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Sign gradations on group ring extensions of graded rings

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Abstract

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For a (group)G-graded ring R and any submonoid H of the center $Z(G)$ containing the identity element e of G we define a G-gradation on the semigroup ring $R[H]$ such that $R[H]_e \cong R^{(H)}$ where $R^{(H)} = \bigoplus_{h\in H} R_h \subseteq R$, and then we give some applications of the new gradations to graded rings and I-adic filtrations.

Introduction

On many occasions information about a $\mathbb{Z}$-graded ring $R = \bigoplus_{n\in\mathbb{Z}} R_n$ may be obtained from properties of the positive part $R^+ = \bigoplus_{n\geq 0} R_n$ and the negative part $R^- = \bigoplus_{n\leq 0} R_n$ combined. When generalizing this technique it turns out that it is not the ordering of the grading group that is important here but in fact the relation with respect to a specified subsemigroup. In this note we first extend this idea to abelian group graded rings. If G is an abelian group, H a subsemigroup of G and R is a G-graded ring then we construct on the semigroup ring RH a G-gradation such that $(RH)_e = R^{(H)}$ where $e \in G$ is the neutral element of G and $R^{(H)} = \bigoplus_{h\in H} R_h$. Moreover, if R is strongly G-graded, that means if for all g and

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h in G we have that $R_h R_g = R_{hg}$, then RH is also strongly G-graded with respect to the new gradation. So we obtain a new tool for deriving properties of $R^{(H)}$ from those of R or vice versa. Particularly interesting are $\mathbb{Z}$-graded rings and the sign gradations on the polynomial rings over these; we include a few applications to the problem of lifting properties from the associated graded ring to an I-adically filtered ring. Throughout we use notation and terminology as in [8]. The main results are Proposition 2.2 and Corollary 2.3, concerning Auslander regularity, yielding new proofs for the regularity of rings of quantum sections (Remark 2.4), and some application to the calculation of K-theory is given in Section 3.

1. Sign gradations

Let G be an abelian group and R a G-graded ring. Consider any subsemigroup H of G and look at the semigroup ring $RH = \bigoplus_{h\in H} R.u_h$, where $u_h u_{h'} = u_{hh'}$ for $h,h' \in H$ and $ru_h = u_h r$ for every $r \in R$. We define a G-gradation on $RH = S$ by putting $S_g = \sum_{h\in H} R_g h.u_h$ for each $g \in G$. We write $R^{(H)}$ for the subring of R obtained by putting $R^{(H)} = \bigoplus_{h\in H} R_h$.

Lemma 1.1. (i) *$S = \bigoplus_{g\in G} S_g$ is a G-graded ring and $S_e = R^{(H)}$.*

(ii) *When R is strongly G-graded, S is strongly G-graded too, so in particular $R^{(H)}$-Mod $\cong$ S-gr_G (cf. [8]).*

Proof. Straightforward. □

Remark 1.2. The construction and the lemma remain unchanged when G is not abelian but $H \subset Z(G)$.

Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a $\mathbb{Z}$-graded ring and consider the polynomial ring $R[y]$ over R in y. For each $k \in \mathbb{Z}$, put

$$R[y]_k^+ = \bigoplus_{n\geq 0} R_{k+n} y^n , \qquad R[y]_k^- = \bigoplus_{n\geq 0} R_{k-n} y^n ;$$

then from the lemma we have the following:

Corollary 1.3. (i) *$R[y] = \bigoplus_{k\in\mathbb{Z}} R[y]_k^+ = \bigoplus_{k\in\mathbb{Z}} R[y]_k^-$.*

(ii) *$\{R[y]_k^+\}_{k\in\mathbb{Z}}$ resp. $\{R[y]_k^-\}_{k\in\mathbb{Z}}$ is a $\mathbb{Z}$-grading on $R[y]$, $R[y]_0^+ = \bigoplus_{n\geq 0} R_n y^n \cong R^+$, $R[y]_0^- = \bigoplus_{n\geq 0} R_{-n} y^n \cong R^-$.*

(iii) *If R is strongly graded then $R[y]$ is strongly graded with respect to the $+$-gradation and the $-$-gradation.*

Remarks 1.4. (1) If G is any ordered abelian group, with positive cone P say, then it is clear how to define a $\pm$-gradation on the semigroup ring RP such that $[RP]_0^- = R^{(-P)}$, $[RP]_0^+ = R^{(P)}$.

(2) In case $G = \mathbb{Z}^n$, one can apply the foregoing to $\mathbb{N}^n = P$. But for a given sign $\sigma = (\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i = \pm 1$ one can also define the σ-gradations on the semigroup ring $R(\mathbb{N} \oplus \cdots \oplus \mathbb{N})$ such that $R(\mathbb{N}^n)_0^\sigma = R^{(\varepsilon_1 \mathbb{N} \oplus \cdots \oplus \varepsilon_n \mathbb{N})}$. All results established for the $\pm$-gradations extend easily to the σ-gradations.

(3) One may consider subgroups H of G. Of course then one may view R as a G/H-graded object over $R^{(H)}$ but for certain properties the torsion that might appear in G/H is lethal. Therefore, if G is torsionfree it is beneficial to view $R^{(H)}$ as the part of degree zero of a G-gradation on RH so that the presentation of the properties considered only has to be verified for torsionfree abelian groups without reverting to torsion grading groups.

(4) If R is strongly G-graded then the strongly graded R-module $R \otimes_{R_e} R^{(H)}$ is a graded ring with respect to the gradation (cf. [8])

$$(R \otimes_{R_e} R^{(H)})_g = R_g \otimes_{R_e} R^{(H)} \cong \bigoplus_{h \in H} (R_g \otimes_{R_e} R_h) \cong \bigoplus_{h \in H} R_{gh}\,, \quad g \in G\,,$$

and hence it is as a graded ring isomorphic to the semigroup ring RH, where RH has the sign gradation as defined in the beginning.

We will apply the sign gradation to some results concerning global dimension and Auslander regularity that have proved to be useful in the theory of filtered rings and $\mathcal{D}$-modules, and also to some facts concerning the K-theory of the rings involved.

2. Some applications to global dimension and Auslander regularity

Proposition 2.1. *Let R be a strongly $\mathbb{Z}$-graded ring. The following are equivalent*:

(1) *R has finite global dimension.*

(2) *R_0 has finite global dimension.*

(3) *R^+ has finite global dimension.*

(4) *R^- has finite global dimension.*

Proof. (1)$\Leftrightarrow$(2) Since R is strongly graded there is a category equivalence between R-gr and R_0-mod (cf. [8, I.3.4]) whence the result follows from the obvious relation between the graded global dimension and the global dimension.

(1)$\Leftrightarrow$(3) The $+$-gradation on $R[y]$ makes this into a strongly graded ring with $R[y]_0^+ = R^+$. Since we know that gl.dim $R[y] = 1 +$ gl.-dim R, the equivalence (1)$\Leftrightarrow$(2) finishes the proof.

(1)$\Leftrightarrow$(4) Similar to (1)$\Leftrightarrow$(3). $\square$

Recall that a (left and right) Noetherian ring R is called *Auslander regular* if R has finite global dimension and it satisfies the following Auslander condition: if M is any nonzero finitely generated R-module, then for every $0 \leq i \leq \text{gl.dim}\, R$ and every nonzero submodule $N \subset \text{Ext}^i_R(M, R)$, we have $\text{Ext}^j_R(N, R) = 0$ for all $j < i$. Auslander regularity of filtered rings and their Rees rings has been studied, e.g., in [5] and [6] it is a property of some interest with respect to the theory of (holonomic, pure) modules over rings of different operators.

Proposition 2.2. *Let R be a strongly $\mathbb{Z}$-graded ring. The following are equivalent*:

(1) *R is Auslander regular.*

(2) *R_0 is Auslander regular.*

(3) *R^+ is Auslander regular.*

(4) *R^- is Auslander regular.*

Proof. If R is Auslander regular then R is also graded Auslander regular (the definition of this is obtained by rephrasing the definition of Auslander regularity intrinsically in the category R-gr). In [5] it has been established, (cf. also [3] and [7]) that the graded Auslander regularity of R already entails the Auslander regularity of R (this is not trivial).

(1)$\Leftrightarrow$(2) The equivalence of R-gr and R_0-mod proves the claim because it suffices to consider the graded Auslander regularity of R. Note also that R is projective (hence flat) as an R_0-module.

(1)$\Leftrightarrow$(3) Since R is Auslander regular $R[y]$ is Auslander regular by [5]. If we use the $+$-gradation on $R[y]$ then it follows from the equivalence (1)$\Leftrightarrow$(2) that R^+ is Auslander regular. Conversely, if R^+ is Auslander regular then by using the $+$-gradation on $R[y]$ again $R[y]$ is graded Auslander regular and it follows from [5] that $R[y]$ is Auslander regular. Hence $R[y, y^{-1}]$ is Auslander regular. But then R is Auslander regular since $R[y, y^{-1}]$ is strongly graded with respect to the natural gradation on it; $R[y, y^{-1}]_n = Ry^n$, $n \in \mathbb{Z}$, and in particular $R[y, y^{-1}]_0 = R$.

(1)$\Leftrightarrow$(4) Similar to (1)$\Leftrightarrow$(3). □

Now consider a ring A and an ideal I of A. For the I-adic filtration on A the associated graded ring is given as $G(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$, so $G(A)_0 = A/I$.

Corollary 2.3. *Suppose that $G(A)$ is isomorphic to R^+ or R^- as graded rings such that $G(A)_0 \cong R_0$ for some strongly $\mathbb{Z}$-graded ring R (e.g. when I is an invertible ideal of A); then*:

(1) *If A/I has finite global dimension and A/I is Noetherian then $G(A)$ has finite global dimension and $G(A)$ is Noetherian.*

(2) *If A/I is Auslander regular then $G(A)$ is Auslander regular.* □

Remarks 2.4. Note that Proposition 2.2 also yields a proof of the regularity of quantum sections as defined in [11] and of gauge algebras as in [2]. Indeed the associated graded ring of a ring of quantum sections is strongly graded, hence its Auslander regularity is characterized by the Auslander regularity of its degree zero part. However, the quantum sections modulo an ideal of the type considered above yield the degree zero part of the associated graded ring (see [11] for full details on these notions) hence Corollary 2.3(2) yields a proof for Auslander regularity of rings of quantum sections.

3. An application to K-groups

We write $K_i(R)$ for the ith K-group in the sense of Quillen. In formally the same way we introduce $K_i(R\text{-Gr})$, that is, the K-theory of the graded category. We write $K_i(R\text{-gr}) = K_i^{g}(R)$, hence obtaining the notation used in [10]. As a rather immediate corollary of Proposition 2.1 and Quillen's theorem, cf. [9], we obtain the following proposition:

Proposition 3.1. *Let R be a strongly $\mathbb{Z}^n$-graded ring and suppose that R_0 has finite global dimension and it is Noetherian, then $K_i^{g}(R) \cong K_i(R_0) \cong K_i(R^\sigma)$ where $R^\sigma = R^{\varepsilon_1\mathbb{N}\oplus\cdots\oplus\varepsilon_n\mathbb{N}}$ as in Remark* 1.4(2). *In particular, if $n = 1$ then $K_i^{g}(R) = K_i(R_0) = K_i(R^+) = K_i(R^-)$.*

Proof. The isomorphism $K_i^{g}(R) \cong K_i(R_0)$ follows again from $R\text{-gr} \cong R_0\text{-mod}$. Since R_0 is Noetherian and has finite global dimension it follows from Proposition 2.1 that R^σ is Noetherian too and $\text{gl.-dim}\, R^\sigma < \infty$. Therefore we may apply Quillen's theorem and conclude $K_i(R_0) \cong K_i(R^\sigma)$. □

Now consider a ring A and an ideal I of A. For the I-adic filtration on A the associated graded ring is given as $G(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$, so $G(A)_0 = A/I$. When I is in the Jacobson radical $J(A)$ and the Rees ring $\tilde{A} = n_{n\geq 0} I^n$ is Noetherian then the filtration is said to be Zariskian (cf. [6, 7] for a detailed treatment of general Zariskian filtrations). Combining foregoing results and the results of [4] one easily derives the following:

Proposition 3.2. *Suppose that $G(A)$ is isomorphic to R^+ or R^-, as graded rings for some strongly $\mathbb{Z}$-graded ring R (e.g. when I is an invertible ideal of A); then:*

(i) *If A/I is Noetherian and has finite global dimension then $G(A)$ is Noetherian and has finite global dimension, hence $K_i(A/I) = K_i(G(A))$.*

(ii) *If the I-adic filtration is Zariskian and A/I has finite global dimension then $K_0(A) \hookrightarrow K_0(A/I) = K_0(G(A))$.*

Proof. (i) Follows from Proposition 2.1 and Proposition 3.1.

(ii) Follows from [4] and (i). □

Corollary 3.3. (i) *With assumptions as in Proposition* 3.2 *we have: if A/I is Auslander regular then $G(A)$ is Auslander regular.*

(ii) *Similarly to Remark* 2.4 *we may observe that Proposition* 3.1 *allows to study the K-theory of rings of quantum sections, cf.* [11], *or generalized gauge algebra's in the sense of* [2] (*extending the notion of Witten's gauge algebra's*). □

We may extend Proposition 3.1 to the situation of Lemma 1.1(ii).

Proposition 3.4. *Let R be a Noetherian, strongly G graded ring having finite global dimension when G is an abelian torsion free group. For any finitely generated subsemigroup H of G without non-trivial units we have $K_i(R_e) \cong K_i^g(R) \cong K_i^g(RH) \cong K_i(R^{(H)})$, where $K_i^g(RH)$ is defined with respect to the sign gradation defined on RH.*

Proof. Since RH is strongly G-graded it follows from RH-gr $\cong R^{(H)}$-mod that $K_i^g(RH) \cong K_i(R^{(H)})$ and a similar argument yields $K_i^g(R) \cong K_i(R_e)$. It is not difficult to prove that RH is gr-Noetherian and it has finite gr-gl.dim. Note that these graded properties do not imply the corresponding upgraded properties in the absence of the finiteness condition on H therefore $R^{(H)}$ is Noetherian and has finite global dimension. Since all the necessary graded properties are available for RH it is possible to use the graded version of Quillen's theorem (that is but a straightforward modification) and conclude that $K_i^g(R) \cong K_i^g(RH)$. □

References

[1] L. Le Bruyn, M. Van den Bergh and F. Van Oystaeyen, Graded Orders (Birkhauser, Basel, 1989).
[2] L. Le Bruyn and F. Van Oystaeyen, Quantum sections and gauge algebras, to appear in the P. Menal Memorial Volume, U. Barcelona Autonoma.
[3] H.-S. Li, Non-commutative Zariskian rings, Thesis, Antwerp, 1990.
[4] H.-S. Li, M. Van den Bergh and F. Van Oystaeyen, Note on K_0-theory of rings with Zariskian filtrations, K-theory 3 (1990) 603–606.
[5] H.-S. Li and F. Van Oystaeyen, Global dimension and Auslander regularity of graded rings and Rees rings, Bull Soc. Math. Belg. Sér. A 43 (1991) 59–87.
[6] H.-S. Li and F. Van Oystaeyen, Zariskian filtrations, Comm. Algebra 17 (1989) 2945–2970.
[7] H.-S. Li and F. Van Oystaeyen, Zariskian filtrations, Monograph, 1993, to appear.
[8] C. Năstăsescu and F. Van Oystaeyen, Graded rings theory, North-Holland Mathematical Library, Vol. 28 (North-Holland, Amsterdam, 1982).
[9] D. Quillen, Higher Algebraic K-theory, Lecture Notes in Mathematics, Vol. 341 (Springer, Berlin, 1973) 85–147.
[10] M. Van den Bergh, A note on graded K-theory, Comm. Algebra 14 (1986) 1561–1564.
[11] F. Van Oystaeyen and R. Sallam, A microstructure sheaf and quantum sections over a projective scheme, J. Algebra, to appear.