\{y\} \in \beta(x)$ and $\{x\} \in \beta(y)$. Denote the set of all $A \in \beta(x)$ that satisfy $A \subseteq S(x)$ as $\beta_c(x)$. Clearly we have $\{x\} \in \beta_c(x)$. Take $A \in \beta_c(y)$ and $\{y\} \in \beta_c(x)$. By definition $A \in \beta(x)$. For each $a \in A$ holds $\{a\} \in \beta(y)$ and $\{y\} \in \beta(a)$. Because $\{y\} \in \beta(x)$ and $\{x\} \in \beta(y)$ this yields $\{a\} \in \beta(x)$ and $\{x\} \in \beta(a)$. Hence $A \in \beta_c(x)$ and β_c is a pointwise bornology.

With the definition used above $1_X : (X, \beta_c) \to (X, \beta)$ is a bounded map. Let $f : (Y, \beta') \to (X, \beta)$ be a bounded map and (Y, β') an extended bornological space. Take $A \in \beta'(x)$. Since f is bounded we have $f(A) \in \beta(f(x))$. For each $a \in A$ we have $\{a\} \in \beta'(x)$ and because β' is extended bornological also $\{x\} \in \beta'(a)$. From this we obtain that $\{f(a)\} \in \beta(f(x))$ and $\{f(x)\} \in \beta(a)$ for all $a \in A$, hence $f(A) \subseteq S(f(x))$ and $f(A) \in \beta_c(f(x))$. In conclusion $f : (Y, \beta') \to (X, \beta_c)$ is bounded and (X, β_c) is the extended bornological coreflection of (X, β). □

Proposition 41. *A set $B \subseteq X$ is bounded in the extended bornological coreflection of a pointwise bornological space (X, β) iff it satisfies*

$$\forall b \in B: \quad B \in \beta(b).$$

Proof. A set $B \subseteq X$ is an element of $\mathcal{B}_{\beta_c}$ if it is an element of some $\beta(x)$ and for all $b \in B$ holds $\{b\} \in \beta(x)$ and $\{x\} \in \beta(b)$. Because $B \in \beta(x)$ and $\{x\} \in \beta(b)$ we obtain $B \in \beta(b)$ for each $b \in B$.

Conversely, if for each $b \in B$ holds $B \in \beta(b)$ and b_0 is an arbitrary element of B, then $B \in \beta(b_0)$ and for all $b \in B$ holds $\{b\} \in \beta(b_0)$ and $\{b_0\} \in \beta(b)$. □

4. Quasi-uniform spaces

We introduced extended and pointwise bornological space in order to generalize the notion of boundedness in $p\infty$-metric and $pq\infty$-metric spaces, so naturally we can associate with each $pq\infty$-metric space (X, d) a pointwise bornological space. We define $\beta_d(x)$ as the set $\{A \subseteq X \mid \exists R > 0\colon\ A \subseteq B_d(x, R)\}$. The couple (X, β_d) is of course a pointwise bornological space. Clearly (X, β_d) is extended bornological if d is symmetric and bornological if d attains only finite values. A quasi-uniformity can be described as a collection of $pq\infty$-metrics. Therefore, we can in a natural way associate with each quasi-uniform space a pointwise bornological space by saying that a set A is bounded according to x iff it is bounded according to x for all uniformly continuous $pq\infty$-metrics. We will see that this pointwise bornology is extended bornological if the quasi-uniform space is a uniform space and that it is bornological if it is a uniform, uniformly connected space.

Definition 42. Let $(X,\mathcal{U})$ be a quasi-uniform space. Define $\beta_{\mathcal{U}}(x)$ for an $x \in X$ as the set of all $A \subseteq X$ for which $A \in \beta_d(x)$ for each uniformly continuous $pq\infty$-metric d. The pair $(X, \beta_{\mathcal{U}})$ is a pointwise bornological space.

Proposition 43. *A set $A \subseteq X$ is in $\beta_{\mathcal{U}}(x)$ iff $\forall U \in \mathcal{U}\ \exists n \in \mathbb{N}$: $A \subseteq U^n(x)$.*

Proof. Suppose A is an element of $\beta_{\mathcal{U}}(x)$ and U is an entourage. Let $(U_{-n})_{n\in\mathbb{N}}$ be a sequence for which $U = U_0$ and $U_{-n-1} \circ U_{-n-1} \circ U_{-n-1} \subseteq U_{-n}$. Define U_n as U^{3n} for $n \in \mathbb{N}_0$. Furthermore, define $\phi(x, y)$ as $\inf\{2^n \mid (x, y) \in U_n\}$ and let d be the $pq\infty$-metric for which $d(x, y)$ is defined as the infimum of all $\sum_{k=1}^n \phi(x_{k-1}, x_k)$, where $x_0 = x$ and $x_n = y$. Now one can prove, in the same way as was done in [4] on pages 212 and 213, that $d \leqslant \phi \leqslant 2d$. The first inequality yields that $U_n \subseteq \{d \leqslant 2^n\}$ and thus that d is uniformly continuous, while we obtain $\{d \leqslant 2^n\} \subseteq U_{n+1}$ from the second inequality. Now A is a subset of $B_d(x, 2^n)$ for some $n \in \mathbb{N}$, hence $A \subseteq U^{3n+3}(x)$.

To prove the converse, let d be a uniformly continuous $pq\infty$-metric on $(X,\mathcal{U})$ and define U as $\{d \leqslant 1\}$. Since there is an $n \in \mathbb{N}$ for which $A \in U^n(x)$ we get that $A \in B_d(x, n)$. □

Proposition 44. *If $(X,\mathcal{U})$ is a uniform space, then $(X, \beta_{\mathcal{U}})$ is extended bornological and the associated extended bornology $\mathcal{B}_{\mathcal{U}}$ consists of all $A \subseteq X$ that satisfy:*

$$\forall U \in \mathcal{U}\ \exists n \in \mathbb{N}: \quad A \times A \subseteq U^n.$$

Proof. Say $\{y\} \in \beta_{\mathcal{U}}(x)$ and take $U \in \mathcal{U}$. Because $(X,\mathcal{U})$ is a uniform space there is an n such that $\{y\} \in (U^{-1})^n(x)$. This implies $\{x\} \in U^n(y)$.

A set A is an element of the associated extended bornology iff it is in some $\beta_{\mathcal{U}}(x)$. This is true iff for each symmetric $U \in \mathcal{U}$ there is an n such that $A \subseteq U^n(x)$, which implies $A \times A \in U^{2n}$. On the other hand, if $A \times A \in U^n$, then $A \in U^n(a)$ for an arbitrary $a \in A$. □

Proposition 45. *Let $(X,\mathcal{U})$ be a uniform space. The components of the extended bornological space $(X, \beta_{\mathcal{U}})$ are exactly the uniform components of $(X,\mathcal{U})$.*

Proof. If $\{y\} \notin \beta_{\mathcal{U}}(x)$, then there is a symmetric $U \in \mathcal{U}$ such that $y \notin \bigcup_{n\in\mathbb{N}} U^n(x)$. Denote $\bigcup_{n\in\mathbb{N}} U^n(x)$ as C. Because $U(C) = C$ and $U(X \setminus C) = X \setminus C$ we get that x and y are not in the same component. Conversely, if x and y are not in the same component and C is the component of x, then there is a symmetric $U \in \mathcal{U}$ such that $U(C) = C$. This yields that for all $n \in \mathbb{N}$ holds $U^n(x) \subseteq C$ and thus that $y \notin \beta_{\mathcal{U}}(x)$. □

Corollary 46. *Let $(X,\mathcal{U})$ be a uniform space. $(X, \beta_{\mathcal{U}})$ is bornological iff $(X,\mathcal{U})$ is uniformly connected.*

Proof. This is true because an extended bornological space is bornological iff it consists of only one component. □

Proposition 47. *If we define $P_U(X,\mathcal{U})$ as $(X, \beta_{\mathcal{U}})$, then P_U is a concrete functor from q**Unif** to **pBor**.*

Proof. Let $f : (X,\mathcal{U}) \to (Y,\mathcal{U}')$ be a uniformly continuous map and take B in $\beta_{\mathcal{U}}(x)$. For each $V \in \mathcal{U}'$ holds $(f \times f)^{-1}(V) \in \mathcal{U}$ and thus there is an $n \in \mathbb{N}$ such that $B \subseteq ((f \times f)^{-1}(V))^n(x)$. This yields $f(B) \subseteq V^n(f(x))$ and we obtain that $f(B)$ is an element of $\beta_{\mathcal{U}'}(f(x))$. □

Proposition 48. *P_U preserves direct sums and products.*

Proof. Let $((X_i,\mathcal{U}_i))_{i\in I}$ be a family of quasi-uniform spaces. If B is bounded according to x in the pointwise bornological space associated with $\coprod_{i\in I}(X_i,\mathcal{U}_i)$, then B and x are in the same summand since $\bigcup_{i\in I}(X_i \times X_i)$ is an entourage in the direct sum structure.

The filter of entourages of the product $\prod_{i\in I}(X_i,\mathcal{U}_i)$ is generated by all sets $(\pi_i \times \pi_i)^{-1}(U_i)$, where U_i is in $\mathcal{U}_i$. Suppose that $\pi_i(A)$ is in $\beta_{\mathcal{U}_i}(\pi_i(a_0))$ for all $i \in I$. We need to show that A is bounded according to a_0 in the pointwise bornology associated with the product of the quasi-uniformities $(\mathcal{U}_i)_{i\in I}$. Let U be equal to $\bigcap_{k=0}^n((\pi_{i_k} \times \pi_{i_k})^{-1}(U_k))$ where each U_k is in $\mathcal{U}_{i_k}$. There is an m such that $\pi_{i_k}(A) \subseteq U_k^m(\pi_{i_k}(a_0))$ for all $0 \leqslant k \leqslant n$. Take an $a \in A$. For each $0 \leqslant k \leqslant n$ there is a sequence $x_0^k, x_1^k, \ldots, x_m^k$ such that $x_0^k = \pi_{i_k}(a_0)$, $x_m^k = \pi_{i_k}(a)$ and each couple (x_l^k, x_{l+1}^k) is in U_k. Take a sequence $a_0, a_1, \ldots, a_{m-1}, a_m = a$ such that each $\pi_{i_k}(a_l)$ is equal to x_l^k. We obtain that each pair (a_l, a_{l+1}) is in $\bigcap_{k=0}^n(\pi_{i_k} \times \pi_{i_k})^{-1}(U_k)$ and thus $a \in U^m(a_0)$. Since we can do this for all $a \in A$ we get $A \subseteq U^m(a_0)$. □

Remark 49. The functor P_U does not preserve quotients and subspaces. Define N as the underlying uniform space of the metric space that is the direct sum of a countable number of copies of the interval $[0,1]$, each endowed with the Euclidian metric. Let $\overline{N}$ be the quotient obtained by glueing all copies of 0 together and denote the projection map as π. It is clear that the entire set $\overline{N}$ is bounded according to 0. However, $\overline{N}$ is not a subset of a finite union of copies of $[0,1]$ and therefore π is no quotient in **pBor**.

Let $[0,1]$ be endowed with the uniformity $\mathcal{U}$ obtained from the Euclidian metric and $\{0,1\}$ with the subspace uniformity $\mathcal{U}'$. The singleton $\{1\}$ is no element of $\beta_{\mathcal{U}'}(0)$, but it is bounded according to 0 in $\{0,1\}$ with the subspace structure obtained from $([0,1], \beta_{\mathcal{U}})$.

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