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ISOTROPY OF 5-DIMENSIONAL QUADRATIC FORMS OVER THE FUNCTION FIELD OF A QUADRIC IN CHARACTERISTIC 2

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ABSTRACT. We complete a classification of quadratic forms over a field of characteristic 2 of type $(1,3)$ that become isotropic over the function field of a quadric.

Keywords: Quadratic forms, function fields of quadrics, isotropy, characteristic two.

Mathematics Subject Classification (MSC 2010): 11E04; 11E81.

1. INTRODUCTION

Throughout this paper F denotes a field of characteristic 2. It is well-known, see [2, (7.32)], that any F-quadratic form φ is isometric to

$$[a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \cdots, c_s\rangle$$

for scalars $a_1, b_1, \cdots, a_r, b_r, c_1, \cdots, c_s \in F$, where $[a, b]$ (*resp.* $\langle c_1, \cdots, c_s\rangle$) denotes the binary quadratic form given by $(x, y) \mapsto ax^2 + xy + by^2$ (*resp.* the diagonal quadratic form given by $(x_1, \ldots, x_s) \mapsto \sum_{i=1}^{s} c_i x_i^2$). In this case the pair (r, s) is unique, and we call it the type of φ. The form $\langle c_1, \cdots, c_s\rangle$ is also unique, we call it the quasilinear part of φ and we denote it by $\mathrm{ql}(\varphi)$. We say that φ is nonsingular (*resp.* singular) if $s = 0$ (*resp.* $s > 0$) and totally singular if $r = 0$. A notion that we will use in the formulation of our results is the domination relation between quadratic forms. Recall a quadratic form $\varphi = (V, q)$ is called dominated by another quadratic form $\psi = (W, p)$, denoted $\varphi \preccurlyeq \psi$, if there exists an injective F-linear map $f : V \longrightarrow W$ such that $q(v) = p(f(v))$ for every $v \in V$. The form φ is called weakly dominated by ψ, denoted $\varphi \preccurlyeq_w \psi$, if $\alpha\varphi \preccurlyeq \psi$ for some $\alpha \in F^*$.

Let φ be an anisotropic F-quadratic form. An important problem in the algebraic theory of quadratic forms is classifying anisotropic F-quadratic forms ψ for which φ becomes isotropic over $F(\psi)$, the function field of the affine quadric given by ψ. This problem has been completely studied by the second author in [8] when φ is of dimension $\leqslant 4$, of dimension 5 and type $(2, 1)$ or an Albert form (i.e., a nonsingular 6-dimensional quadratic form of trivial Arf invariant). The isotropy problem was treated by Faivre for certain forms of dimension $6, 7$ and 8 in [3] and recently the second author and Rehmann studied the isotropy of 5-dimensional quadratic forms of type $(0, 5)$ over function fields of quadrics in [11]. Our aim in this paper is to give a complete answer to the isotropy of 5-dimensional F-quadratic forms over the function field of a quadric for the remaining case, that is forms of type $(1, 3)$.

An important case where the isotropy question is well-known concerns Pfister neighbours. More precisely, if φ is an anisotropic Pfister neighbour of a quadratic Pfister form π, then φ is isotropic over $F(\psi)$ if and only if π is isotropic over ψ, which is equivalent,

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by Theorem 2.1, to $\psi \preccurlyeq_w \pi$. In particular, in this case if the dimension of ψ is greater than half the dimension of π, then ψ is also a Pfister neighbour of π. Here, therefore, we will be most interested in the case where φ is not a Pfister neighbour.

In the following proposition we summarise several cases of forms ψ for which all forms of type $(1,3)$ that are not Pfister neighbours remain anisotropic over $F(\psi)$. These can be deduced from previous results proven by the second author, Hoffmann and Totaro. Here $\triangle(\psi)$ denotes the Arf invariant of ψ (see §2).

Proposition 1.1. *Let φ be an anisotropic F-quadratic form of dimension 5 and type $(1,3)$ that is not a Pfister neighbour, and let ψ be anisotropic F-quadratic form of type (r,s). Then φ is anisotropic over $F(\psi)$ in the following cases:*

(1) $r = 0$ *and* $s = 1$.
(2) $r = 0$ *and* $s \geqslant 5$.
(3) $r = 1$ *and* $s \geqslant 4$.
(4) $r = 2$, $s = 0$ *and* $\triangle(\psi) \neq 0$.
(5) $r = 2$ *and* $s \geqslant 1$.
(6) $r \geqslant 3$.

The following results treat quadratic forms ψ where a quadratic form of type $(1,3)$ that is not a Pfister neighbour may become isotropic over $F(\psi)$. Recall that the norm degree of the totally singular F-quadratic form σ, denoted $\mathrm{ndeg}_F(\sigma)$, is the degree of the field $F^2(xy \mid x, y \in D_F(\sigma))$ over F^2, where $D_F(\sigma)$ is the set of scalars represented by σ. If $\sigma = \langle c_1, \cdots, c_s\rangle$ with $c_1 \neq 0$, then $\mathrm{ndeg}_F(\sigma) = [F^2(c_1c_2, c_1c_3, \cdots, c_1c_s) : F^2]$ (see [5, §8] for more details on the norm degree). In particular, $\mathrm{ndeg}_F(\sigma)$ is always a 2-power and equal to or less than $2^{\dim \sigma}$. Note that if σ is of type $(0,4)$ and $\mathrm{ndeg}_F(\sigma) < 4$, then ψ is isotropic. We write GP_nF for the set of F-quadratic forms similar to n-fold quadratic Pfister forms and $\varphi \sim \psi$ if φ and ψ are Witt equivalent F-quadratic forms (see §2).

Theorem 1.1. *Let φ be an anisotropic F-quadratic form of dimension 5 and type $(1,3)$ that is not a Pfister neighbour and let ψ be anisotropic F-quadratic form.*

(1) *If ψ is of type $(1,1)$ or $(1,2)$, then φ is isotropic over $F(\psi)$ if and only if there exist R_1, R_2 nonsingular F-quadratic forms of dimension 2, scalars $a, b, \alpha, \beta \in F^*$, a nonsingular completion ρ of $\langle 1, a, b\rangle$ and $\pi \in GP_3F$ such that $\psi \preccurlyeq_w \pi$, $\alpha\varphi \cong R_1 \perp \langle 1, a, b\rangle$, $\beta\psi \cong R_2 \perp Q$ and $R_1 \perp R_2 \perp \rho \sim \pi$, where $Q = \langle 1\rangle$ or $\langle 1, a\rangle$ respectively as $\dim\psi = 3$ or 4.*
(2) *If ψ is of type $(2,0)$ and $\triangle(\psi) = 0$ (that is, ψ is similar to a 2-fold Pfister form π), then $\varphi_{F(\psi)}$ is isotropic if and only if $\varphi_{F(\psi')}$ is isotropic, where ψ' is a Pfister neighbour of π of dimension 3. Thus in this case we can reduce to case (1).*
(3) *If ψ is of type $(1,3)$, then φ is isotropic over $F(\psi)$ if and only if φ is similar to ψ.*
(4) *If ψ is of type $(0,3)$ or of type $(0,4)$ and $\mathrm{ndeg}_F(\psi) = 8$, then φ is isotropic over $F(\psi)$ if and only if there exist a form φ' of type $(1,3)$ and a form $\pi \in GP_3(F)$ such that $\varphi \sim \varphi' \perp \pi$, $\psi \preccurlyeq_w \varphi'$ and $\psi \preccurlyeq_w \pi$.*
(5) *If ψ is of type $(0,4)$ and $\mathrm{ndeg}_F(\psi) = 4$, then $\varphi_{F(\psi)}$ is isotropic if and only if $\varphi_{F(\psi')}$ is isotropic, where ψ' is a subform of ψ of dimension 3. Thus in this case we can reduce to case (4).*

The final case not included in Proposition 1.1 and Theorem 1.1 is when ψ is of type $(0,2)$. In this case, that any anisotropic F-quadratic form φ is isotropic over $F(\psi)$ if and only if $\psi \preccurlyeq_w \varphi$ is a classical result, but we include a proof in Lemma 2.1 for completeness.

The proofs of Proposition 1.1 and Theorem 1.1 will be done case-by-case. For Proposition 1.1 we will use some general results on the isotropy of quadratic forms of dimension

$2^n + 1$ over function fields of quadrics proved by the second author and Hoffmann. The proof of Theorem 1.1 is mainly based on the index reduction theorem in characteristic 2 from [14] due to Mammone, Tignol and Wadsworth, and methods specific to totally singular quadratic forms.

2. Definitions and preliminary results

We recall the basic definitions and results we use from the theory of quadratic forms over fields. We refer to [2] as a general reference and for standard notation not explicitly defined here.

By an F-quadratic form we mean a pair (V, q) of a finite dimensional F-vector space V and a map $q : V \to F$ such that $q(\lambda x) = \lambda^2 q(x)$ for all $x \in V$ and $\lambda \in F$, and such that $b_q : V \times V \to F, (x, y) \longmapsto q(x + y) - q(x) - q(y)$ is F–bilinear. We call b_q the polar form of (V, q). By an isometry of F-quadratic forms $\varphi = (V, q)$ and $\psi = (W, p)$ we mean an isomorphism of F–vector spaces $f : V \to W$ such that $q(x) = p(f(x))$ for all $x \in V$. If such an isometry exists, we say φ and ψ are isometric and we write $\varphi \simeq \psi$. We say that φ is similar to ψ if there exists $c \in F^*$ such that $\varphi \simeq c\psi$.

An F-quadratic form $\varphi = (V, q)$ is called isotropic if there exists a nonzero vector $x \in V$ such that $q(x) = 0$, otherwise φ is called anisotropic. We say that a scalar $\alpha \in F$ is represented by $\varphi = (V, q)$ if there exists $x \in V$ such that $q(x) = \alpha$. The set of scalars represented by φ is denoted by $D_F(\varphi)$. Any F-quadratic form φ has a unique decomposition $\varphi \cong \varphi_{an} \perp m \times [0, 0] \perp n \times \langle 0\rangle$, where $m, n \geqslant 0$ are integers and φ_{an} is an anisotropic quadratic form uniquely determined up to isometry, which we call the anisotropic part of φ. The integer m (*resp.* n) is called the Witt index of φ and denoted $i_W(\varphi)$ (*resp.* the defect index of φ and denoted $i_d(\varphi)$). The form $\varphi_{an} \perp m \times [0, 0]$ is also unique.We call it the nondefective part of φ and we denote it by φ_{nd} (see [5, (2.4)]). If $i_d(\varphi) = 0$ then we call φ nondefective.

Two quadratic forms φ_1 and φ_2 are called Witt-equivalent and we write $\varphi_1 \sim \varphi_2$ if $\varphi \perp m \times [0, 0] \cong \varphi_2 \perp n \times [0, 0]$ for some integers $m, n \geqslant 0$. Considering nonsingular quadratic forms up to Witt equivalence gives the Witt group of nonsingular F-quadratic forms, $W_q(F)$. We let $W(F)$ be the Witt ring of regular symmetric F-bilinear forms. There is a natural $W(F)$-module structure on $W_q(F)$ given by the tensor product of a symmetric bilinear form and a quadratic form (see [2, p.51]). Concerning Witt cancellation, we recall the following result:

Proposition 2.1. *(*[7, Prop. 1.2] *for* (1)*;* [5, Lem. 2.6] *for* (2)*) Let μ, ν be F-quadratic forms (possibly singular). Suppose that one of the two following conditions holds:*

(1) $\mu \perp \varphi \cong \nu \perp \varphi$ *for some nonsingular form* φ*.*
(2) μ *and* ν *are nondefective and* $\mu \perp s \times \langle 0\rangle \cong \nu \perp s \times \langle 0\rangle$ *for some integer* s*.*

Then $\mu \cong \nu$*.*

Let $\varphi = (V, q)$ be an F-quadratic form. Let P_φ be the homogeneous polynomial given by φ after a choice of an F-basis of V. The polynomial P_φ is reducible if and only if φ_{nd} is of type $(0, 1)$ or $\varphi_{nd} \cong [0, 0]$, see [14, Prop. 3], and it is absolutely irreducible if φ is not totally singular and $\dim \varphi_{nd} \geqslant 3$, see [4]. When P_φ is irreducible we define the function field $F(\varphi)$ of φ as the field of fractions of the quotient ring

$$F[x_1, \cdots, x_n] / (P_\varphi).$$

We take $F(\varphi) = F$ when P_φ is reducible or $\dim \varphi = 0$. If P_φ is absolutely irreducible and K/F is a field extension, then the compositum $K \cdot F(\varphi)$ coincides with $K(\varphi)$. Note that if $\psi \preccurlyeq \varphi$ and $\dim \psi \geqslant 2$, then $\varphi_{F(\psi)}$ is isotropic. Recall that if φ is nondefective, then $F(\varphi)/F$ is transcendental if and only if φ is isotropic (see [2, (22.9)]).

The following is well-known, but we include a proof for completeness.

Lemma 2.1. *Let φ be an anisotropic F-quadratic form and ψ an anisotropic F-quadratic form over type $(0,2)$. Then $\varphi_{F(\psi)}$ is isotropic if and only if $\psi \preccurlyeq_w \varphi$.*

Proof. Let $\varphi = (V,q)$. We may assume that $\psi = \langle 1, d\rangle$ for some $d \in F^*$. Then $\varphi_{F(\psi)}$ is isotropic if and only if $\varphi_{F(\sqrt{d})}$ is isotropic. If $\varphi_{F(\sqrt{d})}$ is isotropic then there exist vectors $v, v' \in V \setminus \{0\}$ such that $q(v) = dq(v')$ and $b_q(v,v') = 0$. Hence $\psi \preccurlyeq_w \varphi$. The converse is clear. □

A quadratic form ψ is called a subform of another quadratic form φ, denoted by $\psi \subset \varphi$, if there exists an F-quadratic form ψ' such that $\varphi \cong \psi \perp \psi'$. If ψ is nonsingular, then the condition $\psi \preccurlyeq \varphi$ is equivalent to $\psi \subset \varphi$. The domination relation can be viewed as follows:

Proposition 2.2. *(*[5, Lem. 3.1]*) Let φ and ψ be F-quadratic forms. Then $\psi \preccurlyeq \varphi$ if and only if there exist nonsingular forms ψ_r and ρ, nonnegative integers $s' \leqslant s \leqslant s''$, $c_i \in F$ for $i \leqslant s''$, and $d_j \in F$ for $j \leqslant s'$ such that:*

$$\psi \simeq \psi_r \perp \langle c_1, \cdots, c_s\rangle \ ,$$

$$\varphi \cong \psi_r \perp \rho \perp [c_1, d_1] \perp \cdots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1}, \cdots, c_{s''}\rangle \ .$$

The subform theorem will be also needed in our proof of Theorem 1.1:

Theorem 2.1. *(*[5, Th. 4.2]*) Let φ and ψ be F-quadratic forms such that φ is anisotropic and nonsingular and ψ is nondefective. If $\varphi_{F(\psi)}$ is hyperbolic then $ab\psi \preccurlyeq \varphi$ for any $a \in D_F(\varphi)$ and $b \in D_F(\psi)$.*

A nonsingular completion of a totally singular F-quadratic form $\sigma = \langle c_1, \cdots, c_s\rangle$ is nonsingular F-quadratic form isometric to $[c_1, d_1] \perp \cdots \perp [c_s, d_s]$ for some scalars $d_1, \cdots, d_s \in F$. Note that for any nonsingular completion ρ of σ, we have $\rho \perp \sigma \sim \sigma$ because $[c,d] \perp \langle c\rangle \cong [0,0] \perp \langle c\rangle$ for any $c, d \in F$.

Another fact related to the domination relation that we will use is the following result knows as the "Completion Lemma":

Proposition 2.3. *(*[5, Lem. 3.9]*) Let φ and ψ be nonsingular F-quadratic forms and $c_1, \cdots, c_s \in F$ such that $\varphi \perp \langle c_1, \cdots, c_s\rangle \cong \psi \perp \langle c_1, \cdots, c_s\rangle$. For any nonsingular completion ρ of $\langle c_1, \cdots, c_s\rangle$, there exists a nonsingular completion ρ' of $\langle c_1, \cdots, c_s\rangle$ such that $\varphi \perp \rho \cong \psi \perp \rho'$.*

For $n \in \mathbb{N}$, $n > 0$ and $a_1, \cdots, a_n \in F^*$, let $\langle a_1, \ldots, a_n\rangle_b$ denote the n-dimensional symmetric bilinear form given by $((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \sum_{i=1}^n a_i x_i y_i$. A bilinear form isometric to $\langle 1, a_1\rangle_b \otimes \ldots \otimes \langle 1, a_n\rangle_b$ is called an n-fold bilinear Pfister form and denoted by $\langle\langle a_1, \cdots, a_n\rangle\rangle_b$. By a 0-fold bilinear Pfister form, we mean the form $\langle 1\rangle_b$. For $n \in \mathbb{N}$, $n > 0$, an $(n+1)$-fold quadratic Pfister form (or simply just an $(n+1)$-fold Pfister form) is a quadratic form isometric to the tensor product of an n-fold bilinear Pfister form and a nonsingular quadratic form representing 1, where the tensor product is the $W(F)$-module action on $W_q(F)$. Let $P_n(F)$ (*resp.* $GP_n(F)$) denote the set of n-fold quadratic Pfister forms (*resp.* the set $\{\alpha\pi \mid \alpha \in F^* \text{ and } \pi \in P_n(F)\}$). Recall that a quadratic Pfister form is hyperbolic if it is isotropic (see [2, (9.10)]).

An F-quadratic form φ is called a Pfister neighbour if there exists a quadratic Pfister form π such that $2\dim\varphi > \dim\pi$ and $\varphi \preccurlyeq_w \pi$. In this case the form π is unique, and for any field extension K/F, the form φ_K is isotropic if and only if π_K is isotropic. In particular, the forms $\varphi_{F(\pi)}$ and $\pi_{F(\varphi)}$ are isotropic.

For any integer $n \geqslant 1$, let I^nF be the n-th power of the fundamental ideal IF of $W(F)$ (we put $I^0F = W(F)$). Let I_q^nF be the sub-group $I^{n-1}F \otimes W_q(F)$ of $W_q(F)$. This group is additively generated by n-fold quadratic Pfister forms (see [2, §9.B]).

An F-quadratic form $\pi = (V, q)$ is called an n-fold quasi-Pfister form if $q(x) = B(x, x)$ for all $x \in V$, where B is an n-fold bilinear Pfister form. In particular, quasi-Pfister forms are totally singular. A totally singular F-quadratic form σ is called a quasi-Pfister neighbour if there exists an anisotropic quasi-Pfister form π such that $2\dim\sigma > \dim\pi$ and $\sigma \preccurlyeq_w \pi$. As with Pfister neighbours, in this case the form π is unique, and for any field extension K/F, the form σ_K is isotropic if and only if π_K is isotropic and, in particular, the forms $\sigma_{F(\pi)}$ and $\pi_{F(\sigma)}$ are isotropic (see [5, (8.9)]).

Two central simple F-algebras A and B are called Brauer-equivalent, denoted $A \sim B$, if they represent the same class in the Brauer group of F. The degree of a central simple F-algebra A is the integer $\sqrt{\dim_F A}$, and the index of A is the integer $\sqrt{\dim_F D}$, where D is the unique central division F-algebra Brauer-equivalent to A. A central simple algebra of degree two is know as a quaternion algebra. For $a, b \in F$ with $b \neq 0$, we denote by $[a, b)$ the quaternion F-algebra whose standard F-basis $\{1, i, j, k\}$ satisfies the following relation: $i^2 + i = a$, $j^2 = b$, $jij^{-1} = i + 1$ and $k = ij$.

For φ an F-quadratic form, we denote by $C(\varphi)$ (*resp.* $C_0(\varphi)$) the Clifford algebra of φ (*resp.* the even Clifford algebra of φ). If φ is nonsingular, then $C(\varphi)$ is a central simple F-algebra, and the centre of $C_0(\varphi)$ is a separable quadratic F-algebra $Z(\varphi)$ (see [2, §11]). In this case, the Arf invariant of φ, denoted $\triangle(\varphi)$, is the class in the additive group $F/\wp(F)$ of an element $\delta \in F$ satisfying $Z(\varphi) = F[X]/(X^2 + X + \delta)$, where $\wp(F) = \{a^2 + a \mid a \in F\}$. In particular, if $\varphi \cong [a_1, b_1] \perp \cdots \perp [a_r, b_r]$, then $\triangle(\varphi) = a_1b_1 + \cdots + a_rb_r + \wp(F)$ (see [2, §13]).

We will need the following index reduction theorem:

Theorem 2.2. *Let D be a central simple division F-algebra and ψ an F-quadratic form of dimension $\geqslant 2$.*

(1) [14, Th. 4] *If ψ is nonsingular and $\triangle(\psi) \neq 0$, then $D \otimes_F F(\psi)$ is not a division algebra if and only if D contains a sub-algebra isomorphic to $C_0(\psi)$.*
(2) [14, Th. 3] *If $\psi = a_1[1, b_1] \perp \cdots \perp a_n[1, b_n] \perp \langle 1, c_1, \cdots, c_m\rangle$ is anisotropic of dimension $2n + m + 1 \geqslant 2$ with $m \geqslant 0$, then $D \otimes_F F(\psi)$ is not a division algebra if and only if D contains a sub-algebra isomorphic to $[b_1, a_1) \otimes_F \cdots \otimes_F [b_n, a_n) \otimes F(\sqrt{c_1}, \cdots, \sqrt{c_m})$.*

We finish this section with some results needed in the proofs.

Proposition 2.4. *(*[8, Prop. 3.2]*) Let $\varphi = a[1, x] \perp \langle 1, b, c\rangle$ be an anisotropic F-quadratic form. Then φ is a Pfister neighbour if and only if the algebra $[x, a) \otimes_F F(\sqrt{b}, \sqrt{c})$ is split.*

Proposition 2.5. *Let $\varphi = a[1, x] \perp \langle 1, c_1, \cdots, c_s\rangle$ be an anisotropic F-quadratic form. Let K/F be a field extension such that $i_W(\varphi_K) = 1$. Then $[x, a) \otimes_F K(\sqrt{c_1}, \cdots, \sqrt{c_s})$ is split.*

Proof. Let $L = K(\sqrt{c_1}, \cdots, \sqrt{c_s})$. Since $i_W(\varphi_K) = 1$ we have $\varphi_K \cong [0, 0] \perp \langle 1, c_1, \cdots, c_s\rangle_K$. Then $a[1, x]_L \perp \langle 1\rangle_L \perp s \times \langle 0\rangle \cong [0, 0] \perp \langle 1\rangle_L \perp s \times \langle 0\rangle$. By Proposition 2.1(2), we deduce that $a[1, x]_L \perp \langle 1\rangle_L \cong [0, 0] \perp \langle 1\rangle_L$, and thus, taking the even Clifford algebra [14, Lem. 2], we conclude that $[x, a) \otimes_F L$ is split. □

Quadratic forms of dimension 2^n+1 satisfy many properties related to the isotropy problem over the function fields of quadrics. We recall two of these properties.

Proposition 2.6. *(*[10, Cor. 5.11]*) Let φ and ψ be anisotropic F-quadratic forms of type $(1,s)$ and $(1,s')$ respectively. Suppose that $\dim\varphi = 2^n+1$ and $\dim\psi > 2^n+1$ $(n \geqslant 1)$. Then $\varphi_{F(\psi)}$ is anisotropic.*

Proposition 2.7. *(*[6, Th. 1.3]*) Let φ be an anisotropic F-quadratic form of dimension 2^n+1 and ψ an anisotropic totally singular F-quadratic form of dimension $> 2^n$. Then φ is anisotropic over $F(\psi)$.*

We will need some facts about quadratic forms over valued fields. Let K be a field of characteristic 2 which is complete for a discrete valuation ν, A the associated valuation ring, π a uniformiser and $\overline{K}$ the residue field. Let $\varphi = (V,q)$ be a K-quadratic form. The first and the second residue forms of φ are defined as follows (we refer to [1, p.1341] for more details): For any integer $i \geqslant 0$, let $M_i = \{v \in V \mid q(v) \in \pi^i A\}$. M_i is an A-module and clearly $M_0 \supset M_1 \supset M_2$. Let us consider the $\overline{K}$-vector spaces $V_0 = M_0/M_1$ and $V_1 = M_1/M_2$, and define the $\overline{K}$-quadratic forms $\varphi_0 = (V_0,q_0)$ and $\varphi_1 = (V_1,q_1)$ by $q_i(v+M_{i+1}) = \overline{\pi^{-i}q(v)}$ for $i = 0,1$. The forms φ_0 and φ_1 are anisotropic and are called the first and the second residue forms of φ, respectively. These forms may be singular and they satisfy $\dim\varphi = \dim\varphi_0 + \dim\varphi_1$.

We recall the Schwarz inequality [12, p.342] which asserts that for any two vectors $x,y \in V$, we have:

$$\nu(b_q(x,y)^2) \geqslant \nu(q(x)) + \nu(q(y)),$$

where b_q is the polar form of φ.

Example 2.1. We keep the same notations and hypotheses as in the previous paragraph. Let $u,v \in K$ be units and $n \in \mathbb{Z}$ be such that the binary quadratic form $[u, v\cdot\pi^n]$ is anisotropic over K. Then the Schwarz inequality implies that $n \leqslant 0$. If moreover $n < 0$ and even (*resp.* $n < 0$ and odd), then the first and the second residue forms of $[u, v\cdot\pi^n]$ are $\langle\overline{u},\overline{v}\rangle$ and the zero form (*resp.* $\langle\overline{u}\rangle$ and $\langle\overline{v}\rangle$). If $n = 0$ then the first and second residue forms are $[\overline{u},\overline{v}]$ and the zero form.

3. Proof of Proposition 1.1

Let φ be an anisotropic F-quadratic form of dimension 5 and type $(1,3)$ that is not a Pfister neighbour. Let ψ be an anisotropic F-quadratic form of type (r,s). Up to a scalar, we may write $\varphi = x[1,a] \perp \langle 1,b,c\rangle$. Let $L = F(\sqrt{b},\sqrt{c})$. Since φ is not a Pfister neighbour, the algebra $[a,x)\otimes_F L$ is division. If $r = 0$ and $s = 1$ then clearly $\varphi_{F(\psi)}$ is anisotropic.

(i) Suppose that ($r = 0$ and $s \geqslant 5$) or ($r = 1$ and $s \geqslant 4$), then $\varphi_{F(\psi)}$ is anisotropic by Propositions 2.7 and 2.6, respectively.

(ii) Suppose that $\psi = u[1,k] \perp v[1,l]$ and $\triangle(\psi) \neq 0$. If $\varphi_{F(\psi)}$ is isotropic, then $i_W(\varphi_{F(\psi)}) = 1$ because $\langle 1,b,c\rangle_{F(\psi)}$ is anisotropic. It follows from Proposition 2.5 that $[a,x)\otimes_F L(\psi_L)$ is not a division algebra. This implies that ψ is anisotropic over L. Moreover, since $\triangle(\psi_L) \neq 0$, the algebra $[a,x)\otimes_F L$ contains a sub-algebra isomorphic to $C_0(\psi_L)$ by Theorem 2.2(1). By comparing the dimensions of the two algebras, we see that this is not possible.

(iii) Suppose that $r = 2$ and $s \geqslant 1$. Let ψ' be an F-quadratic form dominated by ψ of type $(2,1)$. If $\varphi_{F(\psi)}$ is isotropic, then $\varphi_{F(\psi')}$ is also isotropic because $F(\psi')(\psi)/F(\psi')$

is purely transcendental. Hence $\varphi_{F(\psi')}$ is isotropic, and thus ψ' is not a Pfister neighbour. This implies that $\psi' \cong R \perp \mathrm{ql}(\psi')$ for some nonsingular form R such that $\triangle(R) \neq 0$. Now since $F(R)(\psi')/F(R)$ is purely transcendental, we conclude that $\varphi_{F(R)}$ is isotropic, which is not possible by the case (ii).

(iv) Suppose that $r \geqslant 3$. Let ρ be an F-quadratic form dominated by ψ of type $(2,1)$. If $\varphi_{F(\psi)}$ is isotropic then $\varphi_{F(\rho)}$ is isotropic, which is not possible by the case (iii).

4. Proof of statements (1), (2) and (3) of Theorem 1.1

Let φ be an anisotropic F-quadratic form of dimension 5 and type $(1,3)$ that is not a Pfister neighbour. First let ψ be an anisotropic form similar to a 2-fold Pfister form π over F such that $\varphi_{F(\psi)}$ is isotropic, and let ψ' be a Pfister neighbour of π of dimension 3. Then $\varphi_{F(\psi)}$ is isotropic if and only if $\varphi_{F(\psi')}$ is isotropic, as the field extensions $F(\psi')(\psi)/F(\psi')$ and $F(\psi)(\psi')/F(\psi)$ are transcendental. As ψ' must be of type $(1,1)$, we have reduced Case (2) to Case (1).

Now let ψ be an anisotropic F-quadratic form of type $(1,s)$ with $1 \leqslant s \leqslant 3$ and assume that $\varphi_{F(\psi)}$ is isotropic. Up to a scalar we may write $\psi \cong R_2 \perp \mathrm{ql}(\psi)$, where R_2 is nonsingular of dimension 2 and $\mathrm{ql}(\psi)$ is one of the following forms $\langle 1\rangle$, $\langle 1,a\rangle$ or $\langle 1,a,b\rangle$ as $s=1$, 2 or 3, accordingly. Similarly, up to a scalar, we may write $\varphi \cong R_1 \perp \langle 1,u,v\rangle$. Let Q_1 and Q_2 be the quaternion F-algebras satisfying $C(R_i) \sim Q_i \in \mathrm{Br}(F)$ for $i=1,2$. Since $\varphi_{F(\psi)}$ is isotropic, the algebra $Q_1 \otimes_F F(\psi)(\sqrt{u},\sqrt{v})$ is split (Proposition 2.5).

Claim 1: $\mathrm{ql}(\psi)$ is similar to a subform of $\mathrm{ql}(\varphi)$.

This is trivial if $s=1$. Suppose that $s \geqslant 2$. We have $F(\psi)(\sqrt{u},\sqrt{v}) = F(\sqrt{u},\sqrt{v})(\psi)$ as the nondefective part of $\psi_{F(\sqrt{u},\sqrt{v})}$ is neither of type $(0,1)$ nor isometric to $\mathbb{H}$.

Since $Q_1 \otimes_F F(\sqrt{u},\sqrt{v})(\psi)$ is split, we conclude by Theorem 2.2(2) that ψ is necessarily isotropic over $F(\sqrt{u},\sqrt{v})$. The case $i_W(\psi_{F(\sqrt{u},\sqrt{v})}) > 0$ is excluded otherwise $F(\sqrt{u},\sqrt{v})(\psi)$ would be purely transcendental over $F(\sqrt{u},\sqrt{v})$ and thus $Q_1 \otimes_F F(\sqrt{u},\sqrt{v})$ would be split. Hence $i_d(\psi_{F(\sqrt{u},\sqrt{v})}) > 0$. Moreover, by reasons of dimension, Theorem 2.2(2) implies that $\dim(\mathrm{ql}(\psi)_{F(\sqrt{u},\sqrt{v})})_{an} = 1$. Hence when $s=2$ (*resp.* $s=3$) this implies that $a \in F^2(u,v)$ (*resp.* $a,b \in F^2(u,v)$). Consequently, $\langle\langle u,v\rangle\rangle$ is isotropic over $F(\langle 1,a\rangle)$ or $F(\langle 1,a,b\rangle)$ as $s=2$ or $s=3$, accordingly. In particular, $\langle 1,u,v\rangle$ is isotropic over $F(\langle 1,a\rangle)$ or $F(\langle 1,a,b\rangle)$ as $s=2$ or $s=3$, accordingly. Hence $\mathrm{ql}(\psi)$ is similar to a subform of $\langle 1,u,v\rangle$ (the case $s=2$ is Lemma 2.1 and the case $s=3$ is a consequence of [11, Thm. 1.2]). Hence, up to a scalar, we may suppose that $\mathrm{ql}(\psi)$ is a subform of $\langle 1,u,v\rangle$.

By Claim 1 the nondefective part of $\psi_{F(\sqrt{u},\sqrt{v})}$ is isometric to $(R_2 \perp \langle 1\rangle)_{F(\sqrt{u},\sqrt{v})}$. Since $Q_1 \otimes_F F(\sqrt{u},\sqrt{v})(\psi)$ is split, it follows that $Q_1 \otimes_F F(\sqrt{u},\sqrt{v})(R_2 \perp \langle 1\rangle)$ is also split. Consequently, Theorem 2.2(2) implies that $Q_1 \otimes_F F(\sqrt{u},\sqrt{v})$ is isomorphic to $Q_2 \otimes_F F(\sqrt{u},\sqrt{v})$. This implies that $Q_1 \otimes_F Q_2$ is split over $F(\sqrt{u},\sqrt{v})$. Then there exist $k,l \in F$ such that $Q_1 \otimes_F Q_2 \sim [k,u) + [l,v) \in \mathrm{Br}(F)$. Using the Clifford invariant, we get

$$R_1 \perp R_2 \perp u[1,k] \perp v[1,l] \perp [1, r_1+r_2+u+v] \in I_q^3 F\,,$$

where $\triangle(R_i) = r_i + \wp(F)$ for $i=1,2$. Hence, by [9, Prop. 6.4], there exists $\pi \in GP_3F$ such that

$$R_1 \perp R_2 \perp \rho \sim \pi, \tag{4.1}$$

where $\rho = u[1,k] \perp v[1,l] \perp [1, r_1+r_2+u+v]$ is a nonsingular completion of $\langle 1,u,v\rangle$.

Claim 2: The form π is isotropic over $F(\psi)$. This implies by Theorem 2.1 that $\psi \preccurlyeq_w \pi$.

Since $\psi_{F(\psi)}$ is isotropic, we get $(R_2 \perp \langle 1,u,v\rangle)_{F(\psi)} \cong \mathbb{H} \perp \langle 1,u,v\rangle)_{F(\psi)}$. By the completion lemma (Proposition 2.3), there exist $r,s,t \in F(\psi)$ such that

$$R_2 \perp \rho \cong \mathbb{H} \perp u[1,r] \perp v[1,s] \perp [1,t]\,.$$

Hence we get

$$R_1 \perp R_2 \perp \rho \sim R_1 \perp u[1,r] \perp v[1,s] \perp [1,t] \sim \pi. \tag{4.2}$$

Since $\varphi_{F(\psi)}$ is isotropic, we get $(R_1 \perp \langle 1,u,v\rangle)_{F(\psi)} \cong [0,0] \perp \langle 1,u,v\rangle)_{F(\psi)}$. Again by the completion lemma, there exist $r',s',t' \in F(\psi)$ such that

$$R_1 \perp u[1,r] \perp v[1,s] \perp [1,t] \cong \mathbb{H} \perp u[1,r'] \perp v[1,s'] \perp [1,t']. \tag{4.3}$$

It follows from (4.2) and (4.3) that $u[1,r'] \perp v[1,s'] \perp [1,t'] \sim \pi_{F(\psi)}$, and thus $\pi_{F(\psi)}$ is isotropic. Hence the claim.

Claim 3: If $s=3$, then φ is isometric to ψ.

Without loss of generality, we may suppose that $\pi \in P_3F$. If π is anisotropic then ψ is a Pfister neighbour of π. Hence $\psi_{F(\pi)}$ is also isotropic and $F(\pi)(\psi)/F(\pi)$ is purely transcendental. Consequently φ is isotropic over $F(\pi)$, which is not possible by Proposition 1.1. Hence π is isotropic and thus hyperbolic. It follows from (4.1) that $R_1 \perp \rho \sim R_2$. Hence $R_1 \perp \rho \perp \langle 1,u,v\rangle \sim R_2 \perp \langle 1,u,v\rangle$. Consequently, $R_1 \perp \langle 1,u,v\rangle \sim R_2 \perp \langle 1,u,v\rangle$, which implies that φ is isometric to ψ.

Conversely, suppose that there exist R_1, R_2 nonsingular quadratic forms of dimension 2, scalars $u,v,\alpha,\beta \in F^*$, a nonsingular completion ρ of $\langle 1,u,v\rangle$ and a Pfister form $\pi \in P_3F$ such that: $\psi \preccurlyeq_w \pi$, $\alpha\varphi \cong R_1 \perp \langle 1,u,v\rangle$, $\beta\psi \cong R_2 \perp Q$ and $R_1 \perp R_2 \perp \rho \sim \pi$ such that Q is a subform of $\langle 1,u,v\rangle$. Then $(R_1 \perp R_2 \perp \rho)_{F(\psi)} \sim 0$. In particular, $(R_1 \perp \rho \perp \langle 1,u,v\rangle)_{F(\psi)} \sim (R_1 \perp \langle 1,u,v\rangle)_{F(\psi)} \sim (R_2 \perp \langle 1,u,v\rangle)_{F(\psi)}$. Since $(R_2 \perp \langle 1,u,v\rangle)_{F(\psi)}$ is isotropic, we conclude that $\varphi_{F(\psi)}$ is isotropic.

5. PROOF OF STATEMENTS (4) AND (5) OF THEOREM 1.1

Let φ be an anisotropic F-quadratic form of dimension 5 and type $(1,3)$ that is not a Pfister neighbour and ψ be an anisotropic F-quadratic form of type $(0,3)$ or $(0,4)$. Suppose that $\varphi_{F(\psi)}$ is isotropic. We want to show that there exist a quadratic form φ' of type $(1,3)$ and $\pi \in GP_3(F)$ such that $\varphi \sim \varphi' \perp \pi$ and ψ is weakly dominated by φ' and π. Up to a scalar, we may suppose that $\varphi = \alpha[1,x] \perp \langle 1,u,v\rangle$ for suitable $\alpha,x,u,v \in F^*$.

Case 1. Suppose that ψ is of type $(0,3)$. Up to a scalar, we may suppose that $\psi = \langle 1,a,b\rangle$. We put $\delta = \langle 1,u,v\rangle$.

(a) Suppose that δ is isotropic over $F(\psi)$. Then δ is similar to ψ by [11, Thm. 1.2]. Hence we are done by taking $\varphi' = \varphi$ and π the hyperbolic 3-fold quadratic Pfister form.

(b) Suppose that δ is anisotropic over $F(\psi)$.

Claim. Up to a scalar, we may suppose that a, b or ab is represented by δ.

Since δ is anisotropic over $F(\psi)$, the isotropy of $\varphi_{F(\psi)}$ implies that $i_W(\varphi_{F(\psi)}) = 1$, i.e. $\varphi_{F(\psi)} \cong \mathbb{H} \perp \delta_{F(\psi)}$. Moreover, the isotropy of δ over its own function field implies that $\delta_{F(\delta)} \cong \langle 1,u\rangle_{F(\delta)} \perp \langle 0\rangle$. Hence

$$\varphi_{F(\delta)(\psi)} \sim (\alpha[1,x] \perp \langle 1,u\rangle \perp \langle 0\rangle)_{F(\delta)(\psi)} \sim (\mathbb{H} \perp \langle 1,u\rangle \perp \langle 0\rangle)_{F(\delta)(\psi)}\,.$$

Since the forms $(\alpha[1,x] \perp \langle 1,u\rangle)_{F(\delta)(\psi)}$ and $(\mathbb{H} \perp \langle 1,u\rangle)_{F(\delta)(\psi)}$ are nondefective, it follows from Proposition 2.1 that

$$(\alpha[1,x] \perp \langle 1,u\rangle)_{F(\delta)(\psi)} \sim (\mathbb{H} \perp \langle 1,u\rangle)_{F(\delta)(\psi)}\,.$$

In particular, the form $(\alpha[1,x] \perp \langle 1,u\rangle)_{F(\delta)(\psi)}$ is isotropic.

Note that the form $(\alpha[1,x] \perp \langle 1,u\rangle)_{F(\delta)}$ is anisotropic, otherwise we would get that the algebra $[x,\alpha) \otimes_F F(\delta)(\sqrt{u})$ is split, and thus $[x,\alpha) \otimes_F F(\sqrt{u},\sqrt{v})$ would be split, i.e. φ would be a Pfister neighbour. Hence the Albert form $\gamma := \alpha[1,x] \perp u[1,t^{-1}] \perp [1,x+t^{-1}]$ is anisotropic over $K(\delta)$, where $K = F((t))$ the field of Laurent series. Hence $D := [x,\alpha)_K \otimes_K [t^{-1},u)$ is a division algebra over $K(\delta)$. Since $(\alpha[1,x] \perp \langle 1,u\rangle)_{F(\delta)(\psi)}$ is isotropic, it follows that γ is isotropic over $K(\delta)(\psi)$, and thus the index of the algebra $D := [x,\alpha)_K \otimes_K [t^{-1},u)$ reduces over the extension $K(\delta)(\psi)$. By Theorem 2.2(2), $D_{K(\delta)}$ contains the biquadratic extension $K(\delta)(\sqrt{a},\sqrt{b})$. This implies that the algebra $D_{K(\delta)}$ is isomorphic to the biquaternion algebra of $[r,a) \otimes [s,b)$ for suitable $r,s \in K(\delta)^*$. Hence, using the Jacobson theorem [13], there exists $p \in K(\delta)^*$ such that

$$(5.1) \qquad (\alpha[1,x] \perp u[1,t^{-1}] \perp [1,x+t^{-1}])_{K(\delta)} \cong p(a[1,r] \perp b[1,s] \perp [1,r+s]).$$

We consider the t-adic valuation of the field $K(\delta)$. It is clear that, from the left hand side of (5.1), the first and second residue forms of $\gamma_{K(\delta)}$ are the forms $\alpha[1,x] \perp \langle 1,u\rangle$ and $\langle 1,u\rangle$ respectively. We may suppose that p is square free. Moreover p is a unit, otherwise the second residue form from the right hand side of (5.1) would be of dimension bigger that 2. Using the Schwarz inequality we deduce that the valuations of r, s and $r+s$ are less than or equal to zero (due to the anisotropy of γ over $K(\delta)$, see Example 2.1). Moreover, using Example 2.1 and the fact that the first and the second residue forms of the left hand side of (5.1) are of type $(1,2)$ and $(0,2)$, we conclude that one of the scalars r,s and $r+s$ is a unit and the two other scalars are not units whose valuations are odd. Hence comparing the quasilinear parts of the residue forms, we get that $\langle 1,u\rangle_{F(\delta)}$ is isometric to one of the following forms: $\overline{p}\,\langle 1,a\rangle_{F(\delta)}, \overline{p}\,\langle 1,b\rangle_{F(\delta)}$ or $\overline{p}\,\langle a,b\rangle_{F(\delta)}$, where $\overline{p}$ is the residue class of p. Using the roundness of a quasi-Pfister form ([5, (8.5), (i)]), we conclude that

$$(\star) \qquad \langle 1,u\rangle_{F(\delta)} \cong \begin{cases} \langle 1,a\rangle_{F(\delta)} & \text{or} \\ \langle 1,b\rangle_{F(\delta)} & \text{or} \\ \langle 1,ab\rangle_{F(\delta)}\,, \end{cases}$$

which implies that one of the following three forms is isotropic over $F(\delta)$: $\langle 1,u,a\rangle$, $\langle 1,u,b\rangle$, $\langle 1,u,ab\rangle$. Using [11, Thm. 1.2], and modulo a scalar, we may therefore suppose that δ is isometric to one of the three forms: $\langle 1,u,a\rangle$, $\langle 1,u,b\rangle$, $\langle 1,u,ab\rangle$. Hence the claim.

By the claim above, we may suppose that $\psi = \langle 1,u,w\rangle$ for suitable $w \in F^*$. As before the isotropy of $\varphi_{F(\psi)}$ implies that $[x,\alpha) \otimes F(\sqrt{u},\sqrt{v},\sqrt{w})$ is split. Hence there exist suitable scalars $k,l,m \in F^*$ such that $[x,\alpha)$ is Brauer-equivalent to $[k,u) \otimes_F [l,v) \otimes_F [m,w)$. Using the Clifford invariant, we get that

$$\alpha[1,x] \perp u[1,k] \perp v[1,l] \perp w[1,m] \perp [1,x+k+l+m] \in I_q^3(F)\,.$$

It follows from [9, Prop. 6.4] that

$$(5.2) \qquad \alpha[1,x] \perp u[1,k] \perp v[1,l] \perp w[1,m] \perp [1,x+k+l+m] \sim \pi$$

for some form $\pi \in GP_3(F)$. Using the fact that $\varphi_{F(\psi)} \sim \langle 1,u,v\rangle_{F(\psi)}$ with the completion lemma, we deduce from (5.2) that $(\alpha[1,x] \perp u[1,k] \perp v[1,l] \perp [1,x+k+l+m])_{F(\psi)} \cong \mathbb{H} \perp u[1,k'] \perp v[1,l'] \perp [1,m']$ for suitable $k',l',m' \in F(\psi)^*$. Hence $u[1,k'] \perp v[1,l'] \perp w[1,m] \perp [1,m'] \cong \pi_{F(\psi)}$. In particular, $\psi_{F(\psi)}$ is dominated by $\pi_{F(\psi)}$. This implies that $\pi_{F(\psi)}$ is isotropic, and thus hyperbolic. Hence $\psi \preccurlyeq_w \pi$. Further, since

$$\alpha[1,x] \perp u[1,k] \perp v[1,l] \perp w[1,m] \perp [1,x+k+l+m] \sim \pi\,,$$

we deduce that

$$\alpha[1,x] \perp \langle 1,u,v\rangle \sim w[1,m] \perp \langle 1,u,v\rangle \perp \pi\,.$$

So we take $\varphi' = w[1,m] \perp \langle 1,u,v\rangle$ which dominates ψ.

Conversely, if there exist φ' of type $(1,3)$ and $\pi \in GP_3(F)$ such that $\varphi \sim \varphi' \perp \pi$, ψ is weakly dominated by φ' and π, then $\varphi_{F(\psi)} \sim \varphi'_{F(\psi)}$, and thus $\varphi_{F(\psi)}$ is isotropic.

Case 2. Suppose that ψ is of type $(0,4)$ and $\mathrm{ndeg}_F(\psi) = 8$. We will apply the previous case (i.e., the case of type $(0,3)$) several times. Let ψ' be a subform of ψ of dimension 3. Since $\varphi_{F(\psi)}$ and $\psi_{F(\psi')}$ are isotropic, we get that $\varphi_{F(\psi')}$ is isotropic by [8, Lemme 4.5]. By the claim in the previous case, we may suppose that, up to a scalar, $\varphi \cong \alpha[1,x] \perp \langle 1,u,k\rangle$ for suitable $u,k \in F^*$, and $\psi' = \langle 1,u,b\rangle$. So we write $\psi = \langle 1,u,b,c\rangle$. We put $\delta = \langle 1,u,k\rangle$. Now we repeat the same argument for the form $\psi'' = \langle 1,b,c\rangle$. We conclude as in $(\star)$ that

$$\langle 1,u\rangle_{F(\delta)} \cong \begin{cases} \langle 1,b\rangle_{F(\delta)} \text{ or} \\ \langle 1,c\rangle_{F(\delta)} \text{ or} \\ \langle 1,bc\rangle_{F(\delta)}\,. \end{cases}$$

(a) The first two possibilities give that $\langle 1,u,b\rangle$ or $\langle 1,u,c\rangle$ is isotropic over $F(\delta)$, and thus by [11, Thm. 1.2] this implies that δ is similar to $\langle 1,u,b\rangle$ or $\langle 1,u,c\rangle$.

(b) The third possibility gives that $\langle 1,u,bc\rangle$ is isotropic over $F(\delta)$. The form $\langle 1,u,bc\rangle$ is anisotropic, otherwise we would get that $\mathrm{ndeg}_F(\psi) = 4$. Again by [11, Thm. 1.2] we conclude that δ is similar to $\langle 1,u,bc\rangle$. Hence, up to a scalar, we may suppose that $\varphi \cong \alpha[1,x] \perp \langle 1,u,bc\rangle$ and $\delta = \langle 1,u,bc\rangle$. Now we consider the form $\eta = \left\langle 1,ut^2+b,c\right\rangle$. We know that $F(t)(\eta)$ is isometric to $F(\psi)$. Hence $\varphi_{F(t)(\eta)}$ is isotropic. Again we reproduce the same argument as in $(\star)$ in Case 1 to conclude that

$$\langle 1,u\rangle_{F(t)(\delta)} \cong \begin{cases} \left\langle 1,ut^2+b\right\rangle_{F(t)(\delta)} \text{ or} \\ \langle 1,c\rangle_{F(t)(\delta)} \text{ or} \\ \left\langle 1,uct^2+bc\right\rangle_{F(t)(\delta)}\,. \end{cases}$$

(b.1) In the first possibility, we conclude that $\left\langle 1,u,ut^2+b\right\rangle_{F(t)(\delta)} \cong \langle 1,u,b\rangle_{F(t)(\delta)}$ is isotropic. It follows from [11, Thm. 1.2] that δ is similar to $\langle 1,u,b\rangle$.

(b.2) In the second possibility, we conclude as in case (b.1) that δ is similar to $\langle 1,u,c\rangle$.

(b.3) In the third possibility, we conclude that $\langle 1,u\rangle_{F(t)(\delta)}$ represents uct^2+bc. Since $\langle 1,u\rangle_{F(t)(\delta)}$ represents bc (since δ is isotropic over its own function field), it follows that $\langle 1,u\rangle_{F(t)(\delta)}$ represents uct^2, and in particular it represents uc. Hence $\langle 1,u,uc\rangle_{F(\delta)}$ is isotropic. Consequently $\langle 1,u,c\rangle_{F(\delta)}$ is isotropic because $\langle 1,u,uc\rangle$ and $\langle 1,u,c\rangle$ are quasi-Pfister neighbours of the same quasi-Pfister form $\langle\langle u,c\rangle\rangle$. Hence we get by [11, Thm. 1.2] that δ is similar to $\langle 1,u,c\rangle$.

By cases (a) and (b), we may suppose, up to a scalar, that $\varphi \cong \alpha[1,x] \perp \langle 1,u,v\rangle$ and $\psi = \langle 1,u,v,w\rangle$ for some $w \in F$.

The isotropy of $\varphi_{F(\psi)}$ implies that $[x,\alpha)\otimes F(\sqrt{u},\sqrt{v},\sqrt{w})$ is split. Now we follow the same argument as in Case 1 to conclude the existence of a form φ' of type $(1,3)$, a form $\pi \in GP_3(F)$ such that $\varphi \sim \varphi' \perp \pi$ and ψ is weakly dominated by φ' and π. Conversely, these condition give the isotropy of $\varphi_{F(\psi)}$ as proved in Case 1.

Case 3. Suppose that ψ is of type $(0,4)$ and $\mathrm{ndeg}_F(\psi) = 4$. Let ψ' be a subform of ψ of dimension 3. Since ψ and ψ' are quasi-Pfister neighbour of the same quasi-Pfister form, it follows that $\varphi_{F(\psi)}$ is isotropic if and only if $\varphi_{F(\psi')}$ is isotropic. Hence we have reduced this case to Case 1.

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