University of Antwerp

FACULTY OF APPLIED ECONOMICS

DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCES

On the pricing of options under limited information

Ann De Schepper and Bart Heijnen
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University of Antwerp

Abstract

In spite of the power of the Black & Scholes option pricing method, there are situations in which the hypothesis of a lognormal model is too restrictive. One possibility to deal with this problem, consists of a weaker hypothesis, fixing only successive moments and eventually the mode of the price process of a risky asset, and not the complete distribution. The consequence of this generalization is the fact that the option price is no longer a unique value, but a range of several possible values. We show how to find upper and lower bounds, resulting in a rather narrow range. We give results in case two moments, three moments, or two moments and the mode of the underlying price process are fixed.

Keywords: Black-Scholes, option pricing, limited information.

JEL codes: G130, C190, C650, E400.

1 Introduction

Since the famous paper of Fisher Black and Myron Scholes [1973], the problem of how to determine prices for basic options is solved to a certain extent. Many publications since then still deal with the same problem, mainly in two directions: firstly, the extension of their model to more complex classes of options, and secondly, alternatives or sophistications of their formula taking into account more information about the price process.

In the present paper, we want to make a contribution to this last category, by showing how to price options without imposing a complete model on the underlying price process – as it is the case for the pricing formula of Black & Scholes. The main reason for our approach is the observation of several authors in the past, that — although the lognormal model, which makes up the foundation of the formula of Black & Scholes, can be a rather good model to describe real price processes — there are some shortcomings of the model that can become important, and that can cause (more or less seriously) biased option prices. Without claiming any exhaustivity, we can refer e.g. to interesting contributions of Teichmoeller [1971], Hull and White [1988], Becker [1991], Bakshi, Cao and Chen [1997], Corrado and Su [1998], Gerber...

The method we present in this paper, only uses successive moments of the underlying price process, and not the distribution itself. Thus, the hypothesis used to reach an option price is much weaker than it is the case for the Black & Scholes formula. As a consequence, one unique value for the option price is not possible, since the knowledge of successive moments only gives limited information about the real process, such that expected payoffs can only be calculated approximately. However, we will show that it is possible to construct close (and in some cases very close) absolute upper and lower bounds to the options prices, which eventually can be combined to result into one approximate price.

The paper is organized as follows. We start in section 2 with a description of the problem and of the methodology. Afterwards, in section 3, we present the results for bounds on European option prices when limited information about the price process is available. Proofs of these results are provided in the appendix. Section 4 is meant to illustrate our results numerically and graphically. Afterwards in section 5, we show how the results can be extended to arithmetic Asian options. Section 6 concludes.

2 Description of the problem

2.1 The pricing of European call options

Consider a European call option on a risky asset with current price $S$, that matures at time $T$ with exercise price $K$. In an arbitrage-free setting, the price of this option can be determined as

\[ B_{EC}(T, K, S) = e^{-rT} \mathbb{E}^Q [(S_T - K)_+] , \]  

where $r$ denotes the risk-free interest rate, and where the stochastic process $\{S_t, t \geq 0\}$, starting in $S_0 = S$, describes the price process of the underlying risky asset. We assume that $Q$ is the unique equivalent probability measure, such that the discounted price process is a martingale, or $\mathbb{E}^Q [e^{-rT}S_T] = S$.

Following Merton, we know that the option price must satisfy

\[ \max(0, S - e^{-rT}K) \leq B_{EC}(T, K, S) \leq S, \]  

whatever the real price process of the underlying asset.

An exact option price can only be computed, if the distribution of the price process $\{S_t, t \geq 0\}$ is known for sure – which in reality is not possible. The most commonly used assumption is the Black & Scholes setting, where the price process is assumed to follow a geometric Brownian motion. This means that

\[ S_t = S \cdot e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}, \]  

2
where \( \{W_t, t \geq 0\} \) is a standard Brownian motion. Thus, under the measure \( Q \), the variables \( S_t/S \) are lognormally distributed with mean \( (r - \frac{1}{2} \sigma^2)t \) and variance \( \sigma^2t \). Under these assumptions, the price of a European call can be found according to the well-known Black & Scholes formula (see Black and Scholes [1973]), or

\[
P^{(BS)}_{EC}(T, K, S) = S \cdot \Phi(d_1) - Ke^{-rT} \cdot \Phi(d_2),
\]

with

\[
d_1 = \frac{1}{\sqrt{\sigma^2 T}} \left( \ln(S/K) + (r + \frac{1}{2} \sigma^2)T \right)
\]

\[
d_2 = \frac{1}{\sqrt{\sigma^2 T}} \left( \ln(S/K) + (r - \frac{1}{2} \sigma^2)T \right),
\]

where \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \) denotes the cumulative distribution function of the standard normal distribution.

As mentioned in the introduction, the Black & Scholes pricing formula shows some imperfections in that the whole distribution of the price process is fixed by means of a lognormal model. Although this model performs well in a lot of cases, there can arise wrong prices due to the strong assumptions, if e.g. the tails of the real underlying price process do not correspond perfectly to the tails of the lognormal distribution.

Therefore, we want to weaken this strong assumption, as it was done also in the literature before by several authors. The first of course was Merton [1973], who formulated an upper and lower bound to option prices that require no information about the real price process. Other contributions to this approach are e.g. from Levy [1985], Lo [1987] and Rodriguez [2003]. A brief overview can be found in the last cited reference.

In this paper we want to formulate new option bounds by making use of successive moments of the real price distribution instead of the complete distribution. This was also partially done in Lo [1987]. As a consequence of this milder hypothesis, we arrive at a range of acceptable prices instead of one price, but the range turns out to be rather tight, and with a significantly greater precision than the bounds of Merton. We explain the method for European call options; however, it can be extended to other types of options, e.g. Asian arithmetic options, as will be shown later on.

2.2 Methodology

Suppose that we are dealing with a risk \( X \) for which we are interested in the expected value \( E[(X - K)_+] \). Furthermore, suppose that the exact distribution of the risk \( X \) is not known, but that we have reliable estimates for the mean and for the variance, and preferably also for the mode or for the skewness. Although an exact calculation of the expected value \( E[(X - K)_+] \) in that case is not possible, we can derive boundary values that restrict the possible outcomes for the expected value, taken into account the information about the moments of the risk \( X \). In fact we are looking for

\[
\sup_{F \in \mathcal{B}} \int_0^{+\infty} (x - K)_+ dF(x) \quad \text{and} \quad \inf_{F \in \mathcal{B}} \int_0^{+\infty} (x - K)_+ dF(x),
\]

\[
(6)
\]
where $\mathcal{B}$ is the class of all distribution functions with domain $\mathbb{R}^+$ and with moments (and mode) as given.

Now, if $F \in \mathcal{B}$, and if $P(x)$ is a polynomial of degree 2 (resp. 3) or less, the value of $\int_0^{+\infty} P(x) dF(x)$ only depends on the first two (resp. three) moments of $F$, and thus it is the same for each distribution $F \in \mathcal{B}$ in case these moments are known. Therefore, the problem of (6) is reduced to find such polynomials $P(x)$ greater or smaller than $(x-K)_+$ on $\mathbb{R}^+$ such that for some distribution $G \in \mathcal{B}$ we have

$$\int_0^{+\infty} P(x) dG(x) = \int_0^{+\infty} (x-K)_+ dG(x).$$

(7)

Note that the method is also valid if $(x-K)_+$ is replaced by a more general function $f : \mathbb{R} \to \mathbb{R}$ which is only assumed to be non-negative. More details about this methodology can be found in Heijnen [1989].

### 3 Bounds for European option prices

Since the price of European call options depends on the expected pay-off

$$\mathbb{E}[(S_T - K)_+],$$

(8)

the methodology described in subsection 2.2 can be applied straightforwardly. This results in absolute bounds that restrict the possible outcomes for the expected pay-off, taken into account the information about the moments of the risk $S_T$. With absolute bounds, we mean that the bounds hold for any distribution with given parameters, and that there exists at least one distribution for which the bounds are actually reached.

In order to be able to write the results in function of the current asset price $S$, we will use information about moments of the relative future price $S_T/S$ instead of the moments of $S_T$. We will use the notation $\mu_r$ for the non-central $r$-th moment, i.e.

$$\mu_r = \mathbb{E}[(S_T/S)^r];$$

(9)

for the mode of $S_T/S$, if it exists, we will use the notation $m$.

Note that, in order to guarantee the existence of a distribution function on $\mathbb{R}^+$ with given mode and moments, these “known” parameters cannot be chosen completely arbitrarily. Indeed, the following essential conditions have to be taken into consideration:

$$
\begin{cases}
\mu_2 \geq \mu_1^2 \\
2\mu_1 \geq m \\
\mu_3 \geq \mu_1^2 \mu_1.
\end{cases}
$$

(10)

Remark that –without loss of generality– we can restrict ourselves to the investigation of call options. Indeed, making use of the put-call parity, results for call option prices can always be transformed into prices for put options.
The proofs of the theorems in 3.1, 3.2 and 3.3 infra are provided in the appendix.

3.1 Two moments are known

If two moments of the stochastic variable \( S_T / S \) are known, the European call option price (1) satisfies the boundary conditions

\[
e^{-rT}G_1(S) \leq B_{EC}(T, K, S) \leq e^{-rT}G_2(S),
\]

with

\[
G_1(S) = \begin{cases} 
0 & \text{if } S \leq \frac{1}{\mu_1}K \\
\mu_1S - K & \text{if } S \geq \frac{1}{\mu_1}K 
\end{cases}
\]

and

\[
G_2(S) = \begin{cases} 
\frac{1}{2}(\mu_1S - K) + \frac{1}{2}\sqrt{S^2(\mu_2 - \mu_1^2) + (\mu_1S - K)^2} & \text{if } S \leq \frac{2\mu_1}{\mu_2}K \\
\mu_1S - \frac{\mu_1^2}{\mu_2}K & \text{if } S \geq \frac{2\mu_1}{\mu_2}K.
\end{cases}
\]

3.2 Three moments are known

Define the function \( p : \mathbb{R} \to \mathbb{R} : x \mapsto p(x) \) by

\[
p(x) = (\mu_2 - \mu_1^2)x^2 + (\mu_1\mu_2 - \mu_3)x + (\mu_1\mu_3 - \mu_2^2)
\]

and denote its zeros (which always exist and which are necessarily real and positive) by \( v \) and \( w \), with \( v < w \).

Furthermore, consider the equation

\[
x^3 + Ax^2 + Bx + C = 0
\]

with coefficients

\[
\begin{align*}
A &= -\frac{2\mu_2S + 3\mu_1K}{2\mu_1S} \\
B &= \frac{2\mu_2K}{\mu_1S} \\
C &= -\frac{\mu_2K}{2\mu_1S}
\end{align*}
\]

and denote \( q \) for the unique root of this equation in the interval \([v, +\infty[\).

(1) For the correctness of the formulas, the exercise price \( K \) is assumed to be non-negative.
If three moments of the stochastic variable \( S_T/S \) are known, the European call option price (1) satisfies the boundary conditions

\[
e^{-rT}G_3(S) \leq B_{EC}(T, K, S) \leq e^{-rT}G_4(S),
\]

with

\[
G_3(S) =
\begin{cases}
0 & \text{if } S \leq \frac{\mu_1}{\mu_2}K \\
\mu_1S - K + \frac{-p(K)S^2}{\mu_3S - \mu_2K} & \text{if } \frac{\mu_1}{\mu_2}K \leq S \leq \frac{1}{v}K \\
\mu_1S - K & \text{if } S \geq \frac{1}{v}K
\end{cases}
\]

and

\[
G_4(S) =
\begin{cases}
\frac{\mu_3 \mu_1 - \mu_2^2}{\mu_3 - 2w \mu_2 + w^2 \mu_1} \frac{qS - K}{q} & \text{if } S \leq \frac{3w-v}{2w^2}K \\
\frac{\mu_3 \mu_1 - \mu_2^2}{\mu_3 - 2w \mu_2 + w^2 \mu_1} \frac{wS - K}{w} & \text{if } \frac{3w-v}{2w^2}K \leq S \leq \frac{2}{v+w}K \\
\frac{1}{2}(\mu_1S - K) + \frac{1}{2} \sqrt{S^2(\mu_2 - \mu_1^2) + (\mu_1S - K)^2} & \text{if } \frac{2}{v+w}K \leq S \leq \frac{2\mu_1}{\mu_2}K \\
\mu_1S - \frac{\mu_3}{\mu_2}K & \text{if } S \geq \frac{2\mu_1}{\mu_2}K.
\end{cases}
\]

### 3.3 Two moments and mode are known

The mode can only taken into account, if it satisfies the condition

\[
m \leq \mu_1.
\]

Since we work with right-tailed non-negative distributions, this requirement does not impose any restrictions.

Due to the introduction of the Khinchine transform (see subsection A.3), the results are constructed by means of two transformed moments

\[
\begin{align*}
\nu_1 &= 2\mu_1 - m \\
\nu_2 &= 3\mu_2 - 2m\mu_1.
\end{align*}
\]
The natural condition \( \nu_2 \geq \nu_1^2 \) implies that the mode also has to satisfy the constraints
\[
\mu_1 - \sqrt{3} \sqrt{\mu_2 - \mu_1^2} \leq m \leq \mu_1 + \sqrt{3} \sqrt{\mu_2 - \mu_1^2}.
\]
(23)
Consider the equation
\[
x^3 + Dx^2 + Ex + F = 0
\]
with coefficients
\[
\begin{align*}
D &= -3K \\
E &= (4\nu_1 + 2m)SK - (2m\nu_1 + \nu_2)S^2 \\
F &= 2m\nu_2S^3 - (2m\nu_1 + \nu_2)S^2K.
\end{align*}
\]
(25)

**Case 1.** \( S \leq \frac{1}{m}K \)

If two moments and the mode of the stochastic variable \( S_T / S \) are known, the European call option price (1) satisfies the boundary conditions
\[
e^{-rT}G_5^a(S) \leq B_{EC}(T, K, S) \leq e^{-rT}G_6^a(S),
\]
(26)
with
\[
G_5^a(S) = \begin{cases} 
0 & \text{if } S \leq \frac{1}{\nu_1}K \\
\mu_1S - K + \frac{(K - mS)^2}{2S(\nu_1 - m)} & \text{if } S \geq \frac{1}{\nu_1}K
\end{cases}
\]
(27)
and
\[
G_6^a(S) = \begin{cases} 
\frac{S^2(\nu_2 - \nu_1^2)(y - K)^2}{2(S^2(\nu_2 - \nu_1^2) + (y - \nu_1S)^2)(y - mS)} & \text{if } S \leq \frac{\nu_1(3\nu_2 - 2m\nu_1)}{\nu_2^2}K \\
\frac{\nu_1(\nu_2S - \nu_1K)^2}{2S\nu_2(\nu_2 - m\nu_1)} & \text{if } S \geq \frac{\nu_1(3\nu_2 - 2m\nu_1)}{\nu_2^2}K
\end{cases}
\]
(28)
where \( y \) is the unique root in the interval \([\max(K, \frac{\nu_2}{\mu_1}S), +\infty]\) of equation (24).

**Case 2.** \( S \geq \frac{1}{m}K \)

If two moments and the mode of the stochastic variable \( S_T / S \) are known, the call option price (1) satisfies the boundary conditions
\[
e^{-rT}G_5^b(S) \leq B_{EC}(T, K, S) \leq e^{-rT}G_6^b(S),
\]
(29)
with
\[
G_5^b(S) = \mu_1S - K
\]
(30)
and

\[
G^b_6(S) = \begin{cases} 
\mu_1 S - K + \frac{S^2(\nu_2 - \nu_1^2)(z - K)^2}{2(S^2(\nu_2 - \nu_1^2) + (z - \nu_1 S)^2)(mS - z)} & \text{if } S \leq \frac{2m\nu_1 + \nu_2}{2m\nu_2} K \\
\mu_1 S - K + \frac{K^2 \nu_2 - \nu_1^2}{2mS} & \text{if } S \geq \frac{2m\nu_1 + \nu_2}{2m\nu_2} K,
\end{cases}
\]

where \( z \) is the unique root in the interval \([0, K]\) of equation 24.

4 Numerical and graphical illustrations

In the following subsections, we will compare our pricing bounds with the bounds of Merton and with the Black & Scholes formula.

Note that for a lognormally distributed price process, the moments and mode can be calculated as

\[
\begin{align*}
\mu_1^* &= E[S_T / S] = e^{r T} \\
\mu_2^* &= E[(S_T / S)^2] = e^{2r T + \sigma^2 T} \\
\mu_3^* &= E[(S_T / S)^3] = e^{3r T + 3\sigma^2 T} \\
m^* &= e^{r T - \frac{\sigma^2 T}{2}}.
\end{align*}
\]

For the numerical examples, we will use parameter values \( r = 0.1, \sigma = 0.2, T = 1 \), which are the same as in Rodriguez [2003]. For these choices, the moments can be calculated as

\[
\begin{align*}
\mu_1^* &= 1.10517 \\
\mu_2^* &= 1.27125 \\
\mu_3^* &= 1.52196 \\
m^* &= 1.06184.
\end{align*}
\]

For the exercise price, we will use a value \( K = $50 \).

4.1 Two moments are known

We start by comparing the Black & Scholes price (4) and Merton’s bounds to the absolute bounds of subsection 3.1, based on the knowledge of the two first moments of \( S_T / S \). In order to be able to make this comparison, for the numerical illustrations
we choose the moments as if the price process were lognormally distributed, with
parameters as in (33).

Figure 1 shows the Black & Scholes price $B^{B&S}_{EC}(T, K, S)$ (black line) together with
Merton’s range (pale grey) and our tolerable range $[e^{-rT}G_1(S), e^{-rT}G_2(S)]$ (dark
grey), in function of the current asset price $S$. The functions $G_1$ and $G_2$ are calculated
as in equations (12) and (13), with moments $\mu_1^*$ and $\mu_2^*$ as in (33). It is obvious that
the Black & Scholes price completely fits in with the area that corresponds to the first
two moments for the price process. For all values of $S$, the option price nicely follows
the upper bound $e^{-rT}G_2(S)$. However, for options that are deep in-the-money or deep
out-of-the-money, the range becomes rather narrow, and the Black & Scholes price
seems to be rather low, in that it almost equals the minimal tolerable value. This is
important: indeed, it means that the risk that the Black & Scholes formula overprices
the real price for options that are deep in-the-money or deep out-of-the-money, is very
small, although it is mentioned in the literature, e.g. in Hull [2003]. An underpricing
is far more obvious.

This last consideration can also be seen very clearly, if we compare the Black & Scholes
price with an interpolating pricing formula based on our upper and lower bound, or

$$B^{(\alpha)}_{EC}(T, K, S) = e^{-rT} [G_1(S) + \alpha (G_2(S) - G_1(S))],$$

with $0 \leq \alpha \leq 1$, and with $G_1$ and $G_2$ as defined in equations (12) and (13).

Figure 2 shows the evolution of the value of $\alpha$ for which the price (34) is identical to
the Black & Scholes price (4), in function of the current asset price. For options that
are deep in-the-money, or deep out-of-the-money, the value of $\alpha$ is very low, and the
Black & Scholes price is just above the minimal possible value. The maximal value of
$\alpha$ is reached for a current asset price equal to the present value of the exercise price,
or $S = e^{-rT} K$. 

Figure 1: Price of a European call in function of the current asset price: Black & Scholes price
versus the tolerable area if two moments are known.
Note that an interpolating formula as presented in equation (34) could be useful to price the option if the range of possible values is sufficiently small (see e.g. subsection 4.2). Of course, in that case we are confronted with the difficulty of choosing a plausible value for $\alpha$ in order to end with an acceptable option price.

It is important to remark that for any two given moments $\mu_1$ and $\mu_2$, a lognormal distribution can be fitted. As a consequence, the Black & Scholes price will definitely fall in the tolerable area. However, if the third moment or the mode of the price at maturity time is known, this will not necessarily be compatible with a lognormal distribution for the price process. In that case, the Black & Scholes price can be seriously biased and the possibility even exists of a Black & Scholes price falling outside the tolerable range corresponding to the known moments. We will return to this consideration in the next two subsections.

### 4.2 Three moments are known

As in the previous subsection, we start by comparing the Black & Scholes price (4) to the absolute bounds of subsection 3.2, based on the knowledge of now the first three moments, where the moments are chosen as if the price process were lognormally distributed, with parameters as before, or $r = 0.1$, $\sigma = 0.2$, $T = 1$, $K = $ $50$.

Figure 3 shows the Black & Scholes price $B_{EC}^{BS}(T, K, S)$ (black line), Merton’s range (pale grey), the tolerable area for two moments $[e^{-rTG_1(S)}, e^{-rTG_2(S)}]$ (medium grey), and the tolerable area for three moments $[e^{-rTG_3(S)}, e^{-rTG_4(S)}]$ (dark grey) in function of the current asset price $S$. The functions $G_3$ and $G_4$ are calculated as in equations (18) and (19), with moments $\mu^*_1$, $\mu^*_2$ and $\mu^*_3$ as in (33). Again, in case the third moment is chosen as if the real price process follows a lognormal model, the Black & Scholes price obviously fits in with the area. In figure 4, we depict the deviation of each pricing bound from the Black & Scholes price. This deviation is represented by the vertical distance between the curves and the horizontal axis.
Figure 3: Price of a European call in function of the current asset price: Black & Scholes price versus the tolerable area if two (medium grey) and three (dark grey) moments are known.

Figure 4: Deviation of lower and upper bounds from Black & Scholes price for a European call in function of the current asset price if two (solid line) and three (dotted line) moments are fixed.

dotted line corresponds with the knowledge of three moments, the solid line with two moments.

Remark that in the case of a third moment taken from the lognormal model, the knowledge of the third moment does not reduce fundamentally the tolerable range; the effect of an extra moment then is rather limited.

The previous figures and remarks were made under the assumption that the lognormal model is correct: indeed, we compared the Black & Scholes price with the absolute bounds calculated under the hypothesis that the first three moments for the real price process coincide with the first three moments of the lognormal model. However, if the lognormal model is invalid, the pictures change completely, especially when the “real” third moment is smaller than $\mu_3^*$, i.e. when the price process exhibits a left tail that is much heavier than the right tail.

Figure 5 shows the Black & Scholes price and the tolerable area (for two and three
moments) for two choices of the third moment, different from the third moment $\mu_3^*$ of (33)(2). In the left graph, where $\mu_3 = 1.49$, the Black & Scholes price is situated at the borderlines of the tolerable area: for options that are deep in-the-money, the Black & Scholes price is just too low, while for options that are deep out-of-the-money, the Black & Scholes price is just too high. The right graph shows the situation for $\mu_3 = 1.47$: the assumption of a lognormal distribution as in the Black & Scholes pricing formula now is completely incompatible with the assumption about the skewness, and the Black & Scholes price falls outside the tolerable range for almost all values of $S$. In figure 6, as before we depict the deviation of each pricing bound from the Black & Scholes price for the same choices of the third moment (the dotted line corresponds with the knowledge of three moments, the solid line with two moments). If the third moment of the real price process is equal to $\mu_3 = 1.49$ as in the left graph, the Black & Scholes price is not compatible with reality for a current asset price between $\$36.39$ and $\$44.19$ or above $\$55.42$. If the third moment of the real price process is equal

Note that the minimal possible value for the third moment follows from the formula $\mu_3 \geq \frac{\mu_2^2}{\mu_1}$ (see (10)); for the present parameter choices, this minimal value is 1.46228.
to $\mu_3 = 1.47$ as in the right graph, the Black & Scholes price is not compatible with reality for a current asset price between $29.61$ and $48.96$ or above $51.46$.

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
$\mu_3$ & $S$ & new & new & Black & Scholes & lower bound \& upper bound \\
& & lower bound & upper bound & price & Rodriguez & Rodriguez \\
\hline
1.47 & 30 & 0 & 0.0480 & 0.0538 & 0 & 0.1806 \\
& 40 & 0 & 0.4432 & 1.3950 & 0.7965 & 2.7767 \\
1.49 & 30 & 0 & 0.1686 & 0.538 & 0 & 0.1806 \\
& 40 & 0 & 1.2302 & 1.950 & 0.7965 & 2.7767 \\
& 60 & 15.4681 & 16.9279 & 15.1292 & 14.9845 & 19.1693 \\
1.51 & 30 & 0 & 0.2840 & 0.0538 & 0 & 0.1806 \\
& 40 & 0 & 1.7926 & 1.3950 & 0.7965 & 2.7767 \\
& (-0.460529) & 50 & 5.2266 & 7.9618 & 6.6348 & 6.1724 & 9.8149 \\
1.53 & 30 & 0 & 0.3949 & 0.0538 & 0 & 0.1806 \\
& 40 & 0 & 2.1950 & 1.2950 & 0.7965 & 2.7767 \\
\hline
\end{tabular}
\caption{Results for the prices of a European call option for different choices of the third moment, with $\mu_1^* = 1.10517$ and $\mu_2^* = 1.27125$ as in (33), corresponding with a lognormal distribution with $r = 0.1$ and $\sigma = 0.2$. Note that for the lognormal distribution based on these parameters and with these two moments, the third moment can be calculated as $\mu_3^* = 1.52196$, as in (33), with skewness equal to 0.6136. The other parameters are $T = 1$ and $K = $ 50 as before.}
\end{table}

In Table 1 we compare our absolute bounds for option prices with the results of the Black & Scholes formula and with the bounds in Rodriguez [2003, p.158], for different choices of the third moment. In this respect, it is important to recall that for the bounds of Rodriguez, just as it is the case for the Black & Scholes prices, the asset price is assumed to follow a lognormal distribution. This also means that the third moment is not used for the computation of the values in these columns, which explains the recurring results in the last three columns. Black & Scholes prices that are incompatible with the absolute bounds, are underlined. These values occur when the
skewness\(^{(3)}\) is negative and in absolute value larger than the skewness for the lognormal distribution: this means that the left tail of the real distribution is heavier than it is the case for the lognormal model. Note that our upper bound is more accurate in almost all cases; for the lower bound, our method gives tighter results for higher values of the current asset price in the case of a negative and rather high skewness just as for the Black & Scholes price. The parameter values are the same as before, or \(r = 0.1, \sigma = 0.2, T = 1, K = \$\ 50\).

### 4.3 Two moments and modus are known

We now compare the Black & Scholes price (4) to the absolute bounds of subsection 3.3, based on the knowledge of the first two moments and the mode, where the moments and mode are chosen as if the price process were lognormally distributed, with parameters as before, or \(r = 0.1, \sigma = 0.2, T = 1, K = \$\ 50\).

![Figure 7: Price of a European call in function of the current asset price: Black & Scholes price versus the tolerable area if two moments (medium grey) and the mode (dark grey) are known.](image)

Figure 7 shows the Black & Scholes price \(B_{EC}^{BS}(T,K,S)\) (black line), Merton’s range (pale grey), the tolerable area for two moments \([e^{-rT}G_1(S), e^{-rT}G_2(S)]\) (medium grey), and the tolerable area for two moments and mode \([e^{-rT}G_5(S), e^{-rT}G_6(S)]\) (dark grey) in function of the current asset price \(S\), for parameter choices as before. The functions \(G_5\) and \(G_6\) are calculated as in equations (27), (28), (30) and (31), with moments \(\mu_1^*, \mu_2^*\) and mode \(m^*\) as in (33). One can see that –if the first two moments and the mode of the lognormal model correspond to the values of the “real” price process– the Black & Scholes price seems to be rather high in the neighbourhood of the accrued value or for options that are almost at-the-money, and rather low for options that are deep in-the-money or deep out-of-the-money. In figure 8, we depict

\(^{(3)}\)The skewness in the table is calculated by means of the Fisher formula: skewness = \(\frac{E[(X-\mu)^3]}{E[(X-\mu)^2]^{3/2}}\) or with our notation for the non-central moments: skewness = \(\frac{\mu_3-3\mu_1\mu_2+2\mu_1^3}{(\mu_2-\mu_1)^{3/2}}\).
Figure 8: Deviation of lower and upper bounds from Black & Scholes price for a European call in function of the current asset price if two moments (solid line) and the mode (dotted line) are fixed.

the deviation of each pricing bound from the Black & Scholes price (the dotted line corresponds with the knowledge of two moments and mode, the solid line with two moments).

Figure 9: Price of a European call in function of the current asset price: Black & Scholes price versus the tolerable area if the mode differs from the lognormal value.

Figure 9 shows the Black & Scholes price and the tolerable range (for two moments with and without given mode) for a choice of the mode 0.87, which is different from the mode \( m^* \) of (33)(4). As it was the case in subsection 4.2, the tolerable range becomes rather narrow, and the Black & Scholes price is situated near the borderlines of the range. In figure 10, again we depict the deviation of each pricing bound from the Black & Scholes price, for the same choice for the mode (the dotted line corresponds with

\[(4)\text{Note that the maximal possible value for the mode follows from the condition } m \leq \mu_1 \text{ (see section 3); for the present parameter choices, this maximal value is 1.10517.}\]
Figure 10: Deviation of lower and upper bounds from Black & Scholes price for a European call in function of the current asset price if two moments (solid line) and the mode (dashed line) are fixed, with a mode lower than the lognormal mode.

the knowledge of two moments and mode, the solid line with two moments). Note that the deviation is much less than for the case where the mode of the lognormal model is equal to the real mode of the price process.

5 Extension to arithmetic Asian call options

Due to the fact that our method only requires the knowledge of the first two or three moments of the quantity under investigation, it can be used for a broad range of option types. As an example, we illustrate our results for arithmetic Asian options, for which the pricing is far from evident.

The price of an arithmetic Asian call option (European style) that matures at time $T$ with exercise price $K$ and with $n$ averaging dates, is given by

$$B_{AC}(T, S, K, n) = e^{-rT} \mathbb{E}_Q \left[ \left( \frac{1}{n} \sum_{i=0}^{n-1} S_{T-i} - K \right)^+ \right],$$

where as before $r$ denotes the risk-free interest rate, $Q$ is the unique equivalent probability measure, and the stochastic process $\{S_t, t \geq 0\}$, starting in $S_0 = S$, describes the price process of the underlying risky asset.

Due to the appearance of a sum of dependent variables, an explicit expression for the price of such a call is even not known in a Black & Scholes setting, in that a sum of (dependent) lognormal variables is no longer lognormal. Although a new lognormal distribution can be a good approximation for the distribution of such a sum (see e.g. Levy [1992]) and although this approximation has been used for pricing purposes (see e.g. Levy [1992] and Jacques [1996]), it is shown in Vyncke, Goovaerts and Dhaene [2003] that for options that are out-of-the money, this approach causes
structural underpricing. An approximation by means of Inverse Gaussian distributions (see Jacques [1996] and Milevsky and Posner [1998]) shows exactly the same kind of problems, especially for options that are deep in or out-of-the-money.

Applying the methodology described in subsection 2.2, also bounds for arithmetic Asian options can be calculated, when estimates are known for the first two or three moments. As before, we compare our results with results that make use of the Black & Scholes assumptions. The graphs show the bounds in function of the initial stock price \( S \) for an exercise price \( K = 100 \), time to maturity \( T = 120 \text{ days} \), 30 averaging dates, and parameters \( r = \ln(1.09)/365 \) and \( \sigma = 0.2/\sqrt{365} \). These values are the same as in Jacques [1996] and Vyncke, Goovaerts and Dhaene [2003].

We suggest to calculate the bounds of the option price, starting directly from the moments for the sum \( A_T = \sum_{i=0}^{n-1} S_{T-i} \). If the price process is modelled by a lognormal process, the moments for this sum \( A_T \) can be calculated as

\[
\begin{align*}
\mu_{1}^{**} &= E[A_T/S] = \frac{1}{n} \sum_{i=0}^{n-1} e^{r(T-i)} \\
\mu_{2}^{**} &= E[(A_T/S)^2] = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} e^{r(T-i+T-j)} e^{\sigma^2 \min(T-i,T-j)} \\
\mu_{3}^{**} &= E[(A_T/S)^3] = \frac{1}{n^3} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} e^{r(T-i+T-j+T-k)} e^{2\sigma^2 \min(T-i,T-j,T-k)} e^{\sigma^2 \text{mid}(T-i,T-j,T-k)},
\end{align*}
\]

(36)

where \( \text{mid}(a, b, c) = a + b + c - \min(a, b, c) - \max(a, b, c) \).

For the choices of the parameters as suggested, they are equal to

\[
\begin{align*}
\mu_{1}^{**} &= 1.02522 \\
\mu_{2}^{**} &= 1.06273 \\
\mu_{3}^{**} &= 1.11381.
\end{align*}
\]

(37)

The following graphs compare the tolerable range for the exact price of the arithmetic Asian call option with moments as given above, with estimates of the option price if the underlying price process is assumed to be lognormally distributed. Following the idea of Levy (see Levy [1992]), an estimate of the option price can be derived by fitting a new lognormal distribution to the arithmetic averaging process \( A_T \). For the parameters \( (r, \sigma, T \text{ and } n) \) as before, this new lognormal distribution has parameters \( \tilde{r} = \ln(1.07872)/365 \) and \( \tilde{\sigma} = 0.183054/\sqrt{365} \).

Figure 11 refers to the knowledge of two moments of the sum process \( A_T \), while in figure 12 the relation between the estimated Black & Scholes price and the tolerable area is shown in case three moments are known.

Both figures were made under the assumption that the lognormal model is correct, and we compared the estimated Black & Scholes price with the absolute bounds, calculated
with the first three moments of the fitted lognormal model. However, if the lognormal model is invalid, again the pictures can be very different.

Indeed, figure 13 shows the situation when the third moment differs from the third moment $\mu_3^{**}$ of (37). In the graph, we drew up the area for $\mu_3 = 1.105$; it is clear that now the Black & Scholes price is not acceptable for all values of $S$. Even for options that are almost at-the-money, the estimated Black & Scholes price is not correct.

6 Conclusion

In this paper we showed how close bounds on option prices can be derived if only limited information about the underlying price process is available. The method only uses values for successive moments of the price of the underlying asset at maturity.
time, and not the complete distribution. We showed that in many cases the Black & Scholes option pricing formula performs very well, but that there exist situations (and not only very particular ones) where this famous formula results into prices which are not compatible with specific characteristics of the underlying asset. After an application of our approach to European call options, we showed how to extend the method e.g. to arithmetic Asian call options.

Appendix : Proofs of the results

Call option prices are determined by means of the calculation of expected values of the form $E[(S_T - K)_+]$. As mentioned in subsection 2.2, we want to construct close bounds for such expected values if information is available for the moments of the variable under investigation. For $G$, we will use one, two or three point distributions of the set $B$, while for the polynomial $P(x)$ we will choose that polynomial that matches $(x - K)_+$ in the mass-points of $G$ such that (7) is satisfied.

Following the method, this variable here is $S_T$. However, in order to be able to present the results in function of the (moving) current asset price $S$, which is much more interesting than a result in function of the (fixed) exercise price, we do not use the moments of $S_T$ but of $S_T/S$. Since the results in this paper are based on a method as introduced and developed in the earlier papers Jansen, Haezendonck and Goovaerts [1986] and Heijnen [1989], for each of the proofs, we will first derive the bounds in function of the exercise price, if two or more moments for the price $S_T$ are given. Afterwards, we will transform the results into bounds in function of the current asset price, making use of successive moments of the ratio $S_T/S$. Adding the actualisation factor $e^{-rT}$ completes the proofs.
Since we work with two types of moments, two different notations will be needed. When talking about the ratio $S_T/S$, which is the case in the main part of the paper, we use the notations $\mu_k$ for the moments and $m$ for the mode, or

$$\mu_k = \mathbb{E}[(S_T/S)^k] \tag{A.1}$$

as mentioned in section 3. When working with the original price process, which is the case here in the appendix for the construction of the proofs, we will use the notations $\tilde{\mu}_k$ for the moments and $\tilde{m}$ for the mode, or

$$\tilde{\mu}_k = \mathbb{E}[(S_T)^k]. \tag{A.2}$$

The same principle is used for the transformed moments in case the mode is given, or

$$\nu_1 = 2\mu_1 - \mu, \quad \nu_2 = 3\mu_2 - 2\mu\mu_1 \tag{A.3}$$

and

$$\tilde{\nu}_1 = 2\tilde{\mu}_1 - \tilde{\mu}, \quad \tilde{\nu}_2 = 3\tilde{\mu}_2 - 2\tilde{\mu}\tilde{\mu}_1. \tag{A.4}$$

**A.1 Proof for subsection 3.1**

1. We start by proving that, if two moments $\tilde{\mu}_1$ and $\tilde{\mu}_2$ of the stochastic variable $S_T$ are known, the expected value $\mathbb{E}[(S_T - K)_+]$ satisfies the boundary conditions

$$F_1(K) \leq \mathbb{E}[(S_T - K)_+] \leq F_2(K), \tag{A.5}$$

with

$$F_1(K) = \begin{cases} \tilde{\mu}_1 - K & \text{if } K \leq \tilde{\mu}_1 \\ 0 & \text{if } K \geq \tilde{\mu}_1 \end{cases} \tag{A.6}$$

and

$$F_2(K) = \begin{cases} \frac{(\tilde{\mu}_1 - K) + \frac{\tilde{\mu}_2 - \tilde{\mu}_1^2}{\tilde{\mu}_2} K + \frac{1}{2}(\tilde{\mu}_2 - \tilde{\mu}_1^2)}{2} \tilde{\mu}_1 & \text{if } K \leq \frac{\tilde{\mu}_2}{2\tilde{\mu}_1} \\
\frac{1}{2}(\tilde{\mu}_1 - K) + \frac{1}{2}(\tilde{\mu}_2 - \tilde{\mu}_1^2) + \frac{1}{2}\sqrt{(\tilde{\mu}_2 - \tilde{\mu}_1^2)^2 + (\tilde{\mu}_1 - K)^2}} \end{cases} \tag{A.7}$$

For the upper bounds, the proof can be found in Heijnen [1989]. One just has to replace ‘$d$’ by ‘$K$’ in the formula of the paper, and then taking the limit for $b$ going to infinity (range $[0, +\infty[$ instead of $[0, b]$) results in (A.7).

The proof for the lower bounds (A.6) is obvious since in case $K \leq \tilde{\mu}_1$, $P(x) = x - K$ in (7), and in case $K \geq \tilde{\mu}_1$, the X-axis coincides with (the limiting case for $b$ going to infinity of) $P(x)$.

2. Making use of the relations $\tilde{\mu}_1 = S \cdot \mu_1$ and $\tilde{\mu}_2 = S^2 \cdot \mu_2$, the result can easily be transformed into the bounds of (12) and (13).
A.2 Proof for subsection 3.2

1. We start by proving that, if three moments $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$ of the stochastic variable $S_T$ are known, the expected value $E[(S_T - K)_+]$ satisfies the boundary conditions

$$F_3(K) \leq E[(S_T - K)_+] \leq F_4(K),$$

(A.8)

with

$$F_3(K) = \begin{cases} 
\tilde{\mu}_1 - K & \text{if } K \leq c \\
(\tilde{\mu}_1 - K) + \frac{-\tilde{p}(K)}{\tilde{\mu}_3 - K\tilde{\mu}_2} & \text{if } c \leq K \leq \frac{\tilde{\mu}_2}{\tilde{\mu}_1} \\
0 & \text{if } K \geq \frac{\tilde{\mu}_2}{\tilde{\mu}_1}
\end{cases}$$

(A.9)

and

$$F_4(K) = \begin{cases} 
(\tilde{\mu}_1 - K) + \frac{\tilde{\mu}_2 - \tilde{\mu}_1^2}{\tilde{\mu}_2} K & \text{if } K \leq \frac{\tilde{\mu}_2}{2\tilde{\mu}_1} \\
(\tilde{\mu}_1 - K) + \frac{\sqrt{(\tilde{\mu}_2 - \tilde{\mu}_1^2) + (\tilde{\mu}_1 - K)^2} - (\tilde{\mu}_1 - K)}{2} & \text{if } \frac{\tilde{\mu}_2}{2\tilde{\mu}_1} \leq K \leq \frac{c + c'}{2} \\
\frac{\tilde{\mu}_3\tilde{\mu}_1 - \tilde{\mu}_2^2}{\tilde{\mu}_3 - 2c'\tilde{\mu}_2 + c'^2\tilde{\mu}_1} + \frac{c' - K}{c'} & \text{if } \frac{c + c'}{2} \leq K \leq \frac{2c'^2}{3c' - c} \\
\frac{\tilde{\mu}_3\tilde{\mu}_1 - \tilde{\mu}_2^2}{\tilde{\mu}_3 - 2s\tilde{\mu}_2 + s^2\tilde{\mu}_1} \frac{s - K}{s} & \text{if } K \geq \frac{2c'^2}{3c' - c}
\end{cases}$$

(A.10)

where $c$ and $c'$ are the zeros (which always exist and which are necessarily real and positive) of

$$\tilde{p}(x) = (\tilde{\mu}_2 - \tilde{\mu}_1^2)x^2 + (\tilde{\mu}_1\tilde{\mu}_2 - \tilde{\mu}_3)x + (\tilde{\mu}_1\tilde{\mu}_3 - \tilde{\mu}_2^2)$$

(A.11)

with $c < c'$, and where $s$ is the unique root in the interval $[c', +\infty[$ of the equation

$$2\tilde{\mu}_1 x^3 - (2\tilde{\mu}_2 + 3K\tilde{\mu}_1) x^2 + 4K\tilde{\mu}_2 x - K\tilde{\mu}_3 = 0.$$

(A.12)

For the upper bounds, the proof can be found in Jansen, Haezendonck and Goovaerts [1986]. One just has to replace ‘$t$’ by ‘$K$’ and ‘$a$’ by ‘$0$’ in the formula of the paper, and then taking the limit for $b$ going to infinity results in (A.10).
For the lower bounds, lemma 2 of Jansen, Haezendonck and Goovaerts [1986] can be used to generate two-point and three-point distributions with the given moments $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$. In fact the two-point distribution with mass $(\tilde{\mu}_1 - c')/(c' - s)$ in $c$ and mass $(\tilde{\mu}_1 - c)/c' - c$ in $c'\]$ is the unique two-point distribution on $[0, +\infty]$ with moments $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$ (c and c' the zeros of (A.11)). Therefore, one has to conclude that in the case of $c \geq K$, the best lower bound is $\tilde{\mu}_1 - K$ (take $P(x) = x - K$ in (7)), and in the case of $c' \leq K$, the best lower bound is 0 (take $P(x) = 0$ in (7)).

More calculations only have to be done for $c < K < c'$. We define two sub-cases: $c < K \leq 0' < c'$ and $c < 0' < K < c'$ (with $0' = \tilde{\mu}_2/\tilde{\mu}_1$).

(i) $c < K \leq 0' < c'$

Using the definitions of Jansen, Haezendonck and Goovaerts [1986], we look for the point $s$ for which $u(0, s) = K$\(^{(5)}\). Because $c < K \leq 0'$, we are sure that $s = (\tilde{\mu}_3 - K\tilde{\mu}_2)/(\tilde{\mu}_2 - K\tilde{\mu}_1) > c'$. The polynomial $P(x)$ that is needed in (7) equals:

$$P(x) = \frac{x}{s^2} \left[ s(s - K) + K(x - s) - (x - s)^2 \right].$$ \hspace{1cm} (A.13)

This polynomial passes through the points $(0, 0)$, $(K, 0)$ and $(s, s - K)$, is tangent to $(x - K)_+$ in $s$ and remains lower than or equal to $(x - K)_+$ on $[0, +\infty]$. Therefore the best lower bound in this case is $q_s(s - k)$, with $q_s$ the mass in $s$ as defined in lemma 2 of Jansen, Haezendonck and Goovaerts [1986]. After some calculations one easily finds the lower bound of (18).

(ii) $c < 0' < K < c'$

Now we first have to introduce an upper bound $b$ in the range of the distribution functions. Later on, we will take the limit for $b$ going to infinity.

Using the definitions of Jansen, Haezendonck and Goovaerts [1986] we look for the point $r$ for which $u(r, b) = K$. Because $0' < K < c'$, we are sure that $r = (\tilde{\mu}_3 - (b + K)\tilde{\mu}_2 + bK\tilde{\mu}_1)/(\tilde{\mu}_2 - (b + K)\tilde{\mu}_1 + bK) < c$. The polynomial $P(x)$ that is used in (7) equals:

$$P(x) = \frac{(x - r)^2(x - K)}{(b - r)^2}.$$ \hspace{1cm} (A.14)

This polynomial passes through the points $(r, 0)$, $(K, 0)$ and $(b, b - K)$, is tangent to $(x - K)_+$ in $r$ and remains lower than or equal to $(x - K)_+$ on $[0, b]$. Therefore the best lower bound in this case is $q_b(b - K)$, with $q_b$ the mass in $b$ as defined in lemma 2 of Jansen, Haezendonck and Goovaerts [1986]. After some calculations and after taking

\(^{(5)}\)In Jansen, Haezendonck and Goovaerts [1986] the function $u(r, s)$ is used to generate three-point distributions with the given moments $\tilde{\mu}_1$, $\tilde{\mu}_2$ and $\tilde{\mu}_3$. If $r < c$ and $s > c'$ than there exists such a distribution with spectrum $\{r, u, s\}$, and the masses $q_r$, $q_u$ and $q_s$ can be expressed analytically.
the limit for $b$ going to infinity, this best lower bound becomes zero, which completes the proof of (A.9).

2. Making use of the relations $\tilde{\mu}_1 = S \cdot \mu_1$, $\tilde{\mu}_2 = S^2 \cdot \mu_2$ and $\tilde{\mu}_3 = S^3 \cdot \mu_3$, the results of (A.9) and (A.10) can be transformed into the bounds of (18) and (19). The relation between the zeros $v$ and $w$ of $p(x)$ of (14) and $c$ and $c'$ of $\tilde{p}(x)$ of (A.11) is that $v = S \cdot c$ and $w = S \cdot c'$. The same reasoning holds for the root $q$ of equation (15) and the root $s$ of equation (A.12): $q = S \cdot s$.

A.3 Proof for subsection 3.3

1. If next to the first two moments, also the mode of the underlying variable is known, use can be made of a Khinchine transform, as explained in the following lemma.

Lemma A.1. If a non-negative variable $X$ has mode $m$ and moments $\mu_1$ and $\mu_2$, then there exists a non-negative random variable $Y$ with moments $\nu_1 = 2\mu_1 - m$ and $\nu_2 = 3\mu_2 - 2m\mu_1$, such that for any function $f: \mathbb{R} \to \mathbb{R}$ the following equality holds:

$$\mathbb{E}[f(X)] = \mathbb{E}[g(Y)].$$  \hspace{1cm} (A.15)

The “Khinchine transform” $g: \mathbb{R} \to \mathbb{R}$ is defined as

$$g(y) = \frac{1}{y-m} \int_{y-m}^{y} f(t) \, dt.$$ \hspace{1cm} (A.16)

For a proof, see Feller [1971].

As a consequence, if next to the first two moments also the mode is known, the problem (6) can be transformed as to find

$$\sup_{F \in \mathcal{B}^*} \int_0^{+\infty} g(x) dF(x) \quad \text{and} \quad \inf_{F \in \mathcal{B}^*} \int_0^{+\infty} g(x) dF(x),$$ \hspace{1cm} (A.17)

where $\mathcal{B}^*$ is the class of all distribution functions with domain $\mathbb{R}^+$ and with first two moments $\nu_1$ and $\nu_2$, and where

$$g(y) = \frac{1}{y-m} \int_{y-m}^{y} (t - m - K) \, dt.$$ \hspace{1cm} (A.18)

2. We then start by proving that, if two moments $\tilde{\mu}_1$ and $\tilde{\mu}_2$ and the mode $\tilde{m}$ of the stochastic variable $S_T$ are known, the expected value $\mathbb{E}[(S_T - K)_+]$ satisfies the boundary conditions

$$F_5(K) \leq \mathbb{E}[(S_T - K)_+] \leq F_6(K),$$ \hspace{1cm} (A.19)
with \( F_5(K) \) and \( F_6(K) \) specified as follows:

- if \( K \leq \tilde{m} \)
  \[
  F_5(K) = \tilde{\mu}_1 - K
  \]  
  (A.20)

and

\[
F_6(K) = \begin{cases} 
(\tilde{\mu}_1 - K) + \frac{K^2}{2\tilde{m}} \tilde{\nu}_2 - \tilde{\nu}_1^2}{\tilde{\nu}_2} & \text{if } K \leq \frac{2\tilde{m}\tilde{\nu}_2}{2\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2}

\end{cases}
\]  
(A.21)

where \( z \) is the unique root in the interval \([0, K]\) of the equation

\[
x^3 + Ax^2 + Bx + C = 0
\]  
(A.22)

with coefficients

\[
A = -3K \\
B = (4\tilde{\nu}_1 + 2\tilde{m})K - (2\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2) \\
C = 2\tilde{m}\tilde{\nu}_2 - (2\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2)K.
\]  
(A.23)

- if \( K \geq \tilde{m} \)
  \[
  F_6(K) = \begin{cases} 
(\tilde{\mu}_1 - K) + \frac{(\tilde{\nu}_2 - \tilde{\nu}_1^2)(z - K)^2}{2((\tilde{\nu}_2 - \tilde{\nu}_1^2) + (z - \tilde{\nu}_1)^2)(\tilde{m} - z)} & \text{if } K \geq \frac{2\tilde{m}\tilde{\nu}_2}{2\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2}.

\end{cases}
\]  
(A.24)

where \( y \) is the unique root in the interval \([\max(K, \frac{2\tilde{m}\tilde{\nu}_2}{\tilde{m}\tilde{\nu}_1 + \tilde{\nu}_2}), +\infty]\) of equation (A.22).
Proof of the results for \( K \leq \bar{m} \)

In (A.17) we substitute the Kinchine transform of \((x - K)_+\), i.e.

\[
g(x) = \begin{cases} 
\frac{(\bar{m} - K)^2}{2(\bar{m} - x)} & \text{if } 0 \leq x < K (\leq \bar{m}) \\
\frac{x + \bar{m} - 2K}{2} & \text{if } x \geq K.
\end{cases}
\]  

(A.26)

Because of (20), one always has \( K < 0^* \) \( (K \leq \bar{m} \leq \tilde{\nu}_1 < 0^*) \), so \( g(0^*) = (0^* + \bar{m} - 2K)/2 \) and \( g'(0^*) = 1/2^6 \).

For the upper bounds, the reasoning is completely analogous to subsection 5.2 of Heijnen [1989]. We first look for polynomials of degree 2 for which the equality in (7) holds for a two-point distribution with mass points 0 and 0*. Such polynomial \( P \) has to pass through the points \((0, g(0))\) and \((0^*, g(0^*))\), has to be tangent to (A.26) in 0* and \( P'(0) \geq g'(0) \) must hold. Using (2.1) and (2.2) of Heijnen [1989], this is equivalent to the condition

\[
\frac{1}{2} \left( \frac{1}{2} + \frac{(\bar{m} - K)^2}{2\bar{m}^2} \right) < \frac{1}{0^*} \left( \frac{0^* + \bar{m} - 2K}{2} - \frac{(\bar{m} - K)^2}{2\bar{m}} \right). 
\]  

(A.27)

After some calculations, (A.27) can be transformed into the first condition of (A.21). The best upper bound is then \( q_0 g(0) + q_0^* g(0^*) \), which turns out to be the upper bound (A.21.a).

Then we look for polynomials of degree 2 for which the equality in (7) holds for other two-point distributions with mass points \( r \) and \( r^* \) where \( r \in [0, K] \). Note that firstly because \( K < \tilde{\nu}_1 \) and because the range of the distributions is \([0, +\infty[\), we know that \( r^* \) exists for any \( r \in [0, K] \), and secondly because \( K < 0^* \) it is always true that \( r^* > K \). Such polynomial \( P \) has to pass through the points \((r, g(r))\) and \((r^*, g(r^*))\) and has to be tangent to (A.26) in both \( r \) and \( r^* \). Using (2.1) and (2.2) of Heijnen [1989], the existence of such \( P \) becomes, after some calculations, equivalent to the existence of a unique root in \([0, K]\) of equation (A.22). The left-hand side of (A.22) is a polynomial of degree 3 (let’s call it \( Q \)) with point of inflexion \( K \). The condition in (A.21.b) guarantees that \( Q(0) \leq 0 \). Also \( Q(K) > 0 \) for \( K < \bar{m} \). So the condition in (A.21.b) is sufficient to guarantee a unique root of \( Q \) in \([0, K]\). The best upper bound is then \( q_r g(r) + q_{r^*} g(r^*) \) which, after some calculations, is transformed into the upper bound (A.21.b).

For the best lower bounds, the reasoning is the following. Because of \( K \leq \bar{m} \) and (20), we have \( K \leq \tilde{\mu}_1 \leq \tilde{\nu}_1 \), so \( K^* > K \) (except for the limiting case \( K = \bar{m} =

\text{[6]}\) In Heijnen [1989], \( r^* \) is defined as the mass-point that corresponds to \( r \) to have a two-point distribution with spectrum \( \{r, r^*\} \) with the given moments \( \tilde{\nu}_1 \) and \( \tilde{\nu}_2 \). If \( r < b^* \) or \( r > 0^* \), such distribution exists and the masses \( q_r \) and \( q_{r^*} \) can be expressed analytically. In particular \((r^*)^* = r, 0^* = \tilde{\nu}_2/\tilde{\nu}_1 \) and \( \lim_{b^* \rightarrow +\infty} b^* = \tilde{\nu}_1 \).
\[ \tilde{\mu}_1 = \tilde{\nu}_1 \text{ where the best lower bound is obviously zero}. \]

Taking \( P(x) = (x + \tilde{\mu} - 2K)/2 \) in (7) with the two-point distribution with spectrum \( \{K, K^*\} \), immediately gives \( \tilde{\mu}_1 - K \) as best lower bound.

**Proof of the results for** \( K \geq \tilde{\mu} \)

In (A.17) we substitute the Kintchine transform of \( (x - K)_+ \), i.e.

\[
g(x) = \frac{(x - K)^2}{2(x - \tilde{\mu})} \tag{A.28}
\]

For the upper bounds, the proof can be found in Heijnen [1989]; one just has to replace ‘d’ by ‘K’ and ‘0’ by ‘\( \tilde{\nu}_2/\tilde{\nu}_1 \)’ in the formulae of the paper. Taking the limit for \( b \) going to infinity completes the proof, except for one consideration. In Heijnen [1989] the existence of a unique solution of (A.22) in \( \max(K; \tilde{\nu}_2/\tilde{\nu}_1); +\infty] \) is guaranteed by a condition that is slightly stronger than (20). So we still have to verify that the conditions \( K \geq \tilde{\mu}, (20) \) and the condition in (A.25.b) together are strong enough to guarantee the same. The reasoning is the following. The left-hand side of (A.22) is a polynomial of degree 3, let’s call it again \( Q \).

\( Q \) has its point of inflexion in \( K \) and \( Q(K) < 0 \) for \( K \geq \tilde{\mu} \) while \( Q(+\infty) > 0 \), so (A.22) has a unique real solution in \( \{K, +\infty\} \). Because of conditions (A.25.b) and (20) also \( Q(\tilde{\nu}_2/\tilde{\nu}_1) < 0 \), which implies at least one real solution of (A.22) in \( \{\tilde{\nu}_2/\tilde{\nu}_1; +\infty\} \). Both statements on \( K \) and \( \tilde{\nu}_2/\tilde{\nu}_1 \) together guarantee a unique solution of (A.22) in \( \max(K; \tilde{\nu}_2/\tilde{\nu}_1); +\infty] \).

For the lower bounds we first have to introduce an upper bound in \( b \) in the range of the distribution functions. Later on we will take the limit for \( b \) going to infinity.

Using lemma 3.1 of Heijnen [1989] learns that \( b^* \leq \tilde{\nu}_1 \) if \( b \to +\infty \). So in case \( K < \tilde{\nu}_1 \) we can choose \( b \) large enough to have \( K < b^* < \tilde{\nu}_1 \). Using formula (2.1) of Heijnen [1989] generates a polynomial \( P \) of degree 2 through the points \((b^*, g(b^*))\) and \((b, g(b))\) which is tangent to (A.28) in \( b^* \). This polynomial is useful for (7) if \( P(0) \leq 0 \). Some more calculations show that \( \lim_{b \to +\infty} P(0) < 0 \) if \( \tilde{\mu} \leq K < \tilde{\nu}_1 \), which is the case here. So we can choose \( b \) large enough to guarantee that \( P(0) < 0 \). Then the best lower bound is \( q_0 g(b^*) + q_0 g(b) \) with \( q_0 \) and \( q_0 \) the masses in \( b^* \) and \( b \) as defined in lemma 3.1 of Heijnen [1989]. Taking the limit of this lower bound for \( b \) going to infinity gives \( (\tilde{\nu}_1 - K)^2/2(\tilde{\nu}_1 - \tilde{\mu}) \) as best lower bound for this case, which can easily be transformed into the expression (A.24.a). The best lower bound in case \( K \geq \tilde{\nu}_1 \) is obviously zero, since the best lower bound in case \( K < \tilde{\nu}_1 \) tends to zero for \( K \) going to \( \tilde{\nu}_1 \).

3. Making use of the relations \( \tilde{\mu}_1 = S \cdot \mu_1 \), \( \tilde{\mu}_2 = S^2 \cdot \mu_2 \) and \( \tilde{\mu} = S \cdot m \), the results of (A.20)-(A.21) and (A.24)-(A.25) can be transformed into the bounds of (30)-(31) and (27)-(28). Note that the equations (24) and (A.22) are equivalent.
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Corresponding author

Ann De Schepper
Department of Applied Economics
Prinsstraat 13
B-2000 Antwerp
Belgium
tel : +32-3-220.40.77
fax : +32-3-220.48.17
email: ann.deschepper@ua.ac.be

References