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A proof of the Popov Conjecture for quivers

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Abstract

Let Q be a quiver with dimension vector α. We show that if the space of isomorphism classes of semisimple representations $\mathtt{iss}(Q,\alpha)$ of Q of dimension vector α is not smooth, then the quotient map $\pi:\mathtt{rep}(Q,\alpha) \twoheadrightarrow \mathtt{iss}(Q,\alpha)$ is not equidimensional. In other words, we prove the Popov Conjecture for the natural action of the linear reductive group GL_α on the space $\mathtt{rep}(Q,\alpha)$ of α-dimensional representations of the quiver Q.

1. Introduction

Throughout this paper we will work over an algebraically closed field of characteristic zero, which we will denote by $\mathbb{C}$. Given an affine space V and a (connected) linear reductive group G acting on it, we may consider the quotient variety $V/\!/G$. By $\pi: V \twoheadrightarrow V/\!/G$ we denote the corresponding quotient map. We call a morphism equidimensional if all its fibers have the same dimension. In [1], V.L. Popov made the following conjecture:

Conjecture 1 (Popov's Conjecture). *If π is equidimensional, then $V/\!/G$ is an affine space.*

In this paper, we will give a proof for this conjecture in the case where $V = \mathtt{rep}(Q,\alpha)$ the α-dimensional representations of a quiver Q and $G = \mathrm{GL}_\alpha$ acting on V by basechange. That is, $V/\!/G = \mathtt{iss}(Q,\alpha)$ the space of isomorphism classes of semisimple representations of Q with dimension vector α.

The first step in the proof uses the description obtained by Raf Bocklandt in [2] of quiver settings (Q,α) with a smooth space of isomorphism classes of semisimple representations.

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He introduces three *reduction steps*, removing vertices and arrows that satisfy certain boundary conditions from a quiver setting. The quiver setting (Q', α') obtained when no more vertices satisfy these conditions is called a *reduced quiver setting*. The main result from [2] is that $\mathtt{iss}(Q, \alpha)$ is smooth if and only if $\mathtt{iss}(Q', \alpha')$ is smooth. We will show in Section 3 that

Theorem 1. *Let (Q, α) be quiver setting for which $\mathtt{iss}(Q, \alpha)$ is not smooth and let (Q', α') be its reduced quiver setting, then the quotient map*

$$\pi : \mathtt{rep}(Q, \alpha) \twoheadrightarrow \mathtt{iss}(Q, \alpha)$$

is not equidimensional if for the reduced setting the quotient map

$$\pi' : \mathtt{rep}\big(Q', \alpha'\big) \twoheadrightarrow \mathtt{iss}\big(Q', \alpha'\big)$$

is not equidimensional.

This theorem reduces the question of non-equidimensionality of fibers to a reduced quiver setting.

The next step in the proof is the explicit construction of two fibers of different dimension in the case of a reduced quiver setting. This construction uses the Luna Slice Theorem [3]. More specifically it uses the description of the Luna Slice Theorem for representations of quivers as obtained by Lieven Le Bruyn and Claudio Procesi in [4]. It also uses the fact that the Popov Conjecture was proved for the action of tori on varieties by David Wehlau in [5]. This leads to the following result in Section 4.

Theorem 2. *Let (Q, α) be a reduced quiver setting that has a singular space $\mathtt{iss}(Q, \alpha)$, then the quotient map*

$$\pi : \mathtt{rep}(Q, \alpha) \twoheadrightarrow \mathtt{iss}(Q, \alpha)$$

is not equidimensional.

Combining Theorem 1 with Theorem 2 we obtain

Theorem 3 (the Popov Conjecture for quivers). *Let (Q, α) be a quiver setting, then $\mathtt{iss}(Q, \alpha)$ is an affine space if the quotient map*

$$\pi : \mathtt{rep}(Q, \alpha) \twoheadrightarrow \mathtt{iss}(Q, \alpha)$$

is equidimensional.

2. Preliminaries

2.1. Definitions and notations

We begin by introducing the notions and results we will need throughout the rest of this paper.

Definition 1.

- A *quiver* is a four-tuple $Q = (Q_1, Q_2, h, t)$ consisting of a set of *vertices* Q_1, a set of *arrows* Q_2 and two maps $t : Q_2 \to Q_1$ and $h : Q_2 \to Q_1$ assigning to each arrow its tail, resp. its head:

$$\underset{h(a)}{\bigcirc} \xleftarrow{\quad a \quad} \underset{t(a)}{\bigcirc}$$

- A *dimension vector* of a quiver Q is a map $\alpha : Q_1 \to \mathbb{N}$ and a *quiver setting* is a couple (Q, α) of a quiver and an associated dimension vector. We will call

$$|\alpha| := \sum_{v \in Q_1} \alpha(v)$$

the *length of the dimension vector* α.
- Fix an ordering of the vertices of Q. The *Euler form of a quiver* Q is the bilinear form

$$\chi_Q : \mathbb{N}^{\#Q_1} \times \mathbb{N}^{\#Q_1} \to \mathbb{Z}$$

defined by the matrix having $\delta_{ij} - \#\{a \in Q_2 \mid h(a) = j,\ t(a) = i\}$ as element at location (i, j).
- A quiver is called *strongly connected* if and only if each pair of vertices in its vertex set belongs to an oriented cycle.

A quiver setting is graphically depicted by drawing the quiver and listing in each vertex v the dimension $\alpha(v)$.

Definition 2.

- An α-dimensional representation V of a quiver Q assigns to each vertex $v \in Q_1$ a linear space $\mathbb{C}^{\alpha(v)}$ and to each arrow $a \in Q_2$ a matrix $V(a) \in M_{\alpha(h(a)) \times \alpha(t(a))}(\mathbb{C})$. We denote by $\mathtt{rep}(Q, \alpha)$ the space of all α-dimensional representations of Q. That is,

$$\mathtt{rep}(Q, \alpha) = \bigoplus_{a \in Q_2} M_{\alpha(h(a)) \times \alpha(t(a))}(\mathbb{C}).$$

- We have a natural action of the reductive group

$$\mathrm{GL}_\alpha := \prod_{v\in Q_1} \mathrm{GL}_{\alpha(v)}(\mathbb{C})$$

  on a representation V defined by basechange in the vectorspaces. That is

$$(g_v)_{v\in Q_1}\big(V(a)\big)_{a\in Q_2} = \big(g_{h(a)}V(a)g_{t(a)}^{-1}\big)_{a\in Q_2}.$$

- The quotient space with respect to this action classifies all isomorphism classes of semisimple representations and is denoted by $\mathtt{iss}(Q,\alpha)$. The quotient map with respect to this action will be denoted by

$$\pi : \mathtt{rep}(Q,\alpha) \twoheadrightarrow \mathtt{iss}(Q,\alpha).$$

- The fibre of π in $\pi(0)$ is called the *nullcone* of the quiver setting and is denoted by $\mathtt{Null}(Q,\alpha)$.

For the study of the equidimensionality of the quotient map we introduce

Definition 3. For a given quiver setting (Q,α), we define the *defect of the equidimensionality*

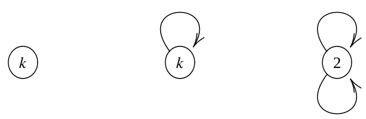
$$\mathrm{def}(Q,\alpha) := \dim \mathtt{Null}(Q,\alpha) - \dim \mathtt{rep}(Q,\alpha) + \dim \mathtt{iss}(Q,\alpha).$$

We then have

Proposition 4 [6, II.4.2, Folgerung 1]. *For a given quiver setting (Q,α) as above, the quotient map π is equidimensional if and only if* $\mathrm{def}(Q,\alpha) = 0$.

2.2. Reducing quiver settings

In [2], Raf Bocklandt introduced three different types of reduction steps on a quiver setting (Q,α). He then continues to show that a quiver setting is coregular (i.e., its quotient space is smooth or, equivalently, is an affine space) if its reduced quiver setting is coregular, where the only reduced coregular quiver settings are

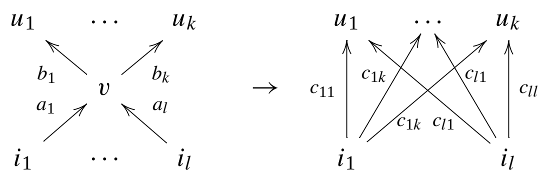


The reduction steps used are

R_I^v: let v be a vertex without loops such that

$$\chi_Q(\alpha,\varepsilon_v) \geqslant 0 \quad \text{or} \quad \chi_Q(\varepsilon_v,\alpha) \geqslant 0.$$

Construct a new quiver setting $(R_I^v(Q), R_I^v(\alpha))$ by removing v and connecting all arrows running through v:



For this step we have $\mathtt{iss}(Q,\alpha) \cong \mathtt{iss}(R_I^v(Q), R_I^v(\alpha))$.

R_{II}^v: let v be a vertex with $\alpha(v) = 1$ and n loops. Let $(R_{II}^v(Q), \alpha)$ be the quiver setting obtained by removing all these loops. We then have

$$\mathtt{iss}(Q,\alpha) \cong \mathtt{iss}\big(R_{II}^v(Q),\alpha\big) \times \mathbb{A}^n.$$

R_{III}^v: let v be a vertex with one loop and $\alpha(v) = n$ such that

$$\chi_Q(\alpha, \varepsilon_v) = -1 \quad \text{or} \quad \chi_Q(\varepsilon_v, \alpha) = -1.$$

Let $(R_{III}^v(Q), \alpha)$ be the quiver setting obtained by removing the loop in v and adding $n-1$ additional arrows between v and its neighbouring vertex with dimension 1 (all having the same orientation as the original arrow). For this step we have

$$\mathtt{iss}(Q,\alpha) \cong \mathtt{iss}\big(R_{III}^v(Q),\alpha\big) \times \mathbb{A}^n.$$

2.3. The Luna Slice Theorem for quivers

In this section we will recall some results by Lieven Le Bruyn and Claudio Procesi on semisimple representations of quivers obtained in [4].

Theorem 4 (Le Bruyn–Procesi). *Let (Q,α) be a quiver setting. Let*

$$\pi : \mathtt{rep}(Q,\alpha) \twoheadrightarrow \mathtt{iss}(Q,\alpha)$$

be the quotient with respect to the natural GL_α-action. Let $S \in \mathtt{iss}(Q,\alpha)$ correspond to the following decomposition in simples

$$S = S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k},$$

with S_i a simple representation of dimension vector α_i (for $1 \leqslant i \leqslant k$).

Define the quiver Q_S as the quiver with k vertices and $\delta_{ij} - \chi_Q(\alpha_i, \alpha_j)$ arrows from vertex i to vertex j. Define α_S as the dimension vector that assigns e_i to vertex i (for $1 \leqslant i \leqslant k$). Then

(1) *there exists an étale isomorphism between an open neighbourhood of S in $\mathtt{iss}(Q,\alpha)$ and an open neighbourhood of the zero representation in $\mathtt{iss}(Q_S,\alpha_S)$;*

(2) *there is an isomorphism as* Gl_α*-varieties*

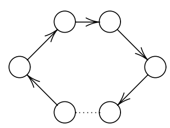
$$\pi^{-1}(S) \cong \mathrm{GL}_\alpha \times^{\mathrm{GL}_{\alpha_S}} \mathtt{Null}(Q_S, \alpha_S).$$

The quiver setting (Q_S, α_S) is called the *local quiver* of S.

In [4] a criterium for the existence of simple representations of dimension vector α was given.

Theorem 5. *Let* (Q, α) *be a quiver setting such that for all vertices* v *we have that* $\alpha(v) \geqslant 1$. *There exist simple representations of dimension vector* α *if and only if*

- Q *has at least two vertices and is of the form*



  and $\alpha(v) = 1$ *for all vertices* v;
- Q *has exactly one vertex, one loop and* $\alpha = 1$;
- *none of the above, but* Q *is strongly connected and*

$$\forall v \in Q_1\colon\ \chi_Q(\alpha, \varepsilon_v) \leqslant 0 \text{ and } \chi_Q(\varepsilon_v, \alpha) \leqslant 0.$$

  Here $\varepsilon_v(w) := \delta_{vw}$ *for all* $w \in Q_1$.

If $\alpha(v) = 0$ *for some vertices* v, (Q, α) *has simple representations if* (Q', α') *has simple representations, where* (Q', α') *is the quiver obtained by removing all vertices* v *with* $\alpha(v) = 0$.

We need one last result from [4]. Let $c = a_1 \dots a_k$ be a cycle in the quiver Q and define the polynomial function

$$f_c : \mathtt{rep}(Q, \alpha) \to \mathbb{C}\colon\ V \mapsto \mathrm{tr}\big(V(a_1) \dots V(a_k)\big);$$

then

Theorem 6. *The ring of polynomial invariants for the action of* GL_α *on* $\mathtt{rep}(Q, \alpha)$

$$\mathbb{C}\big[\mathtt{iss}(Q, \alpha)\big] = \mathbb{C}\big[\mathtt{rep}(Q, \alpha)\big]^{\mathrm{GL}_\alpha}$$

is generated by all f_c, *where* c *is a cycle of length at most* $|\alpha|^2$ *of which all vertices* v *ran through more than once have* $\alpha(v) > 1$.

3. The effects of reduction steps

In this section, we will show that the inverses of the reduction steps introduced by Bocklandt actually increase the defect of the equidimensionality of a singular quiver setting.

First, we turn out attention towards reduction step R_I.

Proposition 5. *Let (Q,α) be a quiver setting containing a vertex v without loops with*

$$\chi_Q\big(\alpha(v),\varepsilon_v\big) \geqslant 0 \quad \textit{or} \quad \chi_Q\big(\varepsilon_v,\alpha(v)\big) \geqslant 0$$

then

$$\operatorname{def}(Q,\alpha) \geqslant \operatorname{def}\big(R_I^v(Q), R_I^v(\alpha)\big).$$

Proof. In [2] the quotient map $\pi : \mathtt{rep}(Q,\alpha) \twoheadrightarrow \mathtt{iss}(Q,\alpha)$ is factored into a projection

$$\pi_v : \mathtt{rep}(Q,\alpha) \twoheadrightarrow \mathtt{rep}\big(R_I^v(Q), R_I^v(\alpha)\big)$$

that maps a representation V onto the representation $\pi_v(V)$ defined as

- $\pi_v(V)(a) = V(a)$ for all arrows a that have $h(a) \neq v$ and $t(a) \neq v$,
- $\pi_v(V)(a) = V(y)V(x)$ if a is the arrow defined in $R_I^v(Q)$ as the composition of arrows y and x,

and the quotient map $\phi : \mathtt{rep}(R_I^v(Q), R_I^v(\alpha)) \twoheadrightarrow \mathtt{iss}(R_I^v(Q), R_I^v(\alpha))$. This implies

$$\begin{aligned}
\operatorname{def}(Q,\alpha) &= \dim \mathtt{Null}(Q,\alpha) - \dim \mathtt{rep}(Q,\alpha) + \dim \mathtt{iss}(Q,\alpha) \\
&= \dim \pi_v^{-1}\big(\phi^{-1}\big(\phi\big(\pi_v(0)\big)\big)\big) - \dim \mathtt{rep}(Q,\alpha) + \dim \mathtt{iss}(Q,\alpha) \\
&= \dim \pi_v^{-1}\big(\mathtt{Null}\big(R_I^v(Q), R_I^v(\alpha)\big)\big) - \dim \mathtt{rep}(Q,\alpha) \\
&\quad + \dim \mathtt{iss}\big(R_I^v(Q), R_I^v(\alpha)\big) \\
&\geqslant \dim \mathtt{Null}\big(R_I^v(Q), R_I^v(\alpha)\big) + \dim \mathtt{rep}(Q,\alpha) - \dim \mathtt{rep}\big(R_I^v(Q), R_I^v(\alpha)\big) \\
&\quad - \dim \mathtt{rep}(Q,\alpha) + \dim \mathtt{iss}\big(R_I^v(Q), R_I^v(\alpha)\big) \\
&= \operatorname{def}\big(R_I^v(Q), R_I^v(\alpha)\big).
\end{aligned}$$

The last inequality here holds by, e.g., [7, ex. 3.22a] applied to the surjective map π_v. □

Reduction step R_{II} is the most straightforward reduction step to verify:

Proposition 6. *Let (Q,α) be a quiver setting containing a vertex v with $\alpha(v) = 1$ and n loops in v, then*

$$\operatorname{def}(Q,\alpha) = \operatorname{def}\big(R_{II}^v(Q),\alpha\big).$$

Proof. Note that a representation $V \in \mathtt{Null}(Q,\alpha)$ has to have all loops in v equal to 0. This means that

$$\begin{aligned}\operatorname{def}(Q,\alpha) &= \dim \mathtt{Null}(Q,\alpha) - \dim \mathtt{rep}(Q,\alpha) + \dim \mathtt{iss}(Q,\alpha)\\ &= \dim \mathtt{Null}\big(R_{II}^v(Q),\alpha\big) - \dim \mathtt{rep}\big(R_{II}^v(Q),\alpha\big) - n\\ &\quad + \dim \mathtt{iss}\big(R_{II}^v(Q),\alpha\big) + n\\ &= \operatorname{def}\big(R_{II}^v(Q),\alpha\big)\end{aligned}$$

which concludes the proof. □

Next, let us turn our attention towards reduction step R_{III}:

Proposition 7. *Let (Q,α) be a quiver setting and let $v \in Q_1$ with one loop such that*

$$\chi_Q(\alpha,\varepsilon_v) = -1 \quad \textit{or} \quad \chi_Q(\varepsilon_v,\alpha) = -1$$

then

$$\operatorname{def}(Q,\alpha) \geqslant \operatorname{def}\big(R_{III}^v(Q),\alpha\big).$$

Proof. We will give the proof for $\chi_Q(\alpha,\varepsilon_v) = -1$; the proof for the other situation is analogously. Let $x \in Q_2$ be the only arrow with $h(x) = v$, let y be the loop in v and let $n = \alpha(v)$. Denote by $x_0,\dots,x_{n-1} \in R_{III}^v(Q)_2$ the n arrows entering v. We let $\phi : \mathtt{rep}(Q,\alpha) \to \mathtt{rep}(R_{III}^v(Q),\alpha)$ be the map defined as

- $\forall a \neq x_0,\dots,x_{n-1} \in R_{III}^v(Q)_2$: $\phi(V)(a) = V(a)$,
- $\forall 0 \leqslant i \leqslant n-1$: $\phi(V)(x_i) = V(y)^i V(x)$.

For $W \in \mathtt{rep}(R_{III}^v(Q),\alpha)$ we set $X_W \in M_n(\mathbb{C})$ as the matrix with columns $W(x_0),\dots,W(x_{n-1})$.

Let $C \subset \mathtt{Null}(R_{III}^v(Q),\alpha)$ be an irreducible component with maximal dimension:

$$\dim C = \dim \mathtt{Null}\big(R_{III}^v(Q),\alpha\big).$$

Let

$$r = \max\big\{\operatorname{rk}(X_W) \mid W \in C\big\}.$$

If $r = 0$, we have a surjective morphism

$$\phi|_{C'} : C' \twoheadrightarrow C,$$

where $C' \subset \mathtt{Null}(Q,\alpha)$ consists of all representations $V \in \mathtt{Null}(Q,\alpha)$ that have $V(x) = 0$, $V(y) = 0$ and for which $\forall a \neq x, y \in Q_2$: $V(a) = W(a)$ for some representation $W \in C$. But then

$$\dim \mathtt{Null}(Q,\alpha) \geqslant \dim C' \geqslant \dim C = \dim\big(\mathtt{Null}\big(R^v_{III}(Q),\alpha\big)\big).$$

If $r > 0$, consider the dense open subset $C_r \subset C$ for which the rth principal leading minor of X_W is not zero (if this subset is empty, reorder the x_i to obtain another irreducible component of maximal dimension where this subset is not empty). Denote by e_i the ith basis element of the canonical basis for $\mathbb{C}^n$, then

$$B = \big\{W(x_0), \ldots, W(x_{r-1}), e_r, \ldots, e_{n-1}\big\}$$

is a basis for all $W \in C_r$ as the rth principal leading minor of X_W is nonzero. We then have a map

$$\psi : C_r \hookrightarrow \mathtt{Null}(Q,\alpha)$$

defined as

- $\forall a \neq x, y \in Q_2$: $\psi(W)(a) = W(a)$,
- $\psi(W)(x) = W(x_0)$,
- $\psi(W)(y)$ is defined with respect to B as the map that sends $W(x_i)$ to $W(x_{i+1})$ for $0 \leqslant i \leqslant r-2$, that sends $W(x_{r-1})$ to 0 and e_i to $W(x_i)$ for $r \leqslant i \leqslant n-1$.

One easily sees this is injective, so

$$\dim \mathtt{Null}\big(R^v_{III}(Q),\alpha\big) = \dim C \leqslant \dim \mathtt{Null}(Q,\alpha).$$

Now note that

$$\dim \mathtt{rep}(Q,\alpha) = \dim \mathtt{rep}\big(R^v_{III}(Q),\alpha\big) + n$$

and

$$\dim \mathtt{iss}(Q,\alpha) = \dim \mathtt{iss}\big(R^v_{III}(Q),\alpha\big) + n,$$

so

$$\begin{aligned}
\operatorname{def}(Q,\alpha) &= \dim \mathtt{Null}(Q,\alpha) - \dim \mathtt{rep}(Q,\alpha) + \dim \mathtt{iss}(Q,\alpha) \\
&\geqslant \dim \mathtt{Null}\big(R^v_{III}(Q),\alpha\big) - \big(\dim \mathtt{rep}\big(R^v_{III}(Q),\alpha\big) + n\big) \\
&\quad + \dim \mathtt{iss}\big(R^v_{III}(Q),\alpha\big) + n \\
&= \operatorname{def}\big(R^v_{III}(Q),\alpha\big). \qquad \square
\end{aligned}$$

Combining Propositions 5, 6, and 7 we obtain

Theorem 7. *Let (Q,α) be a singular quiver setting and let $(R(Q),R(\alpha))$ be its reduced quiver setting, then*

$$\operatorname{def}(Q,\alpha) \geqslant \operatorname{def}\bigl(R(Q),R(\alpha)\bigr).$$

In other words, the quotient map

$$\pi : \mathtt{rep}(Q,\alpha) \twoheadrightarrow \mathtt{iss}(Q,\alpha)$$

is not equidimensional if the quotient map

$$\pi_R : \mathtt{rep}\bigl(R(Q),R(\alpha)\bigr) \twoheadrightarrow \mathtt{iss}\bigl(R(Q),R(\alpha)\bigr)$$

is not equidimensional.

4. The Popov Conjecture for reduced quiver settings

First of all note we have the following straightforward result (by $\mathbf{1}$ we denote the dimension vector that is the constant function with value 1):

Proposition 8. *Let $(Q,\mathbf{1})$ be a reduced quiver setting such that $\mathtt{iss}(Q,\mathbf{1})$ has singularities, then the quotient map*

$$\pi : \mathtt{rep}(Q,\alpha) \twoheadrightarrow \mathtt{iss}(Q,\alpha)$$

is not equidimensional.

Proof. Note that the reductive group $\mathrm{GL}_{\mathbf{1}}$ is a torus. But for tori the Popov Conjecture was proved by Wehlau in [5]. □

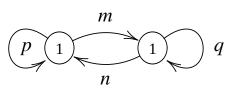
When the dimension vector is greater than 1 on certain vertices and the quiver Q is strongly connected, we can explicitly find the following singularity

Lemma 1. *Let (Q,α) be a reduced strongly connected quiver setting with singular $\mathtt{iss}(Q,\alpha)$. Let v be a vertex with maximal dimension $\alpha(v) \geqslant 2$, then*

(1) *there exist simple representations of dimension vector $\alpha - \varepsilon_v$;*
(2) *the point*

$$S = S_1 \oplus S_2 \in \mathtt{iss}(Q,\alpha),$$

where S_1 has dimension vector $\alpha - \varepsilon_v$ and S_2 has dimension vector ε_v has as local quiver



with $m, n \geqslant 2$.

Proof. We begin with proving (1). By Theorem 5 we have to show that

$$\chi_Q(\alpha - \varepsilon_v, \varepsilon_w) \leqslant 0 \quad \text{and} \quad \chi_Q(\varepsilon_w, \alpha - \varepsilon_v) \leqslant 0$$

for all vertices w of Q. We will show the first inequality, the second inequality is similar.

$$\chi_Q(\alpha - \varepsilon_v, \varepsilon_v) = -\sum_{a \in Q_2,\ h(a)=v} \alpha\bigl(t(a)\bigr) - (l-1)\bigl(\alpha(v) - 1\bigr),$$

where l is the number of loops in v.

- If $l = 0$, we get
$$-n = \chi_Q(\alpha - \varepsilon_v, \varepsilon_v) = \chi_Q(\alpha, \varepsilon_v) - 1 \leqslant -2$$
because (Q, α) is reduced and hence $\chi_Q(\alpha, \varepsilon_v) \leqslant -1$.
- If $l = 1$, we get
$$-n = \chi_Q(\alpha - \varepsilon_v, \varepsilon_v) = \chi_Q(\alpha, \varepsilon_v) \leqslant -2$$
because (Q, α) is reduced and hence $\chi_Q(\alpha, \varepsilon_v) \leqslant -2$ as there is exactly one loop in v.
- If $l \geqslant 2$, we definitely get
$$-n = \chi_Q(\alpha - \varepsilon_v, \varepsilon_v) \leqslant -2$$
as $(l-1)(\alpha(v) - 1) \geqslant 1$, Q is strongly connected and the quiver setting with one vertex, two loops and dimension vector 2 is excluded by the singularity of the quotient.

Now let $w \neq v$ and let x be the number of arrows from v to w. If $x = 0$, we get

$$\chi_Q(\alpha - \varepsilon_v, \varepsilon_w) = \chi_Q(\alpha, \varepsilon_w) - \chi_Q(\varepsilon_v, \varepsilon_w) = \chi_Q(\alpha, \varepsilon_w) \leqslant -1.$$

If $x = 1$, we get

$$\chi_Q(\alpha - \varepsilon_v, \varepsilon_w) = \chi_Q(\alpha, \varepsilon_w) - \chi_Q(\varepsilon_v, \varepsilon_w) = \chi_Q(\alpha, \varepsilon_w) + 1 \leqslant 0.$$

Assume now $x \geqslant 2$, then

$$\begin{aligned}\chi_Q(\alpha - \varepsilon_v, \varepsilon_w) &\leqslant \alpha(w) - x\alpha(v) + x \\ &\leqslant \alpha(v) - x\alpha(v) + x \\ &= (1-x)\alpha(v) + x \\ &\leqslant 0.\end{aligned}$$

Similar arguments show that $\chi_Q(\varepsilon_w, \alpha - \varepsilon_v) \leqslant 0$ for all vertices w in Q, so (1) holds.

In order to prove (2), note we already computed n in the proof of (1) and found that $n \geqslant 2$. Similar computations hold for m. □

This last lemma in combination with the Luna Slice Theorem (Theorem 4) suffices to prove the Popov Conjecture for strongly connected reduced quiver settings

Theorem 8. *Let (Q, α) be a strongly connected reduced quiver setting with singular $\mathtt{iss}(Q, \alpha)$, then the quotient map*

$$\pi : \mathtt{rep}(Q, \alpha) \twoheadrightarrow \mathtt{iss}(Q, \alpha)$$

is not equidimensional.

Proof. If $\alpha = \mathbf{1}$, we proved the non-equidimensionality in Proposition 8. Assume $\alpha \neq \mathbf{1}$. From the dimension formula for morphisms [6, AI.3.3] we know the map π has fibers F of dimension

$$F = \dim \mathtt{rep}(Q, \alpha) - \dim \mathtt{iss}(Q, \alpha) = \dim \mathrm{GL}_\alpha - 1.$$

Now let S be the singularity from Lemma 1, then Theorem 4 yields that

$$\pi^{-1}(S) \cong \mathrm{GL}_\alpha \times^{\mathrm{GL}_{\alpha_S}} \mathtt{Null}(Q_S, \alpha_S).$$

But then

$$\dim \pi^{-1}(S) = \dim \mathrm{GL}_\alpha - \dim \mathrm{GL}_{\alpha_S} + \dim \mathtt{Null}(Q_S, \alpha_S).$$

Now (Q_S, α_S) is a quiver setting with singular $\mathtt{iss}(Q_S, \alpha_S)$ as its reduced form is

$$\begin{tikzcd} \textcircled{1} \arrow[r, bend left, "m"] & \textcircled{1} \arrow[l, bend left, "n"] \end{tikzcd}$$

Moreover, Gl_{α_S} is a torus, so by the Popov Conjecture for tori we have that

$$\pi_S : \mathtt{rep}(Q_S, \alpha_S) \twoheadrightarrow \mathtt{iss}(Q_S, \alpha_S)$$

is not equidimensional. But then

$$\dim \mathtt{Null}(Q_S, \alpha_S) > \dim \mathtt{rep}(Q_S, \alpha_S) - \dim \mathtt{iss}(Q_S, \alpha_S) = \dim \mathrm{GL}_{\alpha_S} - 1.$$

This yields

$$\dim \pi^{-1}(S) > \dim \mathrm{GL}_\alpha - 1 = \dim F$$

and hence π is not equidimensional. □

The statement for quivers that are not strongly connected now follows from the statement for their connected components:

Theorem 9. *Let (Q, α) be a reduced quiver setting with singular $\mathtt{iss}(Q, \alpha)$, then the quotient map*

$$\pi : \mathtt{rep}(Q, \alpha) \twoheadrightarrow \mathtt{iss}(Q, \alpha)$$

is not equidimensional.

Proof. Assume Q has $Q_1, \ldots, Q_s$ strongly connected components (maximal subquivers that are strongly connected) with dimension vectors $\alpha_1, \ldots, \alpha_s$. We know from [4] that

$$\dim \mathtt{iss}(Q, \alpha) = \sum_{i=1}^{s} \dim \mathtt{iss}(Q_i, \alpha_i).$$

We also know that

$$\dim \mathtt{rep}(Q, \alpha) = \sum_{i=1}^{s} \dim \mathtt{rep}(Q_i, \alpha_i) + \dim R,$$

where

$$R = \bigoplus_{a \in Q_2'} M_{\alpha(h(a)) \times \alpha(t(a))}(\mathbb{C})$$

with Q_2' the set of all arrows in Q that are not a part of a cycle. Now note that because of Theorem 6 we have that

$$\mathtt{Null}(Q, \alpha) = \mathtt{Null}(Q_1, \alpha_1) \times \cdots \times \mathtt{Null}(Q_s, \alpha_s) \times R.$$

But then

$$\mathrm{def}(Q, \alpha) = \sum_{i=1}^{s} \mathrm{def}(Q_i, \alpha_i)$$

and as at least one (Q_i, α_i) must have a non-smooth $\mathtt{iss}(Q_i, \alpha_i)$, we have proved the claim. □

Combining Theorem 9 with Theorem 7, we arrive at Theorem 3:

Theorem 3 (the Popov Conjecture for quivers). *Let* (Q, α) *be a quiver setting, then* $\mathtt{iss}(Q, \alpha)$ *is an affine space if the quotient map*

$$\pi : \mathtt{rep}(Q, \alpha) \twoheadrightarrow \mathtt{iss}(Q, \alpha)$$

is equidimensional.

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