AIP | Journal of Mathematical Physics

Saturated Kochen–Specker-type configuration of 120 projective lines in eight-dimensional space and its group of symmetry

Artur E. Ruuge and Freddy van Oystaeyen

Citation: Journal of Mathematical Physics **46**, 052109 (2005); doi: 10.1063/1.1887923
View online: http://dx.doi.org/10.1063/1.1887923
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/46/5?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **46**, 052109 (2005)

Saturated Kochen–Specker-type configuration of 120 projective lines in eight-dimensional space and its group of symmetry

Artur E. Ruuge
Department of Quantum Statistics and Field Theory, Faculty of Physics, Moscow State University , Vorobyovy Gory, 119899 Moscow, Russia and Department of Mathematics and Computer Science, University of Antwerp, Middelheim Campus, Building G, Meddelheimlaan 1, B-2020 Antwerp, Belgium

Freddy van Oystaeyen
Department of Mathematics and Computer Science, University of Antwerp, Middelheim Campus, Building G, Meddelheimlaan 1, B-2020 Antwerp, Belgium

(Received 14 December 2004; accepted 14 February 2005; published online 29 April 2005)

There exists an example of a set of 40 projective lines in eight-dimensional Hilbert space producing a Kochen–Specker-type contradiction. This set corresponds to a known no-hidden variables argument due to Mermin. In the present paper it is proved that this set admits a finite saturation, i.e., an extension up to a finite set with the following property: every subset of pairwise orthogonal projective lines has a completion, i.e., is contained in at least one subset of eight pairwise orthogonal projective lines. An explicit description of such an extension consisting of 120 projective lines is given. The idea to saturate the set of projective lines related to Mermin's example together with the possibility to have a finite saturation allow to find the corresponding group of symmetry. This group is described explicitely and is shown to be generated by reflections. The natural action of the mentioned group on the set of all subsets of pairwise orthogonal projective lines of the mentioned extension is investigated. In particular, the restriction of this action to complete subsets is shown to have only four orbits, which have a natural characterization in terms of the construction of the saturation. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1887923]

I. INTRODUCTION

The analysis of the foundations of quantum theory has two important results, (1) Bell's inequalities; (2) Kochen–Specker theorem. Both of them show that in quantum mechanics a statistical model of a physical experiment may not admit a realization by way of a probability space $(\Omega,\mathcal{F},P)$. The first result shows, that even if one assumes the existence of a measurable space $(\Omega,\mathcal{F})$, one might not be able to consruct a probability measure P. The second result shows that the assumption about the existence of $(\Omega,\mathcal{F})$ may itself be contradictory.

The Bell–Kochen–Specker theory[2,5] restricts itself to considering the measuring devices with only two possible indications. In this case the problem cocerning the existence of $(\Omega,\mathcal{F})$ reduces to a problem about coloring projective lines in a finite dimensional Hilbert space. It is possible to reformulate this problem as follows.[8] Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ of finite dimension $n := \dim_{\mathbb{C}} \mathcal{H}$. Suppose one is given a family of $m \in \mathbb{N}$ orthonormal bases in $\mathcal{H}$: $\{e_i^{(\alpha)}\}_{i=1}^n$, $\alpha=\overline{1,m}$. Choose from each basis an element $e_{i_\alpha}^{(\alpha)}$, $\alpha=\overline{1,m}$, and look at the inner products $(e_{i_\alpha}^{(\alpha)},e_{i_\beta}^{(\beta)})$, $\alpha,\beta=\overline{1,m}$. Is it always possible to make this choice in such a way, that for all α and β the inner product of $e_{i_\alpha}^{(\alpha)}$ and $e_{i_\beta}^{(\beta)}$ does not vanish, i.e., the (projective) lines $\mathbb{C}e_{i_\alpha}^{(\alpha)}$ and $\mathbb{C}e_{i_\beta}^{(\beta)}$ are *not* orthogonal?

One says that a collection of projective lines $\{\mathbb{C}e_i^{(\alpha)}\}_{i,\alpha}$ produces a *Kochen–Specker-type con-*

tradiction if the answer to the formulated question is "No." The original construction by Kochen and Specker in the proof of their theorem generates the first example of this contradiction. Since then several other examples of this type have been found, in particular Refs. 7, 6, 9, 4, and 1. In the present paper we analyze one of them—the example coming from the proof of a no-hidden-variables theorem given by Mermin.[6] Note, that the latter example can be related to the discussion of Einstein–Podolsky–Rosen paradox and Bell's theorem in Ref. 3.

The original construction of Kochen and Specker is quite sophisticated. This is determined by the fact that the authors work in a three-dimensional space. The construction of Mermin and the corresponding proof are much more simple, but the space is eight-dimensional. To be exact, the original paper of Mermin is written in terms of operators on a Hilbert space and the reformulation in terms of projective lines is due to Kernaghan and Peres.[4] In general, the latter consists in the following. One defines a set of 40 projective lines in $\mathbb{C}^8$. The corresponding description is explicit and quite simple. If one views the elements of $\mathbb{C}^8$ as columns of eight complex numbers, then it is possible to represent each of the 40 projective lines by a column with entries 0, 1 or -1. Thus in fact the projective lines can even be viewed as real. The corresponding Kochen–Specker contradiction is then established by some simple arithmetic argument. The link with the Mermin's formulation is as follows. The described set splits into five distinct tuples, each containing eight pairwise orthogonal projective lines. Identifying $\mathbb{C}^8$ with $(\mathbb{C}^2)^{\otimes 3}$ and interpreting each of the five tuples as coming from an orthonormal basis corresponding to some complete set of pairwise commuting orthogonal projectors, one arrives at five complete sets playing a key role in Ref. 6.

There is another well-known example—the "Penrose dodecahedron"—studied in detail by Zimba and Penrose in Ref. 9. In this case one again makes use of 40 projective lines, but this time in $\mathbb{C}^4$. In the corresponding construction one considers a dodecahedron and associates in a certain way to each of its 20 vertices two projective lines in $\mathbb{C}^4$. The whole construction has the symmetry group of a dodecahedron, which naturally acts on the resulting set of 40 projective lines. Comparing the examples of Penrose and of Mermin, it is natural to ask, what can one say about the symmetry of Mermin's example? An additional motivation for this is given by the two examples described in Ref. 1 which also have a high degree of symmetry. One of them is associated to a 120 cell (a four-dimensional analog of dodecahedron), and the other to a 600 cell (a four-dimensional analog of icosahedron). It turns out, that despite of the fact that the projective lines in Mermin's case look quite simple, an answer to this question, as was mentioned in Ref. 4, presents a problem. Its possible solution constitutes the subject of the present paper.

II. SATURATED KOCHEN–SPECKER

Let A denote a set of projective lines in Hilbert space $\mathcal{H}$, $\dim_{\mathbb{C}} \mathcal{H} = n < \infty$. The set A is called *saturated* with respect to orthogonality relation $\perp$ if any of its subsets $B \subset A$ of pairwise orthogonal projective lines can be embedded into a subset $C \subset A$ of n pairwise orthogonal projective lines. Denote $\mathcal{P}_{\perp}(A) := \{B \subset A \mid \forall x, y \in B : x \neq y \Rightarrow x \perp y\}$. Denote $C(A) := \{B \in \mathcal{P}_{\perp}(A) \mid \#B = n\}$. The elements of $C(A)$ will be called *complete* subsets of A. Note that $\mathcal{P}_{\perp}(A)$ contains an empty set and all subsets of cardinality 1. Note that if A produces a Kochen–Specker contradiction, then $C(A)$ is not empty.

If one looks at the mentioned example of Mermin, one observes that the corresponding set is *not* saturated with respect to $\perp$. Intuitively, a saturated set should have a higher degree of symmetry than an unsaturated part of it [an example of a saturated set is the set $\mathbb{P}(\mathcal{H})$ of *all* projective lines in $\mathcal{H}$]. This leads to the idea of how to investigate the symmetry of Mermin's example. One may try to add projective lines to the given set so that to get a saturated set. After that it makes sense to proceed with the symmetry. Naively, such an attempt should look as follows. One takes a subset of pairwise orthogonal lines, tries to find a complete set containing it, and in case there is no such one, invents several other pairwise orthogonal projective lines to make it complete. These new projective lines are added to the initial set, and the whole process is repeated until one reaches a saturation. At each step one solves the problem for the chosen subset, but at the same time one may create other subsets of pairwise orthogonal elements which require a completion. It means, that *a priori* the described algorythm is not even finite.

In the next section we are going to describe a *finite* set A of projective lines in $\mathcal{H}$, $\dim_{\mathbb{C}} \mathcal{H} = 8$, with the following properties: (1) A is saturated with respect to $\perp$; (2) A contains a set of Mermin–Kernaghan–Peres projective lines and due to this, in particular, produces a Kochen–Specker-type contradiction; (3) *every* element of A can be represented by a column with each of the eight entries being 0, 1 or −1.

After that we proceed with the study of the symmetry of the set A. One looks at $Bij(A)$—a group of all bijections of A, and denotes by $Bij_{\perp}(A)$ its subgroup consisting of all bijections which respect the orthogonality relation $\perp$. The set $C(A)$ naturally splits into four disjoint subsets denoted as $C_k(A)$, $k=1,2,4,8$, as will be explained below. We describe a subgroup $\mathcal{G}$ in $Bij_{\perp}(A)$ by giving explicit formulas for a set of its generators and prove, that this group has an action on $C(A)$ such that $C_k(A)$'s conicide with its orbits. It means that one can take any element of $C_k(A)$ and then generate all the other complete subsets belonging $C_k(A)$ by applying the elements of this group. This allows to describe the symmetry of Mermin's example.

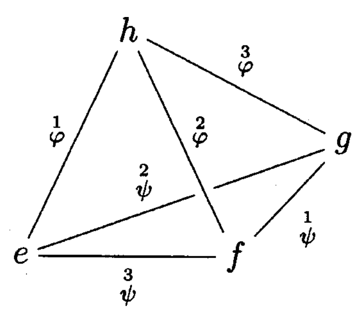
III. 120 PROJECTIVE LINES

We shall now describe a set A which will later be proved to be a saturated extension of Mermin's example. Set $\mathcal{H} := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Recall that one has denoted the set of all projective lines in $\mathcal{H}$ by $\mathbb{P}(\mathcal{H})$. Let V denote a set of four symbols, $V=\{e,f,g,h\}$. We are going to describe a map

$$\xi : \mathcal{P}(V)^{\times} := \mathcal{P}(V) \setminus \{\varnothing\} \to C(\mathbb{P}(\mathcal{H})),$$

such that $\forall U, U' \in \mathcal{P}(V)^{\times} : U \neq U' \Rightarrow \xi(U) \cap \xi(U') = \varnothing$. Note that the latter in particular implies that ξ is injective. Since $\#\mathcal{P}(V)^{\times} = 15$ and $\dim_{\mathbb{C}}(\mathcal{H}) = 8$, the union of all the sets from the image of ξ must yield 120 projective lines. This will produce the set A.

The set $\mathcal{P}(V)^{\times}$ may be visualized as a tetrahedron with vertices labeled as e, f, g, and h. Considering the subsets of V, one identifies the vertices with the subsets of cardinality 1, the edges with subsets of cardinality 2, faces with subsets of cardinality 3, and the body of the tetrahedron with V. Assign to the edges of this graph labels of the form $\overset{\omega}{\varphi}$ or $\overset{\omega}{\psi}$ where φ and ψ are two symbols and ω is a number 1, 2 or 3. Require, that the edges associated to the same vertex have different numbers ω. This defines the labelling of the edges of the graph up to a permutation of labels of vertices. Without loss of generality, one may choose and fix the labeling as shown in



(1)

Note that this tetrahedron appears in Ref. 8.

Choose an arbitrary orthonormal basis $\{\varphi_\alpha\}_{\alpha=0,1}$ in $\mathbb{C}^2$. It will be convenient to view the possible values 0 and 1 of the index α as elements of the group $\mathbb{Z}_2$ with the group operation written additively (addition modulo 2). Consider a matrix $u := \|u(\alpha,\beta)\|_{\alpha,\beta\in\mathbb{Z}_2}$ with entries $u(\alpha,\beta) = -1$ if $(\alpha,\beta)=(1,1)$ and $u(\alpha,\beta)=1$ otherwise. Note that it can be viewed as a character matrix of $\mathbb{Z}_2$. There exist the following properties:

$$(1) \quad u(\alpha,\beta) = u(\beta,\alpha) \in \mathbb{R},$$

$$(2) \quad \sum_{\beta} u(\alpha,\beta)u(\beta,\gamma) = 2\delta_{\alpha,\gamma},$$

$$(3) \quad u(\alpha,1+\alpha) = 1, \tag{2}$$

$$(4) \quad u(\alpha,\beta+\gamma) = u(\alpha,\beta)u(\alpha,\gamma),$$

where α, β, γ run over $\mathbb{Z}_2$. Denote $u_\alpha^\beta := u(\alpha,\beta)/\sqrt{2}$ and define another orthonormal basis $\{\psi_\alpha\}_\alpha$ in $\mathbb{C}^2$: $\psi_\alpha := \Sigma_\beta u_\alpha^\beta \varphi_\beta$.

We shall associate to the mentioned graph and a fixed choice of an orthonormal basis $\{\varphi_\alpha\}_\alpha$ a set A of 120 projective lines in $\mathcal{H}$ expressed via the functions φ_α and the matrix u. It means that there will be a complete set associated to every vertex of the graph (this gives four complete sets), to every edge (this gives six more complete sets), to every face (this gives four complete sets) and to the whole tetrahedron (this gives one complete set). In what follows we use the notation: write $\hat{e}$ instead of $\{e\}$ to refer to a vertex, write ef instead of $\{e,f\}$ to denote an edge, write efg or $\overline{h}$ to denote the face $\{e,f,g\}$, and write $efgh$ instead of $\{e,f,g,h\}$.

Let us start with the description of the complete sets corresponding to vertices. Denote a set of all edges having a common vertex $v \in V$ by E_v, and consider a set

$$S_{\hat{v}} := \mathrm{Maps}(E_v \to \mathbb{Z}_2), \quad v \in V.$$

This set is not empty and contains eight elements. Take any v, say $v=e$, and take any $\sigma \in S_{\hat{e}}$. The labels of the edges eh, eg, and ef from E_e are of the form φ^1, ψ^2, and ψ^3, respectively. Associate to this fact a vector $\Psi_\sigma^{(e)} := \varphi_{\sigma(eh)} \otimes \psi_{\sigma(eg)} \otimes \psi_{\sigma(ef)}$. The vectors $\Psi_\sigma^{(v)}$ for other $v \in V$ are defined in a similar way. As a result one gets for every vertex v an orthonormal basis $\{\Psi_\sigma^{(v)}\}_{\sigma \in S_{\hat{v}}}$. Denote $\Psi_\sigma^v := \mathbb{C}\Psi_\sigma^{(v)}$. It follows, that one has four complete sets of projective lines associated to each vertex. One defines $\xi(\hat{v}) := \{\Psi_\sigma^v\}_{\sigma \in S_{\hat{v}}}$, $v \in V$.

The complete sets associated to edges make use of a slightly more sophisticated indexing. We denote by $E := \{ef, eg, eh, fg, fh, gh\}$ the set of all edges. To define a complete set of projective lines corresponding to an edge, one first takes an arbitrary *ordered* pair of two distinct vertices (v,w). Let z and t denote the remaining two vertices of the graph. Consider a disjoint union of a one-element set $\{vw\}$ and a two-element set $\{z,t\}$ and denote

$$K_{vw} := \mathrm{Maps}(\{vw\} \sqcup \{z,t\} \to \mathbb{Z}_2).$$

Note, that $\#K_{vw}=8$. The complete set of projective lines corresponding to the edge vw will be indexed by the elements of K_{vw}. Take $v=e$, $w=f$ and take arbitrary $\varkappa \in K_{ef}$. Recall that the labels of the edges eh, eg, and ef from E_e are of the form φ^1, ψ^2, and ψ^3, respectively. Denote

$$X_\varkappa^{(e\to f)} := \sum_{\alpha,\mu \in \mathbb{Z}_2} u_\alpha^\mu u_\alpha^{\varkappa(h)} u_\mu^{\varkappa(g)} \varphi_\alpha \otimes \psi_\mu \otimes \psi_{\varkappa(ef)}.$$

One defines the vectors $X_\varkappa^{(v\to w)}$, $\varkappa \in K_{vw}$, corresponding to every ordered pair (v,w), $v \neq w$, in a similar way. Using the properties of the matrix u, one can prove that

(1) the vectors $X_\varkappa^{(v\to w)}$, $\varkappa \in K_{vw}$, are pairwise orthogonal,
(2) the projective lines $\mathbb{C}X_\varkappa^{(v\to w)}$ and $\mathbb{C}X_\varkappa^{(w\to v)}$ coincide.

The latter fact motivates the notation $X_\varkappa^{vw} := \mathbb{C}X_\varkappa^{(v\to w)}$. The first fact implies, that $\{X_\varkappa^{vw}\}_{\varkappa \in K_{vw}}$ is a complete set of projective lines. It follows, that one gets six complete sets of projective lines associated to each edge of the graph. One defines $\xi(\epsilon) := \{X_\varkappa^\epsilon\}_{\varkappa \in K_\epsilon}$, $\epsilon \in E$.

Let us construct the complete sets associated to faces of the tetrahedron. Actually, every face is determined by an opposite vertex of the tetrahedron, and in this sense the complete sets to be constructed can be viewed as associated to vertices. It is simply natural to view them as associated to faces since the role of complete sets associated to vertices is already occupied. Denote

$$R_{\overline{v}} := \mathrm{Maps}(E \setminus E_v \to \mathbb{Z}_2), \quad v \in V.$$

This set is not empty, contains eight elements, and moreover, there is a natural bijection $\nu_v : R_{\overline{v}} \xrightarrow{\sim} S_{\hat{v}}$ established by the formula $\nu_v(\rho)(vw) = \rho(zt)$, where z and t are the vertices complementing $\{v, w\}$ up to V. If there is no risk of confusion, we write ρ^* instead of $\nu_v(\rho)$, $\rho \in R_{\overline{v}}$, as well as σ^* instead of $\nu_v^{-1}(\sigma)$, $\sigma \in S_{\overline{v}}$. Now take $v = e$ and any $\rho \in R_{\overline{e}}$. Recall that the labels of the edges eh, eg, and ef from E_e are of the form φ^1, ψ^2, and ψ^3, respectively. Denote

$$\Phi_\rho^{(e\to f)} := \sum_{\alpha,\mu\in\mathbb{Z}_2} u_\alpha^{\rho^*(eh)+\rho^*(ef)} u_\mu^{\rho^*(eg)+\rho^*(ef)} \varphi_\alpha \otimes \psi_\mu \otimes \psi_{\Delta_\rho+\alpha+\mu},$$

where $\Delta_\rho := \Sigma_{\epsilon\in E\setminus E_e}\rho(\epsilon)$. The vectors $\Phi_\rho^{(v\to w)}$ corresponding to other choices of $v, w \in V, v \neq w$ are defined in a similar way. The properties of the matrix u imply that the vectors $\Phi_\rho^{(v\to w)}, \rho \in R\overline{v}$, are pairwise orthogonal and that $\mathbb{C}\Phi_\rho^{(v\to w)}$ does not depend on the choice of w. Denote, $\Phi_\rho^v := \mathbb{C}\Phi_\rho^{(v\to w)}$. It follows, that to every face $\overline{v}$ one associates a complete set of projective lines $\{\Phi_\rho^v\}_{\rho\in R_{\overline{v}}}$. One defines $\xi(\overline{v}) := \{\Phi_\rho^v\}_{\rho\in R_{\overline{v}}}$, $v \in V$.

Let us finally associate a complete set of projective lines to the whole tetrahedron. Denote

$$\Lambda := \left\{ \pi : V \to \mathbb{Z}_2 \middle| \sum_{v\in V} \pi(v) = 1 \right\}.$$

Note, that $\#\Lambda = 8$. Take an *ordered* pair $(e, f) \in V \times V$. Recall that the labels of the edges eh, eg, and ef from E_e are of the form φ^1, ψ^2, and ψ^3, respectively,

$$F_\pi^{(e\to f)} := \sum_{\alpha,\mu\in\mathbb{Z}_2} u_{\pi(h)}^{\pi(g)+\mu} u_\alpha^{\pi(g)+\mu} \varphi_\alpha \otimes \psi_\mu \otimes \psi_{\pi(e)+\alpha+\mu}.$$

Note, that using the properties of the matrix u one can prove that the expression $u_a^{m+\mu} u_\alpha^{m+\mu}$ remains invariant under the transposition $(a, \alpha) \rightleftarrows (m, \mu)$. The vectors $F_\pi^{(u\to v)}$ corresponding to other ordered pairs are defined in a similar way. Using the properties of u, one can prove, that

(1) the vectors $F_\pi^{(v\to w)}$, $\pi \in \Lambda$, are pairwise orthogonal,
(2) the projective line $\mathbb{C}F_\pi^{(v\to w)}$ does not depend on the choice of the ordered pair (v, w).

The latter fact motivates the notation $F_\pi := \mathbb{C}F_\pi^{(v\to w)}$. The first fact implies, that $\{F_\pi\}_{\pi\in\Lambda}$ is a complete set of projective lines. One defines $\xi(efgh) := \{F_\pi\}_{\pi\in\Lambda}$. This completes the definition of ξ.

Note that the projective lines of the form Ψ_σ^v and F_π have been introduced in Ref. 8, but the set $\{F_\pi\}_\pi$ was not viewed as a complete set associated to the whole tetrahedron (1), since the projective lines of the form $X_\varkappa^{vw}$ and Φ_ρ^v did not exist. One can find the calculations illustrating the mentioned properties of F_π in Ref. 8. The definition of $X_\varkappa^{vw}$ and Φ_ρ^v is new.

Now we have a map $\xi : \mathcal{P}(V)^\times \to C(\mathbb{P}(\mathcal{H}))$. One verifies, that all the described projective lines are distinct. It means, that one gets 15 disjoint complete sets of projective lines in $\mathcal{H}$, for every vertex $v \in V$ a set $\{\Psi_\sigma^v\}_{\sigma\in S_{\hat{v}}}$; for every edge $\epsilon \in E$ a set $\{X_\varkappa^\epsilon\}_{\varkappa\in K_\epsilon}$; for every face $\overline{v}$, $v \in V$, a set $\{\Phi_\rho^v\}_{\rho\in R_{\overline{v}}}$; and for the whole tetrahedron a set $\{F_\pi\}_{\pi\in\Lambda}$. The set $A := \sqcup_{U\in Im(\xi)} U$ has a cardinality $\#A = 120$. We claim, that the set A is saturated with respect to the orthogonality relation $\perp$ and produces a Kochen–Specker-type contradiction.

IV. RELATIONS BETWEEN THE PROJECTIVE LINES

Let us describe the orthogonality relations between the elements of the set A. All these relations follow from the properties (2) of the matrix u. Recall that if $\rho \in R_{\overline{v}}$, then one denotes by ρ^* its image under the natural bijection $\nu_v : R_{\overline{v}} \xrightarrow{\sim} S_{\hat{v}}$. Simillarly, if $\sigma \in S_{\hat{v}}$, one writes σ^* instead of

$\nu_v^{-1}(\sigma) \in R_{\bar{v}}$. If $\sigma \in S_{\hat{v}}$, let $\nabla_\sigma := \sum_{\epsilon \in E_v} \sigma(\epsilon)$. If $\varkappa \in K_\epsilon$, say $\epsilon = ef$, then let $\tilde{\varkappa}$ denote an element of K_{ef} defined by $\tilde{\varkappa}(ef) = 1 + \varkappa(ef) + \varkappa(g) + \varkappa(h)$, $\tilde{\varkappa}(g) = \varkappa(h)$, $\tilde{\varkappa}(h) = \varkappa(g)$. For other ϵ the notation $\tilde{\varkappa}$ for $\varkappa \in K_\epsilon$ is defined in a similar way. Note that $\rho^{**} = \rho$, $\sigma^{**} = \sigma$, and $\tilde{\tilde{\varkappa}} = \varkappa$.

We explicitly describe part of the relations. The others are obtained by permutation of the symbols e, f, g, and h,

(1) $\Psi^e_\sigma \perp \Psi^e_{\sigma'}$ iff $\sigma(\cdot) \neq \sigma'(\cdot)$,
(2) $\Psi^e_\sigma \perp \Psi^f_{\sigma_1}$ iff $\sigma(ef) = 1 + \sigma_1(ef)$,
(3) $\Phi^e_\rho \perp \Phi^e_{\rho'}$ iff $\rho^*(\cdot) \neq \rho'^*(\cdot)$,
(4) $\Phi^e_\rho \perp \Phi^f_{\rho_1}$ iff $\rho^*(ef) = 1 + \rho_1^*(ef)$,
(5) $X^{ef}_\varkappa \perp X^{ef}_{\varkappa'}$ iff $\varkappa(\cdot) \neq \varkappa'(\cdot)$,
(6) $X^{ef}_\varkappa \perp X^{eg}_{\varkappa_1}$ iff $\varkappa(ef) + \varkappa(h) = 1 + \varkappa_1(eg) + \varkappa_1(h)$,
(7) $X^{ef}_\varkappa \perp X^{gh}_{\varkappa_1}$ iff $\varkappa(g) + \varkappa(h) = \varkappa_1(e) + \varkappa_1(f)$,
(8) $F_\pi \perp F_{\pi'}$ iff $\pi(\cdot) \neq \pi'(\cdot)$,
(9) $\Psi^e_\sigma \perp \Phi^e_\rho$ iff $\nabla_\sigma = 1 + \nabla_{\rho^*}$,
(10) $\Psi^e_\sigma \perp \Phi^f_\rho$ iff $\nabla_\sigma + \sigma(ef) = 1 + \nabla_{\rho^*} + \rho^*(ef)$,
(11) $\Psi^e_\sigma \perp X^{ef}_\varkappa$ iff $\sigma(ef) = 1 + \varkappa(ef)$,
(12) $\Psi^e_\sigma \perp X^{fg}_\varkappa$ iff $\sigma(ef) + \sigma(eg) = 1 + \tilde{\varkappa}(e)$,
(13) $\Phi^e_\rho \perp X^{ef}_\varkappa$ iff $\rho^*(ef) = 1 + \tilde{\varkappa}(ef)$,
(14) $\Phi^e_\rho \perp X^{fg}_\varkappa$ iff $\rho^*(ef) + \rho^*(eg) = 1 + \varkappa(e)$,
(15) $\Psi^e_\sigma \perp F_\pi$ iff $\nabla_\sigma = 1 + \pi(e)$,
(16) $\Phi^e_\rho \perp F_\pi$ iff $\nabla_{\rho^*} = 1 + \pi(e)$,
(17) $X^{ef}_\varkappa \perp F_\pi$ iff $\varkappa(g) + \varkappa(h) = 1 + \pi(g) + \pi(h)$.

Note, that there is no 1 in the formula (7). Note that these relations have a self-duality property. Namely, the condition for orthogonality in (5) is equivalent to $\tilde{\varkappa}(\cdot) \neq \tilde{\varkappa}'(\cdot)$, the condition in (6) is equivalent to $\tilde{\varkappa}(ef) + \tilde{\varkappa}(h) = 1 + \tilde{\varkappa}_1(eg) + \tilde{\varkappa}_1(h)$, the condition in (7) is equivalent to $\tilde{\varkappa}(g) + \tilde{\varkappa}(h) = \tilde{\varkappa}_1(e) + \tilde{\varkappa}_1(f)$, and the condition in (17) is equivalent to $\tilde{\varkappa}(g) + \tilde{\varkappa}(h) = 1 + \pi(g) + \pi(h)$. It follows, that if one has a set of pairwise orthogonal projective lines of the form $\{\Psi^{v_i}_{\sigma_i}\}_{i \in I} \cup \{X^{\epsilon_j}_{\varkappa_j}\}_{j \in J} \cup \{\Phi^{w_l}_{\rho_l}\}_{l \in L} \cup \{F_{\pi_m}\}_{m \in M}$, where I, J, L, M are some index sets, $v_i, w_l \in V$, $\epsilon_j \in E$, then by replacing $\Psi^{v_i}_{\sigma_i}$ with $\Phi^{v_i}_{\sigma_i^*}$, $\Phi^{w_l}_{\rho_l}$ with $\Psi^{w_l}_{\rho_l^*}$ and $X^{\epsilon_j}_{\varkappa_j}$ with $X^{\epsilon_j}_{\tilde{\varkappa}_j}$, one obtains a set of projective lines $\{\Phi^{v_i}_{\sigma_i^*}\}_{i \in I} \cup \{X^{\epsilon_j}_{\tilde{\varkappa}_j}\}_{j \in J} \cup \{\Psi^{w_l}_{\rho_l^*}\}_{l \in L} \cup \{F_{\pi_m}\}_{m \in M}$, which are still pairwise orthogonal. It follows, that one has a map $\delta : A \to A$, $\delta^2 = id$, which respects the orthogonality relation $\perp$. Call $\delta \in Bij_\perp(A)$ the *duality map*.

V. LINK TO MERMIN'S EXAMPLE

Let us prove that the set A produces a Kochen–Specker-type contradiction and establish the link with the example of Mermin. Denote

$$\Gamma(A) := \{\phi : C(A) \to A \mid \forall\, B \in C(A) : \phi(B) \in B\},$$

$$\Delta(A) := \{\phi \in \Gamma(A) \mid \forall\, B, B' \in C(A) : B \neq B' \Rightarrow \neg\,(\phi(B) \perp \phi(B'))\}.$$

One must show, that $\Delta(A) = \emptyset$. Suppose the contrary, $\Delta(A) \neq \emptyset$. Denote $B_v := \{\Psi^v_\sigma\}_{\sigma \in S_{\hat{v}}}$ ($v \in V$), $\hat{B} := \{F_\pi\}_{\pi \in \Lambda}$. Take $\phi \in \Delta(A)$. The definition of $\Gamma(A) \supset \Delta(A)$ implies, that for every $v \in V$ one has an element $\phi(B_v) \in B_v$, i.e., $\phi(B_v) = \Psi^v_{\sigma^\phi_v}$, where σ^ϕ_v is some element of $S_{\hat{v}}$. Similarly, $\phi(\hat{B}) = F_{\pi^\phi}$, where π^ϕ is some element of Λ. The definition of $\Delta(A)$ implies, that $\Psi^v_{\sigma^\phi_v}$ is not orthogonal to $\Psi^w_{\sigma^\phi_w}$ (for any $v \neq w$). Using the orthogonality relations one concludes, that $\sigma^\phi_v(vw) = \sigma^\phi_w(vw)$. It means, that a set of functions $\{\sigma^\phi_v\}_{v \in V}$ induces a function $\tau^\phi : E \to \mathbb{Z}_2$ by the formula $\tau^\phi(vw) := \sigma^\phi_v(vw) = \sigma^\phi_w(vw)$ (for any $vw \in E$). Now invoke the fact, that the definition of

$\Delta(A)$ also implies, that for every $v\in V$ the line $\phi(\hat{B})$ should not be orthogonal to $\phi(B_v)$, i.e., $\neg(F_{\pi^\phi}\perp\Psi_{\sigma_v^\phi})$. It follows, that $\forall v\in V:\nabla_{\sigma_v^\phi}=\pi^\phi(v)$. Taking the sum over all $v\in V$ and invoking the definition of Λ, one gets $\Sigma_{v\in V}\nabla_{\sigma_v^\phi}=\Sigma_{v\in V}\pi^\phi(v)=1$. On the other hand,

$$\sum_{v\in V}\nabla_{\sigma_v^\phi}=\sum_{v\in V}\sum_{\epsilon\in E_v}\sigma_v^\phi(\epsilon)=\sum_{\epsilon\in E}(\tau^\phi(\epsilon)+\tau^\phi(\epsilon))=0.$$

Thus one arrives to a contradiction $0=1$. It means that $\nexists\phi\in\Delta(A)$, i.e., $\Delta(A)=\emptyset$.

The link with Mermin's example is established as follows. Let the standard basis in $\mathbb{C}^2$ play the role of the basis $\{\varphi_\alpha\}_{\alpha\in\mathbb{Z}_2}$ involved in the construction of A. Note, that in the proof of $\Delta(A)=\emptyset$ we have used only five complete subsets, B_v $(v=e,f,g,h)$ and $\hat{B}$. The proof of no-hidden-variables theorem by Mermin is given in terms of operators in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$. There are five complete sets of pairwise commuting orthogonal projectors present in that proof. If one looks at the one-dimensional eigenspaces (i.e., the projective lines) associated to each of these sets, one obtains $B_e\sqcup B_f\sqcup B_g\sqcup B_h\sqcup\hat{B}$.

VI. SATURATION PROPERTY, PART 1

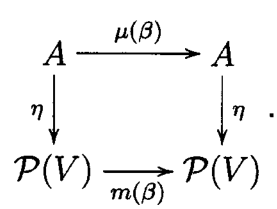
The proof that the set A is saturated with respect to $\perp$ is more bulky. Recall that we have an injective map $\xi:\mathcal{P}(V)^\times\rightarrowtail C(\mathbb{P}(\mathcal{H}))$, where V is a set of 4 symbols $V=\{e,f,g,h\}$. This map has a property $\forall U,U'\in\mathcal{P}(V)^\times,U\neq U':\xi(U)\cap\xi(U')=\emptyset$. It follows, that one can define a map $\eta:A\to\mathcal{P}(V)$ as follows: one chooses $U\in\mathcal{P}(V)^\times$ to be the value $\eta(l)$ of the map η on a projective line $l\in A$ whenever $l\in\xi(U)$, i.e., η is defined from the requirement $\forall U\in\mathcal{P}(V)^\times:l\in\xi(U)\Leftrightarrow\eta(l)=U$. Note, that η induces a surjection onto $\mathcal{P}(V)^\times$.

The projective lines constituting A may be classified as follows. Call $\#\eta(l)$ the *type* of the projective line $l\in A$. There are four types of projective lines. The image $\eta(l)$ is termed the *kind* of the line l. There are 4 kinds in type 1, 6 kinds in type 2, 4 kinds in type 3, and 1 kind in type 4. We shall also refer to projective lines of the types 1, 2, 3, 4, as being projective lines of Ψ type, X type, Φ type, and F type, respectively. In a similar way, if $\eta(l)=\hat{e}$, the line l is said to be of Ψ^e kind, if $\eta(l)=ef$, the line l is said to be of X^{ef} kind, etc.

Naively, in order to prove the saturation property for A one may think of having to do the following: one must take every subset B of A and test if its elements are pairwise orthogonal; if it happens to be so, one must find a complete subset in A containing B. All this appears to be a very boring problem since $\#\mathcal{P}(A)=2^{120}$. There is of course a group of permutations S_4 acting on $\mathcal{P}_\perp(A)$ and an observation about the existence of the duality map δ, but the $\#S_4$ is just $4!=24$ and the order of δ as an element of $Bij_\perp(A)$ is just 2, i.e., $\delta^2=id$. It means, that one must find a more sophisticated approach to prove the saturation.

We have a map $\eta:A\to\mathcal{P}(V)$. It induces a map $\mathcal{P}(\eta):\mathcal{P}(A)\to L:=\mathcal{P}(\mathcal{P}(V))$. How to describe an image of the composition $\mathcal{P}_\perp(A)\rightarrowtail\mathcal{P}(A)\xrightarrow{\mathcal{P}(\eta)}L$, where the first arrow is the canonical injection?

There exists a natural monomorphism of groups m: $Bij(V)\rightarrowtail Bij(\mathcal{P}(V))$, $\beta\mapsto\mathcal{P}(\beta)$. There also exists a natural monomorphism μ: $Bij(V)\rightarrowtail Bij(A)$, such that for every $\beta\in Bij(V)$ there is a commutative diagram



The monomorphism μ is defined as follows. Take any $\beta\in\mathcal{P}(V)$ and describe explicitly the values of $\mu(\beta)(\cdot)$ on the elements of the form Ψ_σ^v, $X_\varkappa^\epsilon$, Φ_ρ^v, and F_π, where $v\in V$, $\epsilon\in E$, $\sigma\in S_{\hat{v}}$, $\varkappa\in K_\epsilon$, $\rho\in R_{\bar{v}}$, and $\pi\in\Lambda$. Let $\mu(\beta)(\Psi_\sigma^v):=\Psi_{\sigma'}^{v'}$, where $v'=\beta(v)$ and $\forall\epsilon_1\in E_{v'}:\sigma'(\epsilon_1)=\sigma(\mathcal{P}(\beta^{-1})(\epsilon_1))$;

$\mu(\beta)(X^{\epsilon}_{\varkappa}) := X^{\epsilon'}_{\varkappa'}$, where $\epsilon' = \mathcal{P}(\beta)(\epsilon)$ and $\varkappa'(\epsilon') = \varkappa(\epsilon)$, $\forall v_1 \in V \setminus \epsilon' : \varkappa'(v_1) = \varkappa(\beta^{-1}(v_1))$; $\mu(\beta)(\Phi^{v}_{\rho}) := \Phi^{v'}_{\rho'}$, where $v' = \beta(v)$ and $\forall \epsilon_1 \in E_{v'} : \rho'^{*}(\epsilon_1) = \rho^{*}(\mathcal{P}(\beta^{-1})(\epsilon_1))$; finally $\mu(\beta)(F_{\pi}) := F_{\pi'}$, where $\forall v_1 \in V : \pi'(v_1) = \pi(\beta^{-1}(v_1))$. Note, that for every $\beta \in Bij(V)$ the map $\mu(\beta)$ respects the orthogonality relation $\perp$ on A, and the fact that this relation has a symmetry with respect to the group of permutations of e, f, g, and h can be expressed as

$$\forall \beta \in Bij(V)\ \forall l, l' \in A : l \perp l' \Rightarrow \mu(\beta)(l) \perp \mu(\beta)(l').$$

It follows, that whenever an element $Q \in L$ stems from some element $B \in \mathcal{P}_{\perp}(A)$, the element of the form $\mathcal{P}(m(\beta))(Q)$, $\beta \in Bij(V)$, also stems from an element of $\mathcal{P}_{\perp}(A)$, namely from $\mathcal{P}(\mu(\beta))(B)$. Factorize L with respect to the equivalence relation $\sim$ induced by permutations,

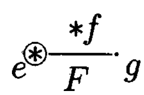
$$Q \sim Q' :\Leftrightarrow \exists\, \beta \in Bij(V) : \mathcal{P}(m(\beta))(Q) = Q'.$$

Denote $\Gamma := L/\sim$. The problem of the description of the image of $\mathcal{P}_{\perp}(A)$ in L is then reduced to describing the image of the composition

$$\mathcal{P}_{\perp}(A) \rightarrowtail \mathcal{P}(A) \xrightarrow{\mathcal{P}(\eta)} L \twoheadrightarrow \Gamma,$$

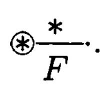
where the first arrow is the canonical injection and the last arrow is the canonical surjection.

It is convenient to introduce graphical notation for the elements of L and Γ. Consider an example. Let $Q = \{\hat{e}, \hat{f}, \overline{e}, eg, efgh\} \in L$. It is represented by a graph,



The general principle is the following. A graph may have up to four vertices labeled by the symbols e, f, g or h. If Q contains $\hat{v}$, introduce a vertex labeled by the symbol v and mark it with *; if Q contains $\overline{v}$, introduce a vertex v and draw a circle around it; if there is $vw \in Q$, introduce two vertices v and w and connect them by an edge; finally if Q contains $efgh$, set a letter F near the corresponding figure. Thus to each $Q \in L$ a graph is associated. Note, that if an element $Q \in L$ stems from some element $B \in \mathcal{P}(A)$, i.e., $\mathcal{P}(\eta)(B) = Q$, then by looking at a graph that represents Q one cannot tell everything about B, but the kinds of projective lines that are present in B can be understood.

The graphs representing the elements of Γ are similar to the graphs representing the elements of L. They are obtained by omitting the labels e, f, g, and h of the vertices. For instance, if $Q \in L$ is as in the example given above, and $[Q] \in \Gamma$ is its image under the natural surjection $L \twoheadrightarrow \Gamma$, then $[Q]$ is represented by the graph



Whenever one has an element of Γ of the form $[\mathcal{P}(\eta)(B)]$, where B is some subset of A, the graph that represents this element is called a *shadow* of B.

Let us introduce more terminology. A graph representing an element of Γ is called *admissible* iff by definition it represents an element of the image of $\mathcal{P}_{\perp}(A) \to \Gamma$; otherwise it is called *non-admissible*. One would like to describe all the admissible graphs. Whenever a graph represents an image of some $Q \in L$ under the canonical surjection $L \twoheadrightarrow \Gamma$, the cardinality $\#Q$ is called the *degree* of this graph. Whenever a graph represents an image of some $B \in \mathcal{P}_{\perp}(A)$ under $\mathcal{P}_{\perp}(A) \to \Gamma$, one says that B *hangs over* this graph. Any $B' \in P_{\perp}(A)$ containing B is called an *extension* of B. It is called a *pure* extension, iff by definition B' and B hang over the same graph. An extension satisfying $B' = B$ is called *trivial*. An extension B' of B is called *complete* iff by definition $\#B' = 8$ [recall that $8 = \dim_{\mathbb{C}}(\mathcal{H})$, $A \subset \mathbb{P}(\mathcal{H})$].

Proposition 1: (1) *The graph* $*$ *is admissible and any element of* $\mathcal{P}_\perp(A)$ *hanging over this graph admits a pure complete extension.*

(2) *The graph* $**$ *is admissible and any element of* $\mathcal{P}_\perp(A)$ *hanging over this graph admits a pure complete extension.*

(3) *The graph* $\triangle$ *is not admissible.*

(4) *The graph* $***$ *is admissible and any element of* $\mathcal{P}_\perp(A)$ *hanging over this graph admits a pure extension up to a set of cardinality 6. Any element of* $\mathcal{P}_\perp(A)$ *of cardinality 6, which hangs over this graph, has a complete extension hanging over* $***\odot$.

(5) *The graph* $***\odot$ *is admissible and any element of* $\mathcal{P}_\perp(A)$ *hanging over this graph admits a pure complete extension.*

(6) *The graph* $****$ *is admissible and any element of* $\mathcal{P}_\perp(A)$ *hanging over this graph does not have nontrivial pure extensions. Any element of* $\mathcal{P}_\perp(A)$ *hanging over this graph has a complete extension hanging over* $\circledast\circledast\circledast\circledast$.

(7) *The graph* $****F$ *is not admissible.*

Proof: (1) A set consisting of one projective line Ψ^e_σ, where σ is some element of $S_{\hat{e}}$, gives an example of a set hanging over $*$. Every set hanging over this graph is of the form $B=\{\Psi^v_\sigma\}_{\sigma\in S}$, where v is some element of V and S is some nonempty subset of $S_{\hat{v}}$. This subset B is always a subset of a complete set $\{\Psi^v_\sigma\}_{\sigma\in S_{\hat{v}}}$.

(2) Take any $a\in\mathbb{Z}_2$ and choose any $\sigma\in S_{\hat{e}}$ such that $\sigma(ef)=a$, and any $\sigma_1\in S_{\hat{f}}$ such that $\sigma_1(ef)=1+a$. Then the projective lines Ψ^e_σ and $\Psi^f_{\sigma_1}$ are orthogonal and one can take them two as a set which hangs over the graph $**$. An arbitrary set B hanging over this graph is always of the form $B=\{\Psi^v_\sigma\}_{\sigma\in S}\cup\{\Psi^w_{\sigma_1}\}_{\sigma_1\in S_1}$, where S and S_1 are some nonempty subsets of $S_{\hat{v}}$ and $S_{\hat{w}}$, respectively, v, $w\in V$, $v\neq w$. One associates to B a parameter $a:=\sigma(vw)=1+\sigma_1(vw)$, where σ is any element of S and σ_1 is any element of S_1. Denote $S':=\{\sigma\in S_{\hat{v}}|\sigma(vw)=a\}$ and $S'_1:=\{\sigma_1\in S_{\hat{w}}|\sigma_1(vw)=1+a\}$. Since $\#S'=\#S'_1=4$, the set $B':=\{\Psi^v_\sigma\}_{\sigma\in S'}\cup\{\Psi^w_{\sigma_1}\}_{\sigma_1\in S'_1}$ gives the required pure complete extension of B.

(3) If the graph $\triangle$ is admissible, then there should exist three pairwise orthogonal projective lines of the form $X^{ef}_\varkappa$, $X^{eg}_{\varkappa_1}$ and $X^{fg}_{\varkappa_3}$, where $\varkappa$, $\varkappa_1$, and $\varkappa_2$ are some elements of K_{ef}, K_{eg}, and K_{fg}, respectively. The orthogonality relations yield three equations,

$$X^{ef}_\varkappa\perp X^{eg}_{\varkappa_1}\Leftrightarrow\varkappa(ef)+\varkappa(h)=1+\varkappa_1(eg)+\varkappa_1(h),$$

$$X^{ef}_\varkappa\perp X^{fg}_{\varkappa_2}\Leftrightarrow\varkappa(ef)+\varkappa(h)=1+\varkappa_2(fg)+\varkappa_2(h),$$

$$X^{eg}_{\varkappa_1}\perp X^{fg}_{\varkappa_2}\Leftrightarrow\varkappa_1(eg)+\varkappa_1(h)=1+\varkappa_2(fg)+\varkappa_2(h).$$

Taking the sum of these three equations one arrives to a contradiction $0=1$. This means, that the mentioned triangle is not admissible.

(4) One can construct an example of three pairwise orthogonal lines of the form Ψ^e_σ, $\Psi^f_{\sigma_1}$, $\Psi^g_{\sigma_2}$ as follows: take any $a,b,c\in\mathbb{Z}_2$ and choose $\sigma\in S_{\hat{e}}$ such that $\sigma(ef)=a$, $\sigma(eg)=b$, any $\sigma_1\in S_{\hat{f}}$ such that $\sigma_1(ef)=1+a$, $\sigma_1(fg)=c$, and $\sigma_2\in S_{\hat{g}}$ such that $\sigma_2(eg)=1+b$, $\sigma_2(fg)=1+c$. Then the orthogonality relations between the mentioned three lines are fulfilled and the set consisting of these three hangs over the graph $***$. An arbitrary set hanging over this graph is of the form $B=\{\Psi^v_\sigma\}_{\sigma\in S}\cup\{\Psi^w_\sigma\}_{\sigma_1\in S_1}\cup\{\Psi^z_\sigma\}_{\sigma_2\in S_2}$, where S, S_1, and S_2 are some nonempty subsets of $S_{\hat{v}}$, $S_{\hat{w}}$, and $S_{\hat{z}}$, respectively, $v,w,z\in V$, $v\neq w$, $v\neq z$, $w\neq z$. Without loss of generality one may specialize v, w and z to e, f, and g, respectively. Associate to B three parameters $a,b,c\in\mathbb{Z}_2$: $a:=\sigma(ef)=1+\sigma_1(ef)$, $b:=\sigma(eg)=1+\sigma_2(eg)$, $c:=\sigma_1(fg)=1+\sigma_2(fg)$, where σ, σ_1 and σ_2 are elements of S, S_1, and S_2, respectively. Denote $S':=\{\sigma\in S_{\hat{e}}|\sigma(ef)=a\&\sigma(eg)=b\}$, $S'_1:=\{\sigma_1\in S_{\hat{f}}|\sigma_1(ef)=1+a\&\sigma_1(fg)=c\}$, and $S'_2:=\{\sigma_2\in S_{\hat{g}}|\sigma_2(eg)=1+b\&\sigma_2(fg)=1+c\}$. Since $\#S'=\#S'_1=\#S'_2=2$, the set

$B' := \{\Psi^v_\sigma\}_{\sigma\in S'} \cup \{\Psi^w_\sigma\}_{\sigma_1\in S_1'} \cup \{\Psi^z_\sigma\}_{\sigma_2\in S_2'}$ is a pure extension of B up to a set of cardinality 6. Now look for a projective line Φ^h_ρ, $\rho\in R_h^-$, which is orthogonal to every element of B'. This yields the following equations:

$$\forall\,\sigma \in S' : \nabla_\sigma + \sigma(eh) = 1 + \nabla_{\rho^*} + \rho^*(eh),$$

$$\forall\,\sigma_1 \in S_1' : \nabla_{\sigma_1} + \sigma_1(fh) = 1 + \nabla_{\rho^*} + \rho^*(fh),$$

$$\forall\,\sigma_2 \in S_2' : \nabla_{\sigma_2} + \sigma_2(gh) = 1 + \nabla_{\rho^*} + \rho^*(gh).$$

Observe that the left-hand sides of these equations may be expressed in terms of parameters a, b, and c as $\nabla_\sigma + \sigma(eh) = \sigma(ef) + \sigma(eg) = a + b$, $\nabla_{\sigma_1} + \sigma_1(fh) = \sigma_1(ef) + \sigma_1(fg) = (1+a) + c$, and $\nabla_{\sigma_2} + \sigma_2(gh) = \sigma_1(eg) + \sigma_1(fg) = (1+b) + (1+c)$. Reduce the equations for ρ to $\rho^*(fh) + \rho^*(gh) = 1 + a + b$, $\rho^*(eh) + \rho^*(gh) = a + c$ and $\rho^*(eh) + \rho^*(fh) = 1 + b + c$. The latter equation is nothing but a sum of the first two and may be dropped. Denote $R' := \{\rho \in R_h^- \,|\, \rho^*(fh) + \rho^*(gh) = 1 + a + b \,\&\, \rho^*(eh) + \rho^*(gh) = a + c\}$. Taking into account that $\#R = 2$, one obtains a set $B'' := B' \cup \{\Phi^h_\rho\}_{\rho\in R'}$, which is a complete extension of B' and hangs over $**{*}\odot$.

(5) The admissibility of $**{*}\odot$ follows from (4). Consider any $B \in \mathcal{P}_\perp(A)$ hanging over this graph. Without loss of generality, one may assume, that $B = \{\Psi^e_\sigma\}_{\sigma\in S} \cup \{\Psi^f_\sigma\}_{\sigma_1\in S_1} \cup \{\Psi^g_\sigma\}_{\sigma_2\in S_2} \cup \{\Phi^h_\rho\}_{\rho\in R}$, where S, S_1, S_2, and R are some nonempty subsets of $S_{\hat e}$, $S_{\hat f}$, $S_{\hat g}$, and R_h^-, respectively. Consider a subset of B consisting of all projective lines of Ψ type and associate to it the parameters $a, b, c \in \mathbb{Z}_2$ in a way as described in the proof of (4). Let S', S_1', S_2', and R' be defined as in the proof of (4). Then the set $\tilde{B} := \{\Psi^e_\sigma\}_{\sigma\in S'} \cup \{\Psi^f_\sigma\}_{\sigma_1\in S_1'} \cup \{\Psi^g_\sigma\}_{\sigma_2\in S_2'} \cup \{\Phi^h_\rho\}_{\rho\in R'}$ is the required pure complete extension of B.

(6) A set of pairwise orthogonal projective lines Ψ^e_σ, $\Psi^f_{\sigma_1}$, $\Psi^g_{\sigma_2}$, and $\Psi^h_{\sigma_3}$, which is required to establish the admissibility of the graph $****$, can be constructed as follows. Take any $\mathbb{Z}_2$-valued function φ on $E := \{ef, eg, eh, fg, fh, gh\}$. Denote $a := \varphi(ef)$, $b := \varphi(eg)$, $c := \varphi(fg)$, $p := \varphi(eh)$, $q := \varphi(fh)$, $r := \varphi(gh)$. Take the following σ, σ_1, σ_2, and σ_3: $\sigma(ef) = a$, $\sigma(eg) = b$, $\sigma(eh) = p$; $\sigma_1(ef) = 1 + a$, $\sigma_1(fg) = c$, $\sigma_1(fh) = q$; $\sigma_2(eg) = 1 + b$, $\sigma_2(fg) = 1 + c$, $\sigma_2(gh) = r$; $\sigma_3(eh) = 1 + p$, $\sigma_3(fh) = 1 + q$, $\sigma_3(gh) = 1 + r$. Then the projective lines Ψ^e_σ, $\Psi^f_{\sigma_1}$, $\Psi^g_{\sigma_2}$, and $\Psi^h_{\sigma_3}$ constitute a set as required. An arbitrary set B hanging over the mentioned graph is of the form $B = \{\Psi^e_\sigma\}_{\sigma\in S} \cup \{\Psi^f_{\sigma_1}\}_{\sigma_1\in S_1} \cup \{\Psi^g_{\sigma_2}\}_{\sigma_2\in S_2} \cup \{\Psi^h_{\sigma_3}\}_{\sigma_3\in S_3}$, where S, S_1, S_2, and S_3 are some nonempty subsets of $S_{\hat e}$, $S_{\hat f}$, $S_{\hat g}$, and $S_{\hat h}$ respectively. To every such B associate φ: $E \to \mathbb{Z}_2$ by setting $\varphi(ef) = \sigma(ef)$, $\varphi(eg) = \sigma(eg)$, $\varphi(eh) = \sigma(eh)$, $\varphi(fg) = \sigma_1(fg)$, $\varphi(fh) = \sigma_1(fh)$, $\varphi(gh) = \sigma_2(gh)$, where σ, σ_1, and σ_2 can be taken to be any elements of S, S_1, and S_2, respectively. The set S is a nonempty subset of $S' := \{\sigma \in S_{\hat e} \,|\, \sigma(ef) = \varphi(ef) \,\&\, \sigma(eg) = \varphi(eg) \,\&\, \sigma(eh) = \varphi(eh)\}$. Since $\#S' = 1$, we see that $\#S = 1$. Similarly, $\#S_1 = \#S_2 = \#S_3 = 1$. In the latter case $S_3 \subset S_3' := \{\sigma_3 \in S_{\hat h} \,|\, \sigma_3(eh) = 1 + \varphi(eh) \,\&\, \sigma_3(fh) = 1 + \varphi(fh) \,\&\, \sigma_3(gh) = 1 + \varphi(gh)\}$. This means that a set hanging over $****$ cannot have nontrivial pure extensions.

Let us now construct a complete extension of a set B, where B hangs over $****$. Let φ denote the function associated to $B = \{\Psi^e_\sigma, \Psi^f_{\sigma_1}, \Psi^g_{\sigma_2}, \Psi^h_{\sigma_3}\} \in \mathcal{P}_\perp(A)$ as described above, and let a, b, c, p, q, and r be its values on the edges ef, eg, fg, eh, fh, and gh, respectively. Looking for an extension which hangs over $\circledast\circledast\circledast\circledast$, we need to construct projective lines of the form Φ^e_ρ, $\Phi^f_{\rho_1}$, $\Phi^g_{\rho_2}$, and $\Phi^h_{\rho_3}$. Define ρ, ρ_1, ρ_2, and ρ_3 by the formulas $\rho^*(ef) = a + b + c + p + q$, $\rho^*(eg) = 1 + a + b + c + p + r$, $\rho^*(eh) = a + b + p + q + r$; $\rho_1^*(ef) = 1 + a + b + c + p + q$, $\rho_1^*(fg) = a + b + c + q + r$, $\rho_1^*(fh) = 1 + a + c + p + q + r$; $\rho_2^*(eg) = a + b + c + p + r$, $\rho_2^*(fg) = 1 + a + b + c + q + r$, $\rho_2^*(gh) = b + c + p + q + r$; $\rho_3^*(eh) = 1 + a + b + p + q + r$, $\rho_3^*(fh) = a + c + p + q + r$, $\rho_3^*(gh) = 1 + b + c + p + q + r$. Straightforward computation establishes that $B' := B \cup \{\Phi^e_\rho, \Phi^f_{\rho_1}, \Phi^g_{\rho_2}, \Phi^h_{\rho_3}\}$ is a complete extension of B.

(7) It is necessary to show that five projective lines of the form Ψ^e_σ, $\Psi^f_{\sigma_1}$, $\Psi^g_{\sigma_2}$, $\Psi^h_{\sigma_3}$, and F_π cannot be pairwise *orthogonal*. Recall that we already know that they cannot be pairwise *nonorthogonal*. The conditions of orthogonality between the projective lines of Ψ type yield a system of

equations, $\sigma(ef)+\sigma_1(ef)=1$, $\sigma(eg)+\sigma_2(eg)=1$, $\sigma(eh)+\sigma_3(eh)=1$, $\sigma_1(fg)+\sigma_2(fg)=1$, $\sigma_1(fh)+\sigma_3(fh)=1$, and $\sigma_2(gh)+\sigma_3(gh)=1$. By summation, one obtains $\nabla_\sigma+\nabla_{\sigma_1}+\nabla_{\sigma_2}+\nabla_{\sigma_3}=0$. On the other hand, the orthogonality conditions with F_π yield the equations, $\pi(e)=1+\nabla_\sigma$, $\pi(f)=1+\nabla_{\sigma_1}$, $\pi(g)=1+\nabla_{\sigma_2}$, and $\pi(h)=1+\nabla_{\sigma_3}$. Recalling that $\Sigma_{v\in V}\pi(v)=1$, and summing the foregoing equations yields $1=\nabla_\sigma+\nabla_{\sigma_1}+\nabla_{\sigma_2}+\nabla_{\sigma_3}$. Hence, the requirement that all five projective lines are pairwise orthogonal leads to a contradiction $1=0$. □

Proposition 2: Let $B\in\mathcal{P}_\perp(A)$ be a set hanging over ∗∗∗∗.

(1) *For every $v\in V$ there exists a unique projective line of Φ^v-kind which is orthogonal to every projective line belonging to B.*

(2) *There exist no extensions of B which contain a projective line of X type or of F type.*

(3) *The complete extension of B is unique and hangs over* ⊛⊛⊛⊛.

Proof: (1) It is sufficient to consider $v=e$. If Ψ^e_σ, $\Psi^f_{\sigma_1}$, $\Psi^g_{\sigma_2}$, and $\Psi^h_{\sigma_3}$ are four pairwise orthogonal projective lines, then the requirement that Φ^e_ρ is orthogonal to each of them yields four equations, $\nabla_{\rho^*}=1+\nabla_\sigma$, $\nabla_{\rho^*}+\rho^*(ef)=1+\nabla_{\sigma_1}+\sigma_1(ef)$, $\nabla_{\rho^*}+\rho^*(eg)=1+\nabla_{\sigma_2}+\sigma_2(eg)$, and $\nabla_{\rho^*}+\rho^*(eh)=1+\nabla_{\sigma_3}+\sigma_3(eh)$. Expressing ∇_{ρ^*} from the first equation and substituting it into the other three, one finds the expressions for the values of ρ^* via σ, σ_1, σ_2, and σ_3.

(2) The fact that B cannot have an extension containing a projective line of F type follows from the nonadmissibility of the graph ∗∗∗∗F. Let us show that the graph ∗—∗ is nonadmissible. This will imply that an extension of B cannot contain an element of X type. Consider three projective lines of the form Ψ^e_σ, $\Psi^f_{\sigma_1}$, and $X^{ef}_\varkappa$, and impose the condition that they are pairwise orthogonal. This yields the equations, $\sigma(ef)=1+\sigma_1(ef)$, $\sigma(ef)=1+\varkappa(ef)$, and $\sigma_1(ef)=1+\varkappa(ef)$. The sum of the second and the third equations yields $\sigma(ef)+\sigma_1(ef)=0$, contradicting the first equation. It means that the mentioned graph is not admissible.

(3) The existence of the extension of B hanging over ⊛⊛⊛⊛ has been proved in the previous proposition. Since an extension of B cannot contain elements of X or F type, it should hang over a graph which may contain only stars and circles. According to the previous proposition, a set hanging over a graph ∗∗∗∗ cannot have nontrivial pure extensions. It follows, that a complete extension of B contains projective lines of the form Φ^v_ρ. Every such projective line is uniquely defined according to (1). It follows that a complete extension of B hangs over ⊛⊛⊛⊛ and is unique. □

VII. GROUP OF SYMMETRY

We have given an *explicit* description of every element of the finite set A and by that we have an opportunity to *construct* the maps φ: $A\to A$ by simply saying for each $l\in A$ which $l'\in A$ corresponds to it under φ. One would like to have a similar opportunity for the set $\mathcal{P}_\perp(A)$, i.e., one needs to *characterize* the elements of $\mathcal{P}_\perp(A)$. In particular, for the set of all complete sets $C(A)\subset\mathcal{P}_\perp(A)$ it would be nice to have some group transitively acting on $C(A)$, so that having found just one complete set, one could automatically generate all the others.

Recall that there is a map η: $A\to\mathcal{P}(V)$, where V is a set of four symbols $V=\{e,f,g,h\}$. We shall describe a group G which acts on the set $L=\mathcal{P}(\mathcal{P}(V))$ and then describe a group $\mathcal{G}$ which acts on $\mathcal{P}_\perp(A)$.

We start with the definition of the group G. Consider the group $Bij(\mathcal{P}(V))$ of all bijections of the power set of V. One has $\#V=4$, $\#\mathcal{P}(V)=16$, $\#Bij(\mathcal{P}(V))=16!$. Associate to each $S\in\mathcal{P}(V)$ a map T_S: $\mathcal{P}(V)\to\mathcal{P}(V)$ defined by the formula

$$T_S(U) := \begin{cases} U, & \text{if } \#(S\cap U) \text{ is even,} \\ \overline{U\Delta S}, & \text{if } \#(S\cap U) \text{ is odd,} \end{cases} \tag{3}$$

where U varies over $\mathcal{P}(V)$, the bar denotes the completion of a set in V and Δ denotes the symmetric difference of two subsets. For any S one has $T_S^2=id$. In particular, T_S is a bijection, $T_S\in Bij(\mathcal{P}(V))$. Define G to be a subgroup in $Bij(\mathcal{P}(V))$ generated by reflections T_S, $S\in\mathcal{P}(V)$:

$$G := \langle\{T_S|S\in\mathcal{P}(V)\}\rangle \subset Bij(\mathcal{P}(V)).$$

Note, that $T_{\varnothing}=T_{efgh}=id$. Note for any S, that $T_S(\varnothing)=\varnothing$.

For a given $S\in\mathcal{P}(V)$, write $U_1\leftrightarrow U_2$ to express that $T_S(U_1)=U_2\,\&\,T_S(U_2)=U_1$, and $U=inv$ to express $T_S(U)=U$. Then, for example, $T_{\hat{e}}$, T_{ef}, and $T_{\overline{e}}$ are explicitly described as follows. $T_{\hat{e}}$ corresponds to

$$\hat{e}\leftrightarrow efgh,\quad \overline{f}\leftrightarrow ef,\quad \overline{g}\leftrightarrow eg,\quad \overline{h}\leftrightarrow eh,$$

$$\varnothing,\overline{e},\hat{f},\hat{g},\hat{h},fg,fh,gh=inv;$$

T_{ef} corresponds to

$$e\leftrightarrow \overline{f},\quad f\leftrightarrow \overline{e},\quad eg\leftrightarrow eh,\quad fg\leftrightarrow fh,$$

$$\varnothing,g,h,\overline{g},\overline{h},ef,gh,efgh=inv;$$

and $T_{\overline{e}}$ corresponds to

$$\overline{e}\leftrightarrow efgh,\quad \hat{f}\leftrightarrow ef,\quad \hat{g}\leftrightarrow eg,\quad \hat{h}\leftrightarrow eh,$$

$$\varnothing,\hat{e},\overline{f},\overline{g},\overline{h},fg,fh,gh=inv.$$

Explicit descriptions of the other $T_{\hat{v}}$, T_{vw}, and $T_{\overline{v}}$ are obtatined via permutations of symbols e, f, g, and h. Note that for every $S\in\mathcal{P}(V)$,

$$T_{m(\beta)(S)}=m(\beta)T_S m(\beta^{-1}),\quad \beta\in Bij(V),$$

where m is the natural monomorphism of groups, m: $Bij(V)\rightarrowtail Bij(\mathcal{P}(V))$, $\beta\mapsto\mathcal{P}(\beta)$.

Recall that there also exists a natural monomorphism μ: $Bij(V)\rightarrowtail Bij(A)$ described in the previous section.

Every element $g\in G$ is a map $g:\mathcal{P}(V)\xrightarrow{\sim}\mathcal{P}(V)$. It induces a map $\mathcal{P}(g):L\xrightarrow{\sim}L$, $L=\mathcal{P}(\mathcal{P}(V))$. It means that there is a natural action of G on L. Recall that we have a map $\eta:A\to\mathcal{P}(V)$. It turns out that the maps T_S, $S\in\mathcal{P}(V)$, can be lifted up to maps $\theta_S:A\xrightarrow{\sim}A$ in such a way that the subgroup of $Bij(A)$ generated by $\{\theta_S\}_S$ has a natural action on the set $\mathcal{P}_{\perp}(A)$.

Proposition 3: For every $S\in\mathcal{P}(V)$ there exists a map $\theta_S:A\to A$ such that

(1) $\forall l,l'\in A: l\perp l'\Rightarrow\theta_S(l)\perp\theta_S(l')$;

(2) *The map θ_S renders the following diagram commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\theta_S} & A \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\ \mathcal{P}(V) & \xrightarrow[T_S]{} & \mathcal{P}(V) \end{array} \tag{4}$$

(3) $\theta_S^2=id$.

Proof: If $S=\varnothing$ or $S=V$, then the corresponding $T_S=id$ and one may take $\theta_S=id$. It means that essentially one must to consider the cases of nonempty proper subsets $S\subset V$. We describe explicit formulas for $\theta_{\hat{e}}$, θ_{ef}, and $\theta_{\overline{e}}$. The other maps are defined by permutations of e, f, g, and h.

We start with $\theta_{\hat{e}}$. Invoking the explicit description for $T_{\hat{e}}$ given above, one has, for example, $\hat{e}\leftrightarrow efgh$. Thus, a projective line of the form Ψ^e_σ, $\sigma\in S_{\hat{e}}$, should map under the action of $\theta_{\hat{e}}$ to a projective line of the form $F_{\pi'}$, where π' is some element of Λ. Denote this by $\Psi^e_\sigma\overset{\theta_{\hat{e}}}{\rightsquigarrow}F_{\pi'}$. A complete description of $\theta_{\hat{e}}$ consists in the description of its actions on the elements of the form

$$\Psi^e_\sigma,\quad \Psi^f_{\sigma_1},\quad \Psi^g_{\sigma_2},\quad \Psi^h_{\sigma_3},$$

$$\Phi^e_\rho, \quad \Phi^f_{\rho_1}, \quad \Phi^g_{\rho_2}, \quad \Phi^h_{\rho_3},$$

$$X^{ef}_\varkappa, \quad X^{eg}_{\varkappa_1}, \quad X^{eh}_{\varkappa_2}, \quad X^{fg}_{\varkappa_3}, \quad X^{fh}_{\varkappa_4}, \quad X^{gh}_{\varkappa_5}, \quad F_\pi.$$

Define $\theta_{\hat{e}}$ as follows:

$$\begin{array}{rr}
\Psi^e_\sigma \overset{\theta_{\hat{e}}}{\rightsquigarrow} F_{\pi'}: & F_\pi \overset{\theta_{\hat{e}}}{\rightsquigarrow} \Psi^e_{\sigma'}: \\
\pi'(e) \coloneqq 1 + \nabla_\sigma, & \sigma'(ef) \coloneqq \pi(f), \\
\pi'(f) \coloneqq \sigma(ef), & \sigma'(eg) \coloneqq \pi(g), \\
\pi'(g) \coloneqq \sigma(eg), & \sigma'(eh) \coloneqq \pi(h). \\
\pi'(h) \coloneqq \sigma(eh). &
\end{array}$$

$$\begin{array}{rr}
\Phi^f_{\rho_1} \overset{\theta_{\hat{e}}}{\rightsquigarrow} X^{ef}_{\varkappa'}: & X^{ef}_\varkappa \overset{\theta_{\hat{e}}}{\rightsquigarrow} \Phi^f_{\rho_1'}: \\
\varkappa'(ef) \coloneqq \nabla_{\rho_1^*}, & \rho_1'^*(ef) \coloneqq \varkappa(ef) + \varkappa(g) + \varkappa(h), \\
\varkappa'(g) \coloneqq \rho_1^*(ef) + \rho_1^*(fg), & \rho_1'^*(fg) \coloneqq \varkappa(ef) + \varkappa(h), \\
\varkappa'(h) \coloneqq \rho_1^*(ef) + \rho_1^*(fh). & \rho_1'^*(fh) \coloneqq \varkappa(ef) + \varkappa(g).
\end{array}$$

$$\begin{array}{rr}
\Phi^g_{\rho_2} \overset{\theta_{\hat{e}}}{\rightsquigarrow} X^{eg}_{\varkappa_1'}: & X^{eg}_{\varkappa_1} \overset{\theta_{\hat{e}}}{\rightsquigarrow} \Phi^g_{\rho_2'}: \\
\varkappa_1'(eg) \coloneqq \nabla_{\rho_2^*}, & \rho_2'^*(eg) \coloneqq \varkappa_1(eg) + \varkappa_1(f) + \varkappa_1(h), \\
\varkappa_1'(f) \coloneqq \rho_2^*(eg) + \rho_2^*(fg), & \rho_2'^*(fg) \coloneqq \varkappa_1(eg) + \varkappa_1(h), \\
\varkappa_1'(h) \coloneqq \rho_2^*(eg) + \rho_2^*(gh). & \rho_2'^*(gh) \coloneqq \varkappa_1(eg) + \varkappa_1(f).
\end{array}$$

$$\begin{array}{rr}
\Phi^h_{\rho_3} \overset{\theta_{\hat{e}}}{\rightsquigarrow} X^{eh}_{\varkappa_2'}: & X^{eh}_{\varkappa_2} \overset{\theta_{\hat{e}}}{\rightsquigarrow} \Phi^h_{\rho_3'}: \\
\varkappa_2'(eh) \coloneqq \nabla_{\rho_3^*}, & \rho_3'^*(eh) \coloneqq \varkappa_2(eh) + \varkappa_2(f) + \varkappa_2(g), \\
\varkappa_2'(f) \coloneqq \rho_3^*(eh) + \rho_3^*(fh), & \rho_3'^*(fh) \coloneqq \varkappa_2(eh) + \varkappa_2(g), \\
\varkappa_2'(g) \coloneqq \rho_3^*(eh) + \rho_3^*(gh). & \rho_3'^*(gh) \coloneqq \varkappa_2(eh) + \varkappa_2(f).
\end{array}$$

$$\begin{array}{rr}
\Psi^f_{\sigma_1} \overset{\theta_{\hat{e}}}{\rightsquigarrow} \Psi^f_{\sigma_1'}: & X^{gh}_{\varkappa_5} \overset{\theta_{\hat{e}}}{\rightsquigarrow} X^{gh}_{\varkappa_5'}: \\
\sigma_1'(ef) \coloneqq \nabla_{\sigma_1}, & \varkappa_5'(gh) \coloneqq \varkappa_5(gh), \\
\sigma_1'(fg) \coloneqq \sigma_1(fg), & \varkappa_5'(e) \coloneqq \varkappa_5(e), \\
\sigma_1'(fh) \coloneqq \sigma_1(fh). & \varkappa_5'(f) \coloneqq 1 + \varkappa_5(e) + \varkappa_5(f).
\end{array}$$

$$\begin{array}{rr}
\Psi^g_{\sigma_2} \overset{\theta_{\hat{e}}}{\rightsquigarrow} \Psi^g_{\sigma_2'}: & X^{fh}_{\varkappa_4} \overset{\theta_{\hat{e}}}{\rightsquigarrow} X^{fh}_{\varkappa_4'}: \\
\sigma_2'(eg) \coloneqq \nabla_{\sigma_2}, & \varkappa_4'(fh) \coloneqq \varkappa_4(fh), \\
\sigma_2'(fg) \coloneqq \sigma_2(fg), & \varkappa_4'(e) \coloneqq \varkappa_4(e), \\
\sigma_2'(gh) \coloneqq \sigma_2(gh). & \varkappa_4'(g) \coloneqq 1 + \varkappa_4(e) + \varkappa_4(g).
\end{array}$$

$$\Psi^h_{\sigma_3} \overset{\theta_{\hat{e}}}{\rightsquigarrow} \Psi^h_{\sigma_3'}: \qquad X^{fg}_{\varkappa_3} \overset{\theta_{\hat{e}}}{\rightsquigarrow} X^{fg}_{\varkappa_3'}:$$
$$\sigma_3'(eh) := \nabla_{\sigma_3}, \qquad \sigma_3'(fg) := \varkappa_3(fg),$$
$$\sigma_3'(fh) := \sigma_3(fh), \qquad \varkappa_3'(e) := \varkappa_3(e),$$
$$\sigma_3'(gh) := \sigma_3(gh). \quad \varkappa_3'(h) := 1 + \varkappa_3(e) + \varkappa_3(h).$$

$$\Phi^e_\rho \overset{\theta_{\hat{e}}}{\rightsquigarrow} \Phi^e_{\rho'}:$$
$$\rho'^*(ef) := 1 + \rho^*(ef),$$
$$\rho'^*(eg) := 1 + \rho^*(eg),$$
$$\rho'^*(eh) := 1 + \rho^*(eh).$$

Note, that the formulas defining $\theta_{\hat{e}}$ have a symmetry with respect to the permutations of symbols f, g, h. It is a straightforward calculation to show that $\theta_{\hat{e}}$ respects the orthogonality relation $\perp$ on A. The commutativity of the mentioned diagram follows directly from the construction of $\theta_{\hat{e}}$. The verification that $\theta_{\hat{e}}$ is indeed a reflection is also straightforward.

Now define θ_{ef},

$$\Psi^e_\sigma \overset{\theta_{ef}}{\rightsquigarrow} \Phi^f_{\rho_1'}: \qquad \Phi^f_{\rho_1} \overset{\theta_{ef}}{\rightsquigarrow} \Psi^e_{\sigma'}:$$
$$\rho_1'^*(ef) := \sigma(ef), \qquad \sigma'(ef) := \rho_1^*(ef),$$
$$\rho_1'^*(fg) := \sigma(ef) + \sigma(eh), \quad \sigma'(eg) := \rho_1^*(ef) + \rho_1^*(fh),$$
$$\rho_1'^*(fh) := \sigma(ef) + \sigma(eg). \quad \sigma'(eh) := \rho_1^*(ef) + \rho_1^*(fg).$$

$$\Psi^f_{\sigma_1} \overset{\theta_{ef}}{\rightsquigarrow} \Phi^e_{\rho'}: \qquad \Phi^e_\rho \overset{\theta_{ef}}{\rightsquigarrow} \Psi^f_{\sigma_1'}:$$
$$\rho'^*(ef) := \sigma_1(ef), \qquad \sigma_1'(ef) := \rho^*(ef),$$
$$\rho'^*(eg) := \sigma_1(ef) + \sigma_1(fh), \quad \sigma_1'(fg) := \rho^*(ef) + \rho^*(eh),$$
$$\rho'^*(eh) := \sigma_1(ef) + \sigma_1(fg). \quad \sigma_1'(fh) := \rho^*(ef) + \rho^*(eg).$$

$$X^{eg}_{\varkappa_1} \overset{\theta_{ef}}{\rightsquigarrow} X^{eh}_{\varkappa_2'}: \qquad X^{eh}_{\varkappa_2} \overset{\theta_{ef}}{\rightsquigarrow} X^{eg}_{\varkappa_1'}:$$
$$\varkappa_2'(eh) := \varkappa_1(f), \qquad \varkappa_1'(eg) := \varkappa_2(f),$$
$$\varkappa_2'(f) := \varkappa_1(eg), \qquad \varkappa_1'(f) := \varkappa_2(eh),$$
$$\varkappa_2'(g) := 1 + \varkappa_1(eg) + \varkappa_1(f) + \varkappa_1(h). \quad \varkappa_1'(h) := 1 + \varkappa_2(eh) + \varkappa_2(f) + \varkappa_2(g).$$

$$X^{fg}_{\varkappa_3} \overset{\theta_{ef}}{\rightsquigarrow} X^{fh}_{\varkappa_4'}: \qquad X^{fh}_{\varkappa_4} \overset{\theta_{ef}}{\rightsquigarrow} X^{fg}_{\varkappa_3'}:$$
$$\varkappa_4'(fh) := \varkappa_3(e), \qquad \varkappa_3'(fg) := \varkappa_4(e),$$
$$\varkappa_4'(e) := \varkappa_3(fg), \qquad \varkappa_3'(e) := \varkappa_4(fh),$$
$$\varkappa_4'(g) := 1 + \varkappa_3(fg) + \varkappa_3(e) + \varkappa_3(h). \quad \varkappa_3'(h) := 1 + \varkappa_4(fh) + \varkappa_4(e) + \varkappa_4(g).$$

$$\Psi^g_{\sigma_2} \overset{\theta_{ef}}{\rightsquigarrow} \Psi^g_{\sigma_2'}: \qquad \Phi^g_{\rho_2} \overset{\theta_{ef}}{\rightsquigarrow} \Phi^g_{\rho_2'}:$$

$$\begin{aligned} \sigma_2'(eg) &\coloneqq \sigma_2(eg) + \sigma_2'(gh), & \rho_2'^*(eg) &\coloneqq \rho_2^*(eg) + \rho_2^*(gh),\\ \sigma_2'(fg) &\coloneqq \sigma_2'(fg) + \sigma_2'(gh), & \rho_2'^*(fg) &\coloneqq \rho_2^*(fg) + \rho_2^*(gh),\\ \sigma_2'(gh) &\coloneqq \sigma_2'(gh). & \rho_2'^*(gh) &\coloneqq \rho_2^*(gh). \end{aligned}$$

$$\Psi^h_{\sigma_3} \overset{\theta_{ef}}{\rightsquigarrow} \Psi^h_{\sigma_3'}: \qquad \Phi^h_{\rho_3} \overset{\theta_{ef}}{\rightsquigarrow} \Phi^h_{\rho_3'}:$$

$$\begin{aligned} \sigma_3'(eh) &\coloneqq \sigma_3(eh) + \sigma_3(gh), & \rho_3'^*(eh) &\coloneqq \rho_3^*(eh) + \rho_3^*(gh),\\ \sigma_3'(fh) &\coloneqq \sigma_3(fh) + \sigma_3(gh), & \rho_3'^*(fh) &\coloneqq \rho_3^*(fh) + \rho_3^*(gh),\\ \sigma_3'(gh) &\coloneqq \sigma_3(gh). & \rho_3'^*(gh) &\coloneqq \rho_3^*(gh). \end{aligned}$$

$$X^{ef}_{\varkappa} \overset{\theta_{ef}}{\rightsquigarrow} X^{ef}_{\varkappa'}: \qquad X^{gh}_{\varkappa_5} \overset{\theta_{ef}}{\rightsquigarrow} X^{gh}_{\varkappa_5'}:$$

$$\begin{aligned} \varkappa'(ef) &\coloneqq 1 + \varkappa(ef) + \varkappa(g) + \varkappa(h), & \varkappa_5'(gh) &\coloneqq \varkappa_5(gh),\\ \varkappa'(g) &\coloneqq \varkappa(g), & \varkappa_5'(e) &\coloneqq \varkappa_5(e),\\ \varkappa'(h) &\coloneqq \varkappa(h). & \varkappa_5'(f) &\coloneqq \varkappa_5(f). \end{aligned}$$

$$F_{\pi} \overset{\theta_{ef}}{\rightsquigarrow} F_{\pi'}:$$

$$\begin{aligned} \pi'(e) &\coloneqq \pi(f), & \pi'(f) &\coloneqq \pi(e),\\ \pi'(g) &\coloneqq \pi(g), & \pi'(h) &\coloneqq \pi(h). \end{aligned}$$

The verification that θ_{ef} satisfies all the conditions of the proposition is straightforward just as in the case with $\theta_{\hat{e}}$. Note, that the formulas for θ_{ef} are invariant under the transposition of symbols e and f and under the transposition of symbols g and h.

Now define $\theta_{\overline{e}}$,

$$\Phi^e_{\rho} \overset{\theta_{\overline{e}}}{\rightsquigarrow} F_{\pi'}: \qquad F_{\pi} \overset{\theta_{\overline{e}}}{\rightsquigarrow} \Phi^e_{\rho'}:$$

$$\begin{aligned} \pi'(e) &\coloneqq 1 + \nabla_{\rho^*}, & \rho'^*(ef) &\coloneqq \pi(f),\\ \pi'(f) &\coloneqq \rho^*(ef), & \rho'^*(eg) &\coloneqq \pi(g),\\ \pi'(g) &\coloneqq \rho^*(eg), & \rho'^*(eh) &\coloneqq \pi(h).\\ \pi'(h) &\coloneqq \rho^*(eh). \end{aligned}$$

$$\Psi^f_{\sigma_1} \overset{\theta_{\overline{e}}}{\rightsquigarrow} X^{ef}_{\varkappa'}: \qquad X^{ef}_{\varkappa} \overset{\theta_{\overline{e}}}{\rightsquigarrow} \Psi^f_{\sigma_1'}:$$

$$\begin{aligned} \varkappa'(ef) &\coloneqq 1 + \sigma_1(ef), & \sigma_1'(ef) &\coloneqq 1 + \varkappa(ef),\\ \varkappa'(g) &\coloneqq \sigma_1(ef) + \sigma_1(fh), & \sigma_1'(fg) &\coloneqq 1 + \varkappa(ef) + \varkappa(h),\\ \varkappa'(h) &\coloneqq \sigma_1(ef) + \sigma_1(fg). & \sigma_1'(fh) &\coloneqq 1 + \varkappa(ef) + \varkappa(g). \end{aligned}$$

$$\Psi^g_{\sigma_2} \overset{\theta_{\overline{e}}}{\rightsquigarrow} X^{eg}_{\varkappa_1'}: \qquad X^{eg}_{\varkappa_1} \overset{\theta_{\overline{e}}}{\rightsquigarrow} \Psi^g_{\sigma_2'}:$$

$$\begin{aligned} \varkappa_1'(eg) &\coloneqq 1 + \sigma_2(eg), & \sigma_2'(eg) &\coloneqq 1 + \varkappa_1(eg),\\ \varkappa_1'(f) &\coloneqq \sigma_2'(eg) + \sigma_2(gh), & \sigma_2'(fg) &\coloneqq 1 + \varkappa_1(eg) + \varkappa_1(h),\\ \varkappa_1'(h) &\coloneqq \sigma_2(eg) + \sigma_2(fg). & \sigma_2'(gh) &\coloneqq 1 + \varkappa_1(eg) + \varkappa_1(f). \end{aligned}$$

$$\Psi^h_{\sigma_3} \overset{\theta_{\bar{e}}}{\rightsquigarrow} X^{eh}_{\varkappa_2'}: \qquad X^{eh}_{\varkappa_2} \overset{\theta_{\bar{e}}}{\rightsquigarrow} \Psi^h_{\sigma_3'}:$$

$$\varkappa_2'(eh) \coloneqq 1 + \sigma_3(eh), \qquad \sigma_3'(eh) \coloneqq 1 + \varkappa_2(eh),$$

$$\varkappa_2'(f) \coloneqq \sigma_3(eh) + \sigma_3(gh), \quad \sigma_3'(fh) \coloneqq 1 + \varkappa_2(eh) + \varkappa_2(g),$$

$$\varkappa_2'(g) \coloneqq \sigma_3(eh) + \sigma_3(fh). \quad \sigma_3'(gh) \coloneqq 1 + \varkappa_2(eh) + \varkappa_2(f).$$

$$\Phi^f_{\rho_1} \overset{\theta_{\bar{e}}}{\rightsquigarrow} \Phi^f_{\rho_1'}: \qquad X^{gh}_{\varkappa_5} \overset{\theta_{\bar{e}}}{\rightsquigarrow} X^{gh}_{\varkappa_5'}:$$

$$\rho_1'^*(ef) \coloneqq \nabla_{\rho_1^*}, \qquad \varkappa_5'(gh) \coloneqq 1 + \varkappa_5(gh) + \varkappa_5(f),$$

$$\rho_1'^*(fg) \coloneqq \rho_1^*(fg), \qquad \varkappa_5'(e) \coloneqq 1 + \varkappa_5(e) + \varkappa_5(f),$$

$$\rho_1'^*(fh) \coloneqq \rho_1^*(fh). \qquad \varkappa_5'(f) \coloneqq \varkappa_5(f).$$

$$\Phi^g_{\rho_2} \overset{\theta_{\bar{e}}}{\rightsquigarrow} \Phi^g_{\rho_2'}: \qquad X^{fh}_{\varkappa_4} \overset{\theta_{\bar{e}}}{\rightsquigarrow} X^{fh}_{\varkappa_4'}:$$

$$\rho_2'^*(eg) \coloneqq \nabla_{\rho_2^*}, \qquad \varkappa_4'(fh) \coloneqq 1 + \varkappa_4(fh) + \varkappa_4(g),$$

$$\rho_2'^*(fg) \coloneqq \rho_2^*(fg), \qquad \varkappa_4'(e) \coloneqq 1 + \varkappa_4(e) + \varkappa_4(g),$$

$$\rho_2'^*(gh) \coloneqq \rho_2^*(gh). \qquad \varkappa_4'(g) \coloneqq \varkappa_4(g).$$

$$\Phi^h_{\rho_3} \overset{\theta_{\bar{e}}}{\rightsquigarrow} \Phi^h_{\rho_3'}: \qquad X^{fg}_{\varkappa_3} \overset{\theta_{\bar{e}}}{\rightsquigarrow} X^{fg}_{\varkappa_3'}:$$

$$\rho_3'^*(eh) \coloneqq \nabla_{\rho_3^*}, \qquad \varkappa_3'(fg) \coloneqq 1 + \varkappa_3(fg) + \varkappa_3(h),$$

$$\rho_3'^*(fh) \coloneqq \rho_3^*(fh), \qquad \varkappa_3'(e) \coloneqq 1 + \varkappa_3(e) + \varkappa_3(h),$$

$$\rho_3'^*(gh) \coloneqq \rho_3^*(gh). \qquad \varkappa_3'(h) \coloneqq \varkappa_3(h).$$

$$\Psi^e_{\sigma} \overset{\theta_{\bar{e}}}{\rightsquigarrow} \Psi^e_{\sigma'}:$$

$$\sigma'(ef) \coloneqq 1 + \sigma(ef),$$

$$\sigma'(eg) \coloneqq 1 + \sigma(eg),$$

$$\sigma'(eh) \coloneqq 1 + \sigma(eh).$$

In order to obtain formulas for $\theta_{\bar{e}}$ one may take the formulas defining $\theta_{\hat{e}}$ and perform the replacements of the symbols $\sigma \leftrightarrow \rho^*$, $\varkappa \to \tilde{\varkappa}$, $\pi \equiv \pi$, and similar for $(\cdot)'$ symbols. The verification that $\theta_{\bar{e}}$ satisfies the three conditions of the proposition is again straightforward.

The other $\theta_{\hat{v}}$, θ_{vw}, and $\theta_{\bar{v}}$ $(v,w \in V, v \neq w)$ are defined from $\theta_{\hat{e}}$, θ_{ef}, and $\theta_{\bar{e}}$ via the permutations of symbols e, f, g, and h, i.e., in such a way that for every $S \in \mathcal{P}(V)$,

$$\theta_{m(\beta)(S)} = \mu(\beta)\theta_S \mu(\beta^{-1}), \quad \beta \in Bij(V),$$

where μ is the natural monomorphism, μ: $Bij(V) \rightarrowtail Bij(A)$. □

Denote by $Bij_\perp(A)$ the subgroup of $Bij(A)$ consisting of all bijections of A which respect the orthogonality relation $\perp$ on A. We have constructed a family of reflections $\theta_S \in Bij_\perp(A)$, $S \in \mathcal{P}(V)$. Denote by $\mathcal{G}$ the subgroup of $Bij_\perp(A)$ generated by these reflections,

$$\mathcal{G} \coloneqq \langle \{\theta_S | S \in \mathcal{P}(V)\} \rangle \subset Bij_\perp(A).$$

Note that the correspondence $T_S \mapsto \theta_S$ *does not* define a homomorphism from G to $\mathcal{G}$, since, for example, the order of an element $T_{\bar{f}} T_{\hat{e}} \in G$ is 2, and the order of $\theta_{\bar{f}} \theta_{\hat{e}} \in G$ is 4.

Let us mention some properties of the groups G and $\mathcal{G}$. First of all, recall that we have natural monomorphisms, m: $Bij(V)\rightarrowtail Bij(\mathcal{P}(V))$, μ: $Bij(V)\rightarrowtail Bij(A)$. It turns out, that the images of these monomorfisms are in fact contained in G and $\mathcal{G}$, respectively, i.e., each of the two groups contains a copy of S_4. Denote by t_{ef} the bijection $V\tilde{\rightarrow}V$ which interchanges the symbols e and f, i.e., $t_{ef}:e\mapsto f,f\mapsto e,g\mapsto g,h\mapsto h$. Let us write (ef) instead of $m(t_{ef})$ and $(\widetilde{ef})$ instead of $\mu(t_{ef})$. One defines in a similar way the transformations (vw) and $(\widetilde{vw})$ for all $v,w\in V,v\neq w$.

Proposition 4: For all $v,w,z\in V$, $v\neq w$, $v\neq z$, $w\neq z$,

$$T_{vw}T_{uz}T_{uw}=(wz),$$

$$\theta_{vw}\theta_{vz}\theta_{vw}=(\widetilde{wz}). \tag{5}$$

Proof: It is sufficient to verify that $T_{ef}T_{eg}T_{ef}=(fg)$ and that $\theta_{ef}\theta_{eg}\theta_{ef}=(\widetilde{fg})$. The latter is established by a straightforward computation. □

Consider a product $D:=(ef)(gh)T_{ef}T_{gh}$. For every $U\in\mathcal{P}(V)$, $D(U)=\overline{U}$ if $\#U$ is odd, and $D(U)=U$ if $\#U$ is even. Hence the map D is obtained by

$$\hat{e}\leftrightarrow\overline{e}, \quad \hat{f}\leftrightarrow\overline{f}, \quad \hat{g}\leftrightarrow\overline{g}, \quad \hat{g}\leftrightarrow\overline{h},$$

$$\emptyset,ef,eg,eh,fg,fh,gh,efgh=inv.$$

Note, that for any $\epsilon\in E$, $D=(\epsilon)(\overline{\epsilon})T_{\epsilon}T_{\overline{\epsilon}}$. Consider an analog of D in $\mathcal{G}$, the product $\delta:=(\widetilde{ef})(\widetilde{gh})\theta_{ef}\theta_{gh}$. Observe that δ is just the duality transformation mentioned in the section describing the orthogonality relation on A, $\delta(\Psi^{v}_{\sigma})=\Phi^{v}_{\sigma^*}$, $\delta(X^{\epsilon}_{\varkappa})=X^{\epsilon}_{\tilde{\varkappa}}$, $\delta(\Phi^{v}_{\rho})=\Psi^{v}_{\rho^*}$, and $\delta(F_{\pi})=F_{\pi}$ (notation σ^*, ρ^*, and $\tilde{\varkappa}$ as in that section). More generally, for any $\epsilon\in E$ one has $\delta=(\tilde{\epsilon})(\tilde{\overline{\epsilon}})\theta_{\epsilon}\theta_{\overline{\epsilon}}$. The transformations $D\in G$ and $\delta\in\mathcal{G}$ allow to obtain $T_{\overline{v}}$ and $\theta_{\overline{v}}$ $(v\in V)$ from $T_{\hat{v}}$ and $\theta_{\hat{v}}$ according to $T_{\overline{v}}=DT_{\hat{v}}D$ and $\theta_{\overline{v}}=\delta\theta_{\hat{v}}\delta$. Any transformation T_{ϵ} $(\epsilon\in E)$ commutes with D, $DT_{\epsilon}=T_{\epsilon}D$, and any transformation θ_{ϵ} $(\epsilon\in E)$ commutes with δ, $\delta\theta_{\epsilon}=\theta_{\epsilon}\delta$.

Note that $(ev)T_{\hat{e}}(ev)=T_{\hat{v}}$, $v\in V$, $v\neq e$. Since every transformation of the form (vw) and the transformation D belong to a subgroup G_2 of G generated by $\{T_{\epsilon}\}_{\epsilon\in E}$, any set generating G_2 appended with an element $T_{\hat{e}}$, generates the whole group G. Similarly, if one denotes by $\mathcal{G}_2$ the subgroup of $\mathcal{G}$ generated by $\{\theta_{\epsilon}\}_{\epsilon\in E}$, then any set of generators of $\mathcal{G}_2$ appended with an element $\theta_{\hat{e}}$ generates the whole group $\mathcal{G}$.

The groups G_2 and $\mathcal{G}_2$ should be investigated in more detail. We start with the group G_2. It is convenient to consider $W_{vw}:=(vw)T_{vw}$ $(v,w\in V,v\neq w)$. Denote by G_2' the subgroup of G_2 generated by (vw)'s, and by G_2'' a subgroup of G_2 generated by W_{vw}'s. Together G_2' and G_2'' generate the whole G. The explicit description of W_{ef} is

$$\hat{e}\leftrightarrow\overline{e}, \quad \hat{f}\leftrightarrow\overline{f}, \quad eg\leftrightarrow fh, \quad eh\leftrightarrow fg,$$

$$\emptyset,\hat{g},\hat{h},\overline{g},\overline{h},ef,gh,efgh=inv.$$

and the explicit descriptions of the other W_{vw} are similar. One verifies that $W_{ef}W_{fg}=W_{eg}$. More generally, for any v, w, $z\in V$, $v\neq w$, $v\neq z$, $w\neq z$,

$$W_{vw}W_{wz}=W_{vz}.$$

It follows, that G_2'' consists of all elements of the form W_{ϵ}, $\epsilon\in E$, an element $D=W_{ef}W_{gh}$ and a unit element of G. Since every element $b\in G_2'$ preserves the cardinality [i.e., $\#b(U)=\#U$, $U\in\mathcal{P}(V)$], and no element $w\in G_2''$ except the unit preseves the cardinality, it follows that the intersection of G_2' and G_2'' is trivial. Moreover, the group G_2'' is normal in G_2 since $(ef)W_{ef}(ef)=W_{ef}$, $(eg)W_{ef}(eg)=W_{fg}$, and $(gh)W_{ef}(gh)=W_{ef}$, and there exists a natural action of G_2' on G_2'' defined as follows. From the explicit description of W_{vw} one observes that G_2'' is isomorphic to a group Λ_0 of

$\mathbb{Z}_2$-valued functions ϕ on V satisfying a condition $\Sigma_{v\in V}\phi(v)=0$, i.e., it is a sample of $(\mathbb{Z}_2)^3$. An element $\beta\in Bij(V)$ acts on Λ_0 by the formula $\phi\mapsto\phi\circ\beta^{-1}$. This induces an action of $G_2'\simeq Bij(V)$ on $G_2''\simeq\Lambda_0$. Since $Bij(V)\simeq S_4$ and $\Lambda_0\simeq(\mathbb{Z}_2)^3$, it follows that one may view the group G_2 as a semidirect product $(\mathbb{Z}_2)^3\rtimes S_4$.

The considerations about the group $\mathcal{G}_2$ are similar to the considerations about G_2. In particular, the elements $\omega_{vw}:=(\widetilde{vw})\theta_{vw}$ have the properties similar to the properties of W_{vw}. As a result, one gets that $\mathcal{G}_2\simeq(\mathbb{Z}_2)^3\rtimes S_4$ as well.

G is a group generated by the following five elements: (ef), (fg), (gh), $T_{\hat{e}}$, and W_{ef}. The corresponding Coxeter matrix is defined by

$$\mathrm{ord}((ef)T_{\hat{e}})=3,\quad \mathrm{ord}((fg)T_{\hat{e}})=2,\quad \mathrm{ord}((gh)T_{\hat{e}})=2,$$

$$\mathrm{ord}((ef)W_{ef})=2,\quad \mathrm{ord}((fg)W_{ef})=4,\quad \mathrm{ord}((gh)W_{ef})=2,$$

$$\mathrm{ord}((ef)(fg))=3,\quad \mathrm{ord}((ef)(gh))=2,\quad \mathrm{ord}((fg)(gh))=3,$$

$$\mathrm{ord}(W_{ef}T_{\hat{e}})=3,$$

where $\mathrm{ord}(\cdot)$ denotes the order of a group element. One verifies that the Coxeter matrix associated to the original set of generators $\{T_S\}_S$, $S\in\mathcal{P}(V)$, is defined by the formula

$$\mathrm{ord}(T_{S_2}T_{S_1})=\begin{cases}2, & \text{if } \#(S_1\setminus S_2) \text{ is even and } \#(S_2\setminus S_1) \text{ is even},\\ 3, & \text{if } \#(S_1\setminus S_2) \text{ is odd and } \#(S_2\setminus S_1) \text{ is odd},\\ 4, & \text{otherwise}.\end{cases}$$

Note, that the group G contains other reflections besides the ones already mentioned. In particular, there exist reflections which interchange $ef\leftrightarrow efgh$, for example, $T_{\hat{f}}W_{ef}T_{\hat{e}}W_{ef}$. At least some of the reflections can be generated starting from $\{T_S\}_S$ by using the following facts: whenever R_1 and R_2 are two reflections, $R_2R_1R_2$ is again a reflection; if R_1 and R_2 commute, then their product R_2R_1 is again a reflection.

Note that there is another way of expressing (vw) and D in G. Verify that additioned to (5) there is also a formula $T_{\hat{e}}T_{\hat{f}}T_{\hat{e}}=(ef)$ and $T_{\hat{e}}T_{\bar{e}}T_{\hat{e}}=D$. After replacing the left-hand and right-hand sides of these equalities by their analoges in $\mathcal{G}$, one observes, that $\theta_{\hat{e}}\theta_{\bar{e}}\theta_{\hat{e}}=\delta$, but $\theta_{\hat{e}}\theta_{\hat{f}}\theta_{\hat{e}}\neq(\widetilde{ef})$. What is the deviation of the value of $(\widetilde{ef})\theta_{\hat{e}}\theta_{\hat{f}}\theta_{\hat{e}}$ from identity? We need more notation to express that. Consider an Abelian group $\mathcal{F}$ of all $\mathbb{Z}_2$-valued functions on the set of all edges $E=\{\epsilon\subset V \mid \#\epsilon=2\}$. We shall associate to every $\varphi\in\mathcal{F}$ a transformation $I_\varphi\in Bij(A)$ and then show that in fact I_φ falls into the group $\mathcal{G}$. The product $(\widetilde{vw})\theta_{\hat{v}}\theta_{\hat{w}}\theta_{\hat{v}}$ will be equal to I_φ where φ is some element of $\mathcal{F}$.

Take any $\varphi\in\mathcal{F}$. Denote $a:=\varphi(ef)$, $b:=\varphi(eg)$, $c:=\varphi(fg)$, $p:=\varphi(eh)$, $q:=\varphi(fh)$, $r:=\varphi(gh)$. The transformation I_φ will not change the kind of a projective line and we will describe its action on Ψ^e_σ, $X^{ef}_\varkappa$, Φ^e_ρ, and F_π. The other cases are obtained by permutation of the symbols e, f, g, h. A projective line Ψ^e_σ is mapped by I_φ to $\Psi^e_{\sigma'}$ with $\sigma'(ef)=\sigma(ef)+a$, $\sigma'(eg)=\sigma(eg)+b$, $\sigma'(eh)=\sigma(eh)+p$. A projective line $X^\epsilon_\varkappa$ maps to $X^\epsilon_{\varkappa'}$ with $\varkappa'(ef)=\varkappa(ef)+a$, $\varkappa'(g)=\varkappa(g)+p+q$, $\varkappa'(h)=\varkappa(h)+b+c$. A projective line Φ^e_ρ maps to $\Phi^e_{\rho'}$ with $\rho'(\cdot)$ defined by $\rho'^*(ef)=\rho^*(ef)+b+c+p+q+r$, $\rho'^*(eg)=\rho^*(eg)+a+c+p+q+r$, $\rho'^*(eh)=\rho^*(eh)+a+b+c+q+r$. Finally, the projective line F_π is mapped by I_φ to $F_{\pi'}$, where $\pi'(\cdot)$ is defined as $\pi'(e)=\pi(e)+a+b+p$, $\pi'(f)=\pi(f)+a+c+q$, $\pi'(g)=\pi(g)+b+c+r$, $\pi'(h)=\pi(h)+p+q+r$. Note, that since $\Sigma_{v\in V}\pi(v)=1$, one gets $\Sigma_{v\in V}\pi'(v)=1$. The difference $\pi'-\pi$ satisfies $\Sigma_{v\in V}(\pi'-\pi)(v)=0$.

We have defined a collection $\{I_\varphi\}_{\varphi\in\mathcal{F}}$ of maps $A\to A$, such that $I_\varphi^2=id$. This implies, in particular, that I_φ is a bijection, and one may consider the subgroup in $Bij(A)$ generated by

$\{I_\varphi\}_{\varphi\in\mathcal{F}}$. Since for every $\varphi_1, \varphi_2 \in \mathcal{F}$ we have $I_{\varphi_1} I_{\varphi_2} = I_{\varphi_1+\varphi_2}$, this subgroup is Abelian. Denote by χ_ϵ ($\epsilon \in E$) the element of $\mathcal{F}$ which has a value 1 on the edge ϵ and a value 0 on all other edges. Straightforward computation establishes that

$$(\widetilde{vw})\,\theta_{\hat{v}}\theta_{\hat{w}}\theta_{\hat{v}} = I_{\chi_{vw}} \tag{6}$$

(for every $v, w \in V$, $v \neq w$). Since every map I_φ, $\varphi \in \mathcal{F}$, may be represented as a composition of maps of the form I_{χ_ϵ}, $\epsilon \in E$, it follows from (6) that every map I_φ is in $\mathcal{G}$. It follows that the set $\{I_\varphi\}_{\varphi\in\mathcal{F}}$ generates some Abelian subgroup $\mathcal{N}$ in $\mathcal{G}$.

Proposition 5: The group $\mathcal{N}$ generated by $\{I_\varphi\}_{\varphi\in\mathcal{F}}$ is a normal subgroup of $\mathcal{G}$.

Proof: We define for $S \in \mathcal{P}(V)$ a morphism τ_S, $\mathcal{F} \to \mathcal{F}$, such that $\forall \varphi \in \mathcal{F}: \theta_S I_\varphi = I_{\tau_S(\varphi)}\theta_S$. Since $\theta_\emptyset = \theta_V = id$, set $\tau_\emptyset := id$ and $\tau_V := id$. The set of formulas for the other cases of S will have a symmetry with respect to the permutations of e, f, g, and h, and in fact the nontrivial part of the proof will consist in providing the definitions of $\tau_{\hat{e}}$, τ_{ef}, and $\tau_{\bar{e}}$.

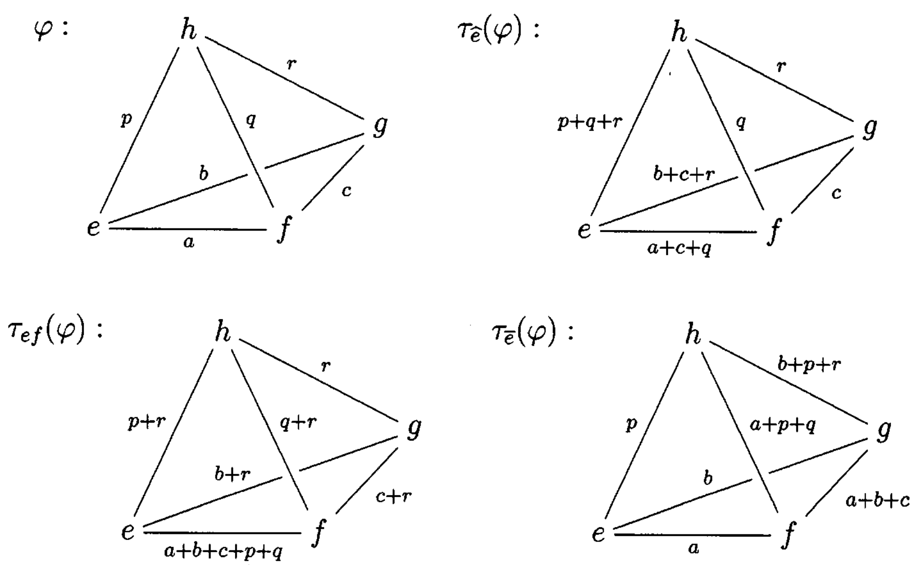
There exists a natural monomorphism ν: $Bij(V) \rightarrowtail End(\mathcal{F})$ defined as follows: for every $\beta \in Bij(V)$ the morphism $\nu(\beta)$: $\mathcal{F} \to \mathcal{F}$ is defined by $\varphi \mapsto \varphi'$, $\varphi'(\epsilon) := \varphi(m(\beta^{-1})(\epsilon))$ for all $\epsilon \in E$, where m is the natural monomorphism $Bij(V) \rightarrowtail Bij(\mathcal{P}(V))$. Recall that there also exists a natural monomorphism μ: $Bij(V) \rightarrowtail Bij(A)$.

The I_φ are defined in such a way, that $\forall \beta \in Bij(V): \mu(\beta) I_\varphi \mu(\beta^{-1}) = I_{\nu(\beta)(\varphi)}$. Recall that for every $\beta \in Bij(V)$ and every $S \in \mathcal{P}(V)$, $\mu(\beta)\theta_S\mu(\beta^{-1}) = \theta_{m(\beta)(S)}$. Hence,

$$\begin{aligned} I_{\tau_{m(\beta)(S)}(\varphi)} &= \theta_{m(\beta)(S)} I_\varphi \theta^{-1}_{m(\beta)(S)} = \mu(\beta)\theta_S\mu(\beta^{-1}) I_\varphi [\mu(\beta)\theta_S\mu(\beta^{-1})]^{-1} = \mu(\beta)\theta_S I_{\nu(\beta^{-1})(\varphi)}\theta_S^{-1}\mu(\beta^{-1}) \\ &= \mu(\beta) I_{(\tau_S\nu(\beta^{-1}))(\varphi)}\mu(\beta^{-1}) = I_{(\nu(\beta)\tau_S\nu(\beta^{-1}))(\varphi)}, \end{aligned}$$

where $\varphi \in \mathcal{F}$. It follows, that the collection $\{\tau_S\}_{S\in\mathcal{P}(V)}$ should satisfy $\nu(\beta)\tau_S\nu(\beta^{-1}) = \tau_{m(\beta)(S)}$, $\beta \in Bij(V)$. Hence it is necessary to describe just three morphisms $\tau_{\hat{e}}, \tau_{ef}, \tau_{\bar{e}}: \mathcal{F} \to \mathcal{F}$.

It is convenient to represent an element of $\mathcal{F}$ by a graph, which is a tetrahedron with vertices e, f, g, h, and equip its edges with the values of the considered element of $\mathcal{F}$ on the corresponding edge. Take any $\varphi \in \mathcal{F}$ and denote by a, b, c, p, q, and r the values of φ on the edges ef, eg, fg, eh, fh, and gh, respectively. Define $\tau_{\hat{e}}$, τ_{ef}, and $\tau_{\bar{e}}$ as follows:



(7)

The explicit descriptions of the other τ_S are induced by permutations of labels of vertices e, f, g, and h. Recall that $\tau_{\varnothing}=\tau_V=id$. Straightforward computation establishes that $\theta_{\hat{e}}I_{\varphi}=I_{\tau_{\hat{e}}(\varphi)}\theta_{\hat{e}}$, $\theta_{ef}I_{\varphi}=I_{\tau_{ef}(\varphi)}\theta_{ef}$, and $\theta_{\bar{e}}I_{\varphi}=I_{\tau_{\bar{e}}(\varphi)}\theta_{\bar{e}}$. This completes the proof that the group $\mathcal{N}$ is a normal subgroup of the group $\mathcal{G}$. □

Note that now one has three types of transformations indexed by $S\in\mathcal{P}(V)$, a bijection $T_S\in Bij(\mathcal{P}(V))$ [refer to (3)], a bijection $\theta_S\in Bij_{\perp}(A)$ [refer to (4)], and an automorphism $\tau_S\in Aut(\mathcal{F})$ [refer to (7)].

The groups G and $\mathcal{G}$ will play a key role in the proof of the saturation property of A. In particular, it will be shown below that the image of the composition of maps $\mathcal{P}_{\perp}(A)\rightarrowtail\mathcal{P}(A)\xrightarrow{\mathcal{P}(\eta)}L$, where the first arrow is a canonical injection, is invariant under the G action. It will be shown, that an image of a complete set under this composition can have a cardinality only 1, 2, 4 or 8. This induces a partition of $C(A)$ into four subsets. The $\mathcal{G}$ action will fix each of these subsets and it will turn out that $\mathcal{G}$ acts transitively on each one of them.

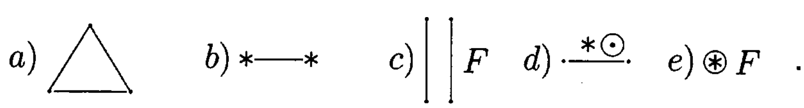
VIII. SATURATION PROPERTY, PART 2

We have constructed a group G and an action of G on L. Let H be a subgroup of G. Two graphs corresponding to some elements of Γ are called *H-equivalent* iff they can be represented in L by elements of the same H orbit.

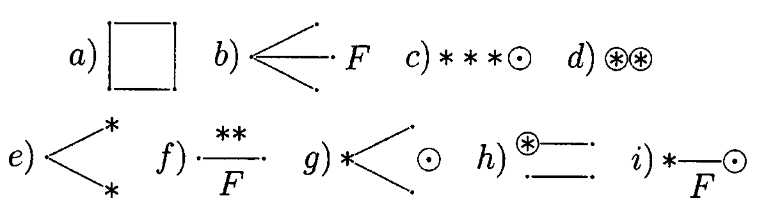
In the group G there is an element D. Its action on L induces a map $\partial:\Gamma\to\Gamma$, which in terms of graphs replaces a star $*$ by a circle $\odot$ around the same vertex and a circle $\odot$ by a star $*$ at the same vertex; the edges $\cdot\!\!-\!\!\cdot$ and the symbol F remain untouched. It means that if one is given a graph, then by applying if necessary the transformation ∂, it is possible to produce a graph with the number of stars $*$ greater or equal to the number of circles $\odot$. We shall call a graph satisfying this condition, *primary*, and a graph not satisfying this condition, *secondary*. For example, a graph $F*\circledast\!-\!\cdot$ is primary, and a graph $F\odot\circledast\!-\!\cdot$ is secondary. If the graphs represent the elements of Γ related to one another by the transformation ∂, one calls these graphs mutually *dual*. If a graph coincides with its dual, it is called *self-dual*. For example, the graph $*\!-\!\odot$ is self-dual.

Note, that any G-equivalence class is invariant under the transformation induced by ∂. The set of elements constituting a G-equivalence class, is completely determined by a list of all primary elements belonging to it; the other elements are obtained by duality.

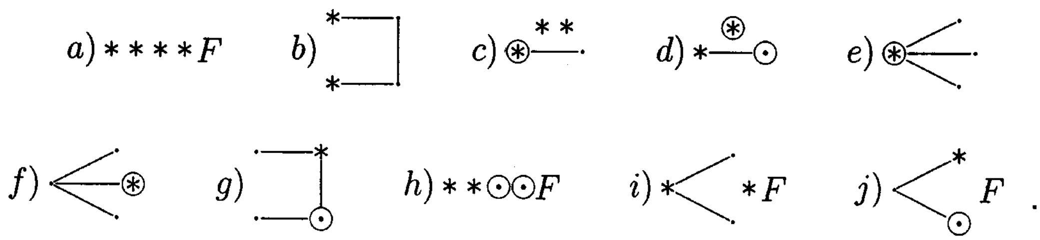
Proposition 6: (1) The complete list of primary graphs from the G-equivalence class of the graph $\triangle$ is of the form



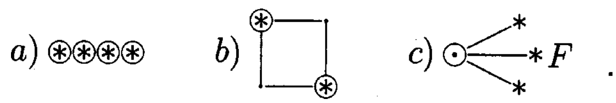
(2) The complete list of primary graphs from the G-equivalence class of the graph $***\odot$ is of the form:



(3) The complete list of primary graphs from the G-equivalence class of the graph $****F$ is of the form:

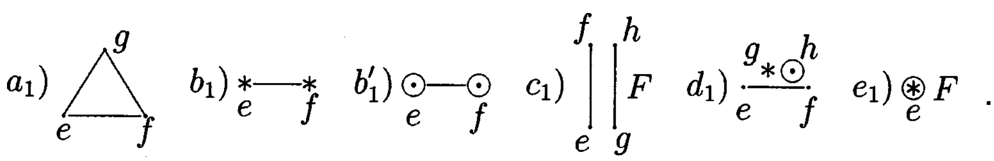


(4) The complete list of primary elements of the G-equivalence class of the graph $\circledast\circledast\circledast\circledast$ *is of the form*



Proof: The proof is straightforward. Let us consider the proof of (1) in more detail. Note that all the graphs listed in (1) except the graph (b) are self-dual. Denote a graph obtained by duality from (b) as (b'); it is of the form $\odot$—$\odot$. Denote the set of graphs (1) appended with (b') as N_3 (N stands for nonadmissible and 3 stands for the degree).

That the G-equivalence class of the triangle indeed coincides with N_3, follows from two facts: (i) the list N_3 is complete, i.e., there is no other graph not present in the list, which is G-equivalent to some member of the list; (ii) any two members of the list N_3 are G-equivalent. In order to establish (i), let us choose and fix some representative in L for each element of N_3. This is equivalent to assigning some labels to the vertices of the graphs and can be done, for example, as follows:



Now for each of these labeled graphs calculate the result of the action on them of the transformations induced by $T_{\hat{v}}$ and T_{vw} $(v,w \in V, v \neq w)$, and after that delete the labels e, f, g, h. For example, if one takes the graph (a_1), then the transformation associated to $T_{\hat{e}}$ followed by the deletion of symbols e, f, g, and h, generates a graph $\odot$—$\odot$. If instead of $T_{\hat{e}}$ one takes $T_{\hat{h}}$, then the result will be the unlabeled triangle. Performing similar calculations, in each case one obtains an element of N_3, i.e., N_3 is complete.

Now let us establish (ii), i.e., the fact that N_3 is just one G-equivalence class. It is convenient to do this in several steps by taking bigger and bigger subgroups H of G and splitting N_3 into H-equivalence classes. Denote by G_1 the subgroup of G generated by $\{T_{\hat{v}}\}_{v\in V}$. Then N_3 splits into three G_1-equivalence classes: the first class consists of (a) and (b'); the second class consists of (b) and (e); and the third class consists of (c) and (d). Denote by $G_{1,3}$ the subgroup of G generated by $T_{\hat{v}}$ and $T_{\overline{v}}$, $v \in V$. Recall that $T_{\overline{v}}=DT_{\hat{v}}D$ where $D=T_{ef}T_{gh}$. Hence, $G_{1,3}$ is generated by the $\{T_{\hat{v}}\}_{v\in V}$ and D. The list N_3 splits into two $G_{1,3}$-equivalence classes, the first class contains the graphs (a), (b), (b'), and (e); and the other (c) and (d). Finally, the action associated to T_{eg} on the graph $e*—*f$ gives a graph, $e—h*f\odot g$. Hence, one obtains a transition from (b) to (d), i.e., a link between the two mentioned $G_{1,3}$-equivalence classes. It follows, that any two elements of N_3 are G-equivalent and this completes the proof of the first part of the proposition.

The proof of (2), (3), and (4) is similar. □

We have calculated some of the G-equivalence classes of some examples of graphs representing the elements of Γ. It will be interesting to describe for every given degree the sets of G-equivalence classes of admissible graphs.

Proposition 7: If γ_1 is an admissible graph and γ_2 is a graph G-equivalent to γ_1, then it is also admissible. Moreover, if B_1 hangs over γ_1, then there exists a set B_2 which hangs over γ_2 and has the same cardinality as B_1.

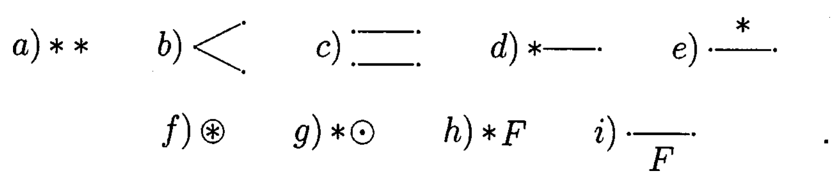
Proof: Choose a representative Q_1 for γ_1 in L and a representative Q_2 for γ_2 in L. Hence γ_1 is associated to $[Q_1]\in\Gamma$ and γ_2 is associated to $[Q_2]\in\Gamma$. The assumption that γ_1 and γ_2 are G-equivalent implies that Q_1 and Q_2 are related as $Q_2=\hat{T}(Q_1)$, where $\hat{T}:=T_{S_1}T_{S_2}\cdots T_{S_m}\in G$, m is some natural number and $S_1,S_2,\ldots,S_m$ are some nonempty proper subsets of $V=\{e,f,g,h\}$. Since Q_1 is admissible, choose a set $B_1\in\mathcal{P}_\perp(A)\subset\mathcal{P}(A)$, such that $\mathcal{P}(\eta)(B_1)=Q_1$, where $\eta:A\to\mathcal{P}(V)$ is the natural map. For every $l\in B_1$ look at $l':=\hat{\theta}(l)$, where $\hat{\theta}:=\theta_{S_1}\theta_{S_2}\ldots\theta_{S_m}\in\mathcal{G}$. As l runs over the whole B_1, l' sweeps up some set $B_2\subset A$. Note, that since every θ_S is a bijection, $\hat{\theta}$ is a bijection as well, and the sets B_1 and B_2 have the same cardinality. Using the commutative diagram relating T_S and θ_S, $S\in\mathcal{P}(V)$, and the fact that every θ_S respects $\perp$, one concludes that $B_2\in\mathcal{P}_\perp(A)$ and that $\mathcal{P}(\eta)(B_2)=Q_2$. In particular, B_2 hangs over γ_2, and by that provides a realization of the admissiblity of γ_2. □

The graph * is an admissible graph of degree 1. The complete list of all primary elements from its G-equivalence class is of the form

$$a)*\qquad b)\cdot\!\!-\!\!\cdot\qquad c)F.$$

It means, that the G-equivalence class of the graph $*$ coincides with the set of all graphs of degree 1.

Similarly, one has an admissible graph $**$ of degree 2. The complete list of all primary elements from its G-equivalence class is of the form

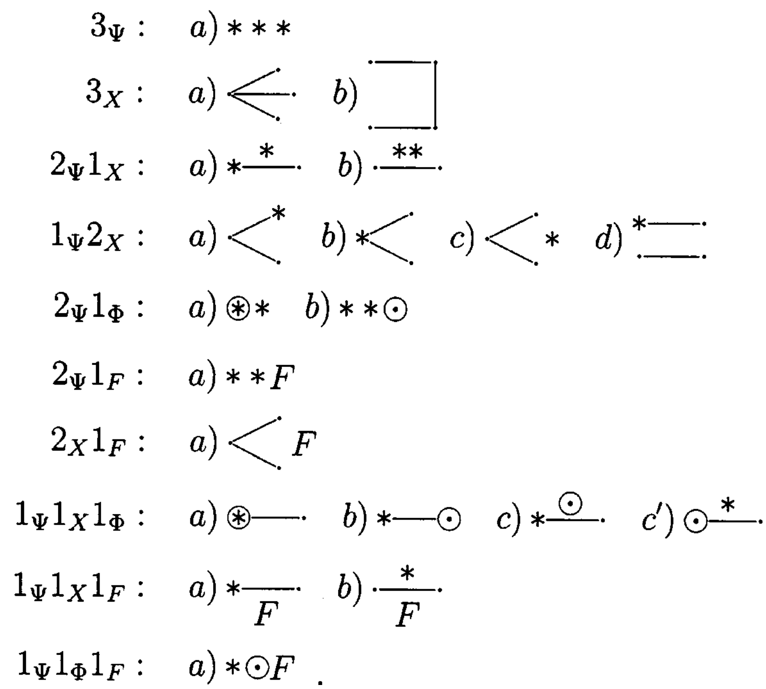


This G-equivalence class coincides with the set of all graphs of degree 2.

In degree 3 there exist admissible, as well as nonadmissible graphs.

Proposition 8: The set of all admissible graphs of degree 3 coincides with the G-equivalence class of the graph $***$.

Proof: Let us generate a list of all primary graphs of degree 3. It is convenient to present it in a systematic way. Denote by Γ_{k_1,k_2,k_3,k_4} a subset of Γ consisting of all the elements which are associated to the graphs which have k_1 stars, k_2 edges, k_3 circles, and k_4 instances of the symbol F. Look at all decompositions of 3 into a sum of four non-negative integers, $3=3+0+0+0=2+1+0+0=1+1+1+0$. Each of the mentioned three variants corresponds to some Γ_{k_1,k_2,k_3,k_4} with $k_1\geqslant k_2\geqslant k_3\geqslant k_4$, one gets $\Gamma_{3,0,0,0}$, $\Gamma_{2,1,0,0}$, and $\Gamma_{1,1,1,0}$. After that one generates the other Γ_{k_1,k_2,k_3,k_4} by permuting the arguments k_1, k_2, k_3, k_4 in the obtained three variants. Finally, one deletes from the list all the entries which do not satisfy $k_1\geqslant k_3$ and $k_4\leqslant 1$. Inside each of the sets Γ_{k_1,k_2,k_3,k_4} one generates the corresponding graphs by exploring the different variants. After excluding the known nonadmissible graphs of degree 3, i.e., the graphs which are G-equivalent to a triangle, the list becomes



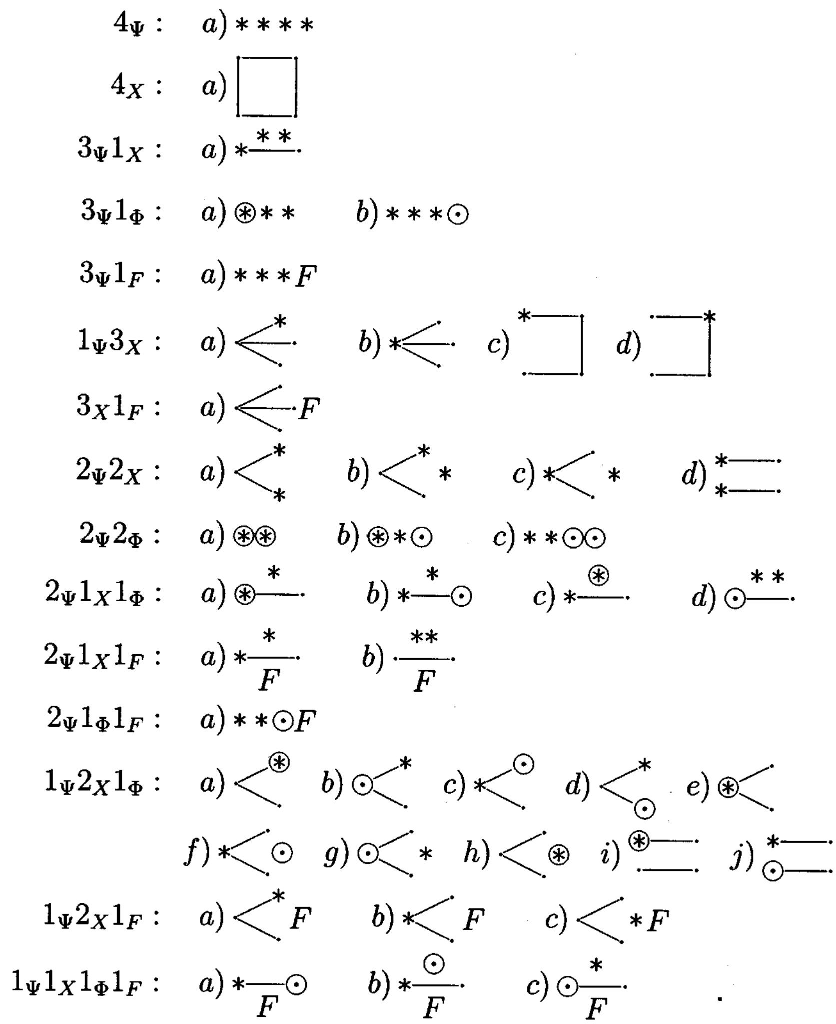
Here we use the labels of the form $1_\Psi 2_X$, $1_\Psi 1_X 1_F$, etc., to classify the graphs. The label $1_\Psi 2_X$ is associated to admissible graphs from $\Gamma_{1,2,0,0}$, $1_\Psi 1_X 1_F$ is associated to admissible graphs from $\Gamma_{1,1,0,1}$, etc. In a way, similar to the proof of the previous proposition, check that all the presented graphs are in fact a set of all primary graphs of a G-equivalence class of $***$. □

Now let us look at the graphs of degree 4. Note, that any admissible graph of degree 4 should satisfy a *necessary* condition, it does not contain a nonadmissible graph of degree 3. More precisely, one says that a graph γ_1 is *contained* in the graph γ_2 by definition iff these two graphs can be represented in L by Q_1 and Q_2, respectively, in such a way that $Q_1 \subset Q_2$.

Proposition 9: (1) A graph of degree 4 is admissible iff it does not contain a nonadmissible graph of degree 3.

(2) The set of all admissible graphs of degree 4 splits into two G-equivalence classes, one of the graph $***\odot$ *and the other of the graph* $****$.

Proof: Consider the set of *all* graphs of degree 4, which do not contain a nonadmissible graph of degree 3. The set of all *admissible* graphs of degree 4 is a subset of this set. One generates the required list in analogy with the case of degree 3. There are five ways to decompose 4 into a sum of four non-negative integers, $4=4+0+0+0=3+1+0+0=2+2+0+0=2+1+1+0=1+1+1+1$. It means, in particular, that one must have a series of graphs marked by labels 4_Ψ, $3_\Psi 1_X$, $2_\Psi 2_X$, $2_\Psi 1_X 1_\Phi$, and $1_\Psi 1_X 1_\Phi 1_F$. The label 4_Ψ generates 4_X, 4_Φ, and 4_F. Since a graph cannot contain more than one symbol F, delete 4_F. Since it suffices to consider only primary graphs, exclude the case 4_Φ. Treating the other labels in a similar fashion, one arrives at the following: $3_\Psi 1_X$ generates $3_\Psi 1_\Phi$, $3_\Psi 1_F$, $1_\Psi 3_X$, and $3_X 1_F$; $2_\Psi 2_X$ generates $2_\Psi 2_\Phi$; $2_\Psi 1_X 1_\Phi$ generates $2_\Psi 1_X 1_F$, $2_\Psi 1_\Phi 1_F$, $1_\Psi 2_X 1_\Phi$, and $1_\Psi 2_X 1_F$; finally, $1_\Psi 1_X 1_\Phi 1_F$ does not generate more labels. Having a set of all possible labels classifying the mentioned graphs, one generates a set of graphs present under each label by exploring different variants. The resulting complete list of primary graphs is given below,



Note, that the existence of nonadmissible graphs of degree 3 excludes many variants of the graphs of degree 4. Now exclude all the graphs G-equivalent to $***\odot$. It is easily verified, that all the remaining graphs constitute in fact the set of all primary graphs from the G-equivalence class of $****$. □

We have shown, that the set of all admissible graphs of degree 4 splits into two G-equivalence classes, one containing the graph $****$ and the second, the graph $***\odot$. The elements from the first class will be referred to as *singlets* and the elements of the second class as *doublets*.

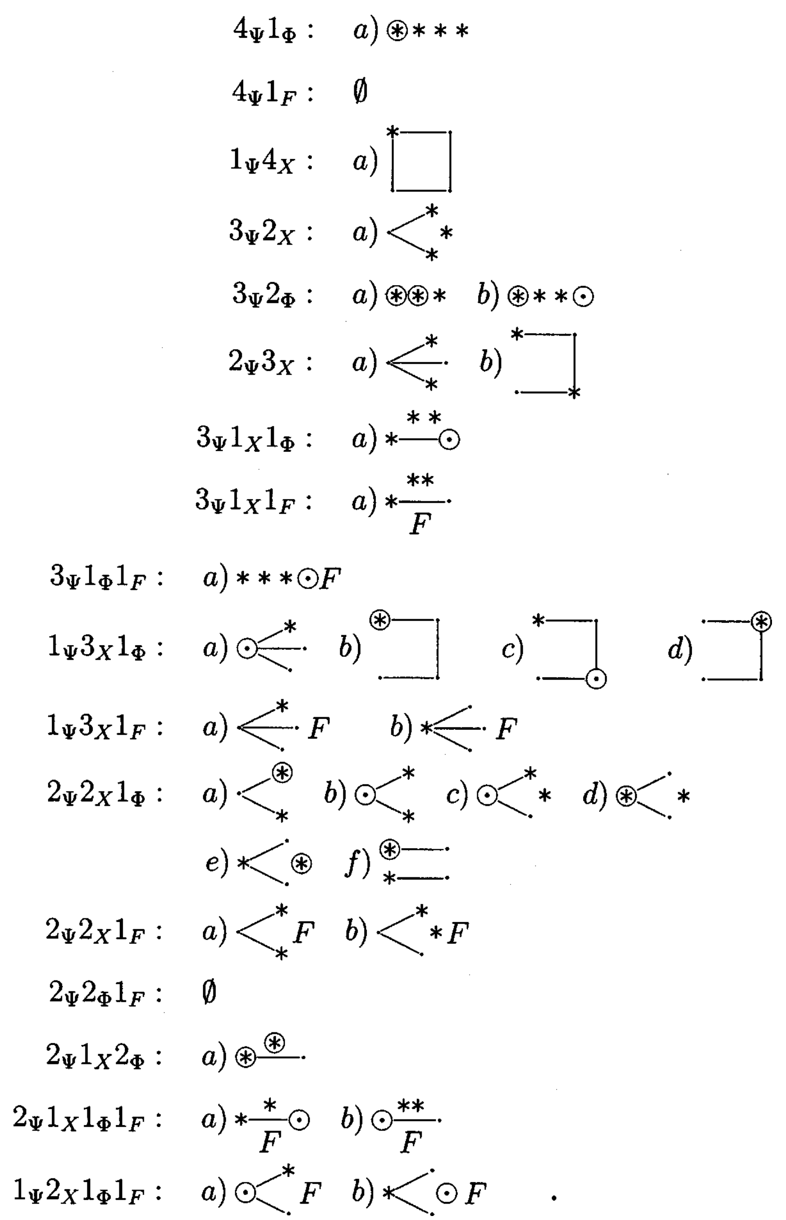
Now consider the case of the graphs of degree 5. Recall that one knows at least some of the nonadmissible graphs of this degree, these are the graphs from the G-equivalence class of the graph $****F$.

Proposition 10: (1) A graph of degree 5 is admissible iff it does not contain a nonadmissible graph of degree 3 and does not belong to the G-equivalence class of the graph $****F$;

(2) *Every admissible graph of degree 5 is G-equivalent to the graph* $\circledast***$.

Proof: Let us generate the list of all graphs of degree 5 which do not contain a non-admissible subgraph of degree 3. There exist six ways to decompose 5 into a sum of four non-negative integers, $5=5+0+0+0=4+1+0+0=3+2+0+0=3+1+1+0=2+2+1+0=2+1+1+1$. This gives six labels, 5_Ψ, $4_\Psi 1_X$, $3_\Psi 2_X$, $3_\Psi 1_X 1_\Phi$, $2_\Psi 2_X 1_\Phi$, $2_\Psi 1_X 1_\Phi 1_F$. By permuting the symbols Ψ, X, Φ, and F, one generates the other labels. Excluding from the resulting set of labels the ones, which do not satisfy the conditions $k_3 \leqslant k_1 \leqslant 4$ and $k_4 \leqslant 1$, where k_1, k_3, and k_4 are the numbers in a label associated to the symbols Ψ, Φ, and F, respectively, one arrives at the following: 5_Ψ generates 5_X, but itself is deleted from the list; $4_\Psi 1_X$ generates $4_\Psi 1_\Phi$, $4_\Psi 1_F$, $4_X 1_F$, and $1_\Psi 4_X$; $3_\Psi 2_X$ generates

$3_\Psi 2_\Phi$, $2_\Psi 3_X$; $3_\Psi 1_X 1_\Phi$ generates $3_\Psi 1_X 1_F$, $3_\Psi 1_\Phi 1_F$, $1_\Psi 3_X 1_\Phi$, and $1_\Psi 3_X 1_F$; $2_\Psi 2_X 1_\Phi$ generates $2_\Psi 2_X 1_F$, $2_\Psi 2_\Phi 1_F$, and $2_\Psi 1_X 2_\Phi$; finally, $2_\Psi 1_X 1_\Phi 1_F$ generates $1_\Psi 2_X 1_\Phi 1_F$. This yields 20 labels in total. Inside each label the corresponding graphs are generated by exploring all the possible variants. Note, that in some of the cases one inevitably obtains a graph containing a nonadmissible part of degree 3, a graph under the label 5_X should contain a triangle since there are maximum four vertices on a graph; a graph under the label $4_\Psi 1_X$ always contains $*—*$; and a graph under the label $4_X 1_F$ contains either a triangle or a graph $\|F$. It means that the labels 5_X, $4_\Psi 1_X$, and $4_X 1_F$ can be omitted. After deleting from the list the known nonadmissible graphs of degree 5, i.e., the graphs which are G-equivalent to the graph $****F$, the following list is obtained:



One verifies, that this list coincides with the set of all primary graphs belonging to the G-equivalence class of the graph $***\circledast$. Since the latter graph is known to be admissible, all these graphs are admissible. □

One could proceed in a similar way and investigate the cases of the graphs of degree 6, 7, and

8, but it turns out that one does not have to do it. Recall that the aim is to prove that the set of projective lines A is saturated with respect to the orthogonality relation $\perp$.

Theorem 1: *The set A is saturated with respect to $\perp$. Moreover, for every $B \in \mathcal{P}_{\perp}(A)$, one has the following*:

(1) *If B has a shadow of degree* 1 *or* 2, *then it admits a pure complete extension.*
(2) *If B has a shadow of degree 3, then it admits a complete extension hanging over a doublet.*
(3) *If the shadow of B is a doublet, then B has a unique pure complete extension.*
(4) *If the shadow of B is a singlet, then B has a unique complete extension; this extension has a shadow of degree 8.*
(5) *If the degree of the shadow of B is* $\geqslant 5$, *then B has a unique complete extension; this extension has a shadow of degree 8.*

Proof: Take any $B \in \mathcal{P}_{\perp}(A)$ and denote $Q \coloneqq \mathcal{P}(\eta)(B) \in L$. Let γ denote the shadow of B, i.e., the graph associated to $[Q] \in \Gamma$, and d denote the degree of γ.

(1) Suppose that $d=1$ or $d=2$. Then γ is G-equivalent to $*$, or respectively, $**$. Represent the corresponding latter graph by some $Q' \in L$. There exists $\hat{T} \coloneqq T_{S_1} T_{S_2} \dots T_{S_m}$, where $S_1, S_2, \dots, S_m$ are some nonempty proper subsets of V (m is some natural number), such that $Q' = \hat{T}(Q)$. Denote $B' \coloneqq \hat{\theta}(B)$, where $\hat{\theta} \coloneqq \theta_{S_1}\theta_{S_2} \cdots \theta_{S_m}$. Choose a complete set B'' containing B' such that $\mathcal{P}(\eta)(B'') = \mathcal{P}(\eta)(B')$. The set $\hat{B} \coloneqq \hat{\theta}^{-1}(B)$ yields the required pure complete extension of B.

(2) The case of $d=3$ is similar and the difference is that γ is now G-equivalent to $***$. Let one choose Q' and construct $\hat{T}$, $\hat{\theta}$ and B' by analogy with the previous case. The complete set $B'' \supset B'$ cannot be chosen now to have the same shadow as B', but it can have a shadow $***\odot$. Denote $Q'' \coloneqq \mathcal{P}(\eta)(B'')$. The set $\hat{B} \coloneqq \hat{\theta}^{-1}(B'')$ gives the required extension of B. This extension hangs over a graph associated to $[\hat{Q}]$, where $\hat{Q} \coloneqq \hat{T}^{-1}(Q'')$, which is G-equivalent to $***\odot$ and by that is a doublet.

(3) Suppose that $d=4$ and γ is a doublet. Then it is G-equivalent to $***\odot$. Choose Q' and define $\hat{T}$, $\hat{\theta}$, and B' in analogy with the two previous cases. There exists a unique pure extension B'' of B'. The required unique pure complete extension $\hat{B}$ of B will be of the form $\hat{B} \coloneqq \hat{\theta}^{-1}(B'')$. Note that not all the complete extensions of a set B have to be pure.

(4) Suppose that $d=4$ and γ is a singlet. Then γ is G-equivalent to $****$. Choose Q' and construct $\hat{T}$, $\hat{\theta}$, and B' in analogy with the three previous cases, i.e., we have $Q' = \hat{T}(Q)$, $B' = \hat{\theta}(B)$, $\eta \circ \hat{\theta} = \hat{T} \circ \eta$. The set B' has a unique complete extension B'' and this extension hangs over $\circledast\circledast\circledast\circledast$. The set $\hat{B} \coloneqq \hat{\theta}^{-1}(B'')$ is the unique complete extension of the original set B. The shadow of $\hat{B}$ is of degree 8 and is given by the graph associated to $[\hat{T}^{-1}(Q'')]$, where $Q'' \coloneqq \mathcal{P}(\eta)(B'')$.

(5) Suppose that $d \geqslant 5$. Recall that the set of all admissible graphs of degree 5 is the G-equivalence class of the graph $***\circledast$. This graph contains a singlet $****$. It follows, that every admissible graph of degree 5 contains a singlet, since a singlet can be G-equivalent only to a singlet. Whenever a set B_0 hangs over a singlet $****$, the corresponding complete extension exists and is unique. At the same time, for every $v \in V$ there exists a unique projective line of the form Φ_{ρ}^{v}, $\rho \in R_{v}^{-}$ which is orthogonal to every element of B_0; there exist no projective lines of X or F type, which are orthogonal to every element of B_0. Thus the construction of the complete extension of B_0 may be viewed as a step-by-step appending of the mentioned unique Φ_{ρ}^{v} to the set B_0 as v runs over V. One concludes, that whenever one has some sets B_1, B_2, B_3, and B_4 hanging over the graphs $***\circledast$, $**\circledast\circledast$, $*$, and $\circledast\circledast\circledast\circledast$, respectively, one may extract from each of them a part $B_i^0 \subset B_i$ ($i=1,2,3,4$) hanging over $****$; the unique complete extension $\hat{B}_i^0$ of B_i^0 at the same time plays a role of a unique complete extension of B_i and one has $B_i^0 \subset B_i \subset \hat{B}_i^0$, $i=1,2,3,4$.

Now let $B \in \mathcal{P}_{\perp}(A)$ have an arbitrary shadow γ of degree $d \geqslant 5$. Every such γ should contain an admissible graph $\overline{\gamma}$ of degree 5 and it is possible to choose in B a subset $\overline{B}$ hanging over $\overline{\gamma}$. Using the lists of graphs from the proofs of the two previous propositions, one verifies in a straightforward way that every admissible graph of degree 5 contains a singlet. It means, that one can always find in $\overline{\gamma}$ some singlet γ_0 and choose $B_0 \subset \overline{B}$ which hangs over this singlet. The graph γ_0, as any other singlet, is G-equivalent to $****$. Denote $Q_0 := \eta(B_0)$ and choose any $Q_0' \in L$ representing the $****$. There exists a collection of nonempty proper subsets $S_i \subset V$, $i = 1, 2, \ldots, m$ (m is some integer), such that $Q_0' = \hat{T}(Q_0)$, where $\hat{T} := T_{S_1} T_{S_2} \cdots T_{S_m} \in G$. Denote $\hat{\theta} := \theta_{S_1} \theta_{S_2} \cdots \theta_{S_m}$. The set $\hat{\theta}(B_0)$ has a shadow $****$, the set $\hat{\theta}(\overline{B})$ has a shadow $\circledast***$ and the set $\hat{\theta}(B)$ has a shadow consisting of four stars $*$ and $d-4$ circles $\odot$. A unique complete extension $\widetilde{B}$ of $\hat{\theta}(B_0)$ is at the same time a unique complete extension for $\hat{\theta}(\overline{B})$ and $\hat{\theta}(B)$. The shadow of $\widetilde{B}$ has a degree 8 and is of the form $\circledast\circledast\circledast\circledast$. The set $\hat{B} := \hat{\theta}^{-1}(\widetilde{B})$ has a shadow of degree 8 as well and provides the required unique complete extension of the set B. □

IX. TRANSITIVE ACTION

We have the set A of 120 projective lines in $\mathcal{H} \simeq \mathbb{C}^8$ which produces a Kochen–Specker-type contradiction and is saturated with respect to the orthogonality relation $\perp$. Note, that if one extracts a subset A_0 from A consisting of all projective lines of Ψ and Φ type, one can still prove that A_0 is saturated with respect to $\perp$, but A_0 will not produce a Kochen–Specker-type contradiction. Consider the set $C(A)$ of all complete subsets of A and denote by $C_d(A)$ the subset of $C(A)$ consisting of all the elements which have a shadow of degree $d \in \mathbb{N}$.

Theorem 2: (1) *The set* $C_d(A) \neq \emptyset$ *iff* d *is equal to* 1, 2, 4 *or* 8;
(2) *The group* $\mathcal{G}$ *acts transitively on each of the* $C_d(A)$, $d = 1, 2, 4, 8$.

Proof: (1) The statement that d cannot be other than 1, 2, 4 or 8 whenever $C_d(A) \neq \emptyset$ follows from the fact that a set hanging over a graph of degree 3 cannot have pure complete extensions and the fact that if a set hangs over a graph of degree $\geqslant 5$, then its complete extension always has a shadow of degree 8. The examples of realizations of all four mentioned possibilities have been given in the proof of Proposition 1.

(2) Recall that $\mathcal{G}$ is a subgroup of $Bij(A)$ and the action of $\theta \in \mathcal{G}$ on $B \in C(A)$ is given by $\mathcal{P}(\theta)(B)$. Let us start with the component $C_8(A)$. Every element $B \in C_8(A)$ can be viewed as $B = \mathcal{P}(\theta)(B_0)$, where B_0 is some element of $C_8(A)$ with a shadow $\circledast\circledast\circledast\circledast$ and θ is some element of $\mathcal{G}$. Denote by $C_8^0(A) \subset C_8(A)$ the set of all complete subsets with the specified shadow. It follows, that the problem is reduced to the following: for every two $B, B' \in C_8^0(A)$ show that there exists $\theta \in \mathcal{G}$ such that $\mathcal{P}(\theta)(B) = B'$. Every element of $C_8^0(A)$ is determined by its part which hangs over a singlet $****$. There are as many elements in $C_8^0(A)$ as the sets hanging over this singlet. Take any $B, B' \in C_8^0(A)$ and denote by $B_1 \subset B$ and by $B_1' \subset B'$ their parts hanging over $****$. One associates in the way described in part (6) of the proof of Proposition 1 to B_1 and B_1' some functions φ and φ', respectively, $\varphi, \varphi' : E \to \mathbb{Z}_2$, where E is the set of all edges of the tetrahedron representing $\mathcal{P}(V)^{\times}$. One verifies, that $B_1' = \mathcal{P}(I_{\varphi+\varphi'})(B_1)$. This implies that the action of $\mathcal{G}$ on $C_8(A)$ is transitive.

Now consider the case of $C_4(A)$. This set consists of all those complete subsets of A hanging over a doublet. Denote by $C_4^0(A) \subset C_4(A)$ the set of all complete subsets with shadow $***\odot$. In analogy to the case of $C_8(A)$, the original problem reduces to the problem to show that for every $B, B' \in C_4^0(A)$ such that $\mathcal{P}(\eta)(B) = \mathcal{P}(\eta)(B')$ [recall that $\eta : A \to \mathcal{P}(V)$ denotes the natural map], there exists $\theta \in \mathcal{G}$ such that $\mathcal{P}(\theta)(B) = B'$. Take any of the mentioned B and B' and assume without loss of generality that $\mathcal{P}(\eta)(B) \in L$ is visualized by a graph $*_e *_f *_g \odot_h$. One associates in the way as pointed out in part (5) of the proof of Proposition 1 to B a triple of parameters a, b, and c. Let $\psi : \{ef, eg, fg\} \to \mathbb{Z}_2$ denote the function, which has values on the edges ef, eg, and fg given by a, b, and c, respectively. In a similar way a function $\psi' : \{ef, eg, fg\} \to \mathbb{Z}_2$ is associated to the set B'. Choose any $\varphi, \varphi' : E \to \mathbb{Z}_2$ such that their restrictions to $\{ef, eg, fg\}$ coincide with ψ and ψ', respectively. It is clear that $B' = \mathcal{P}(I_{\varphi+\varphi'})(B)$. This completes the proof for the case $C_4(A)$.

The investigation of the case $C_2(A)$ is similar and contains a graph $**$ and a pair of $\mathbb{Z}_2$-valued functions on just one edge. The case $C_1(A)$ involves a graph $*$ and does not require a similar construction of $\mathbb{Z}_2$-valued functions. □

ACKNOWLEDGMENTS

The present work has been supported by the European Science Foundation programme on Noncommutative Geometry.

[1] Aravind, P. K. and Lee-Elkin, F., "Two noncolourable configurations in four dimensions illustrating the Kochen–Specker theorem," J. Phys. A **31**, 9829–9834 (1998).
[2] Bell, J. S., "On the problem of hidden variables in quantum mechanics," Rev. Mod. Phys. **38**, 447–452 (1966).
[3] Greenberger, D. M., Horne, M. A., Shimony, A., and Zeilinger, A., "Bell's theorem without inequalities," Am. J. Phys. **58**, 1131–1143 (1990).
[4] Kernaghan, M. and Peres, A., "Kochen–Specker theorem for eight-dimensional space," Phys. Lett. A **198**, 1–5 (1995).
[5] Kochen, S. and Specker, E. P., "The problem of hidden variables in quantum mechanics," J. Math. Mech. **17**, 59–87 (1967).
[6] Mermin, D. N., "Hidden variables and the two theorems of John Bell," Rev. Mod. Phys. **65**, 803–815 (1993).
[7] Peres, A., "Two simple proofs of the Kochen–Specker theorem," J. Phys. A **24**, L175–L178 (1991).
[8] Ruuge, A. E., "Indeterministic objects in the category of effect algebras and the passage to the semiclassical limit," Int. J. Theor. Phys. **43**, 2325–2354 (2004).
[9] Zimba, J. and Penrose, R., "On Bell nonlocality without probabilities: More curious geometry," Stud. Hist. Philos. Sci **24**, 697–720 (1993).