Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT
Journal of Algebra 267 (2003) 756–772
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Hopf filtrations and Larson-type orders in Hopf algebras

Farahat S. Aly [a] and Freddy Van Oystaeyen [b,*]

[a] *Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo 11884, Egypt*
[b] *Department of Mathematics and Computer Science, University of Antwerp, Universiteitsplein 1, 2610 Antwerp, Belgium*

Received 12 September 2001
Communicated by Susan Montgomery

0. Introduction

The theory of orders and maximal orders over Dedekind domains or valuation-rings is an important ingredient in the study of arithmetical properties of algebras, in particular central simple algebras over fields. In [2], Larson introduced certain Hopf orders in a group algebra KG, where G is a finite group. The essential tool is a Zassenhaus valuation of G. Because of the multiplicative nature of that theory, those so-called Larson orders cannot be defined in other Hopf algebras without essential modification of the theory. In this paper we present the idea to replace the valuation filtration by a more general filtration, obtaining a bijective correspondence between the "Hopf filtrations" thus obtained and Hopf orders appearing as the degree zero part in such filtration. Just like in the group algebra case it is then possible to relate a Hopf order to an arithmetical object, i.e., a function $\xi : H \to \mathbb{R}_+^m \cup \{\infty\}$. These functions exhibit specific properties comparable to the ones expected for a valuation-like order function, i.e., so that $-\xi$ should rightfully be called a "Hopf valuation." We point out that $-\xi$ can be expressed as a function defined on subspaces in H so that it may be thought of as a generalized place (pseudo-place) in the sense of [4].

In Section 2 we introduce the deformation of a Hopf (valuation) filtration and establish how it can be used to construct new Hopf orders. This explains in a general Hopf algebra framework how Larson-type orders come into being. We provide several examples, e.g., for Sweedler's 4-dimensional Hopf algebra, Taft algebras, etc. in Section 4.

In a forthcoming paper we apply the theory of Hopf orders to the theory of orders in H-module algebras.

* Corresponding author.
E-mail addresses: farahata@yahoo.com (F.S. Aly), voyst@uia.ua.ac.be (F. Van Oystaeyen).

0021-8693/$ – see front matter
doi:10.1016/S0021-8693(03)00141-8

1. Hopf filtrations

Throughout we consider a Hopf algebra H over K, the latter being a commutative field. We consider a valuation ring O_v of K, for notational convenience we shall write $D = O_v$ (in fact the theory can be extended to the case where Dedekind domains D are being considered, by rather straightforward local–global methods). Let us write v for the valuation (order) function associated to O_v and Γ for its totally ordered Abelian value group; thus $v : K^* \twoheadrightarrow \Gamma$. Note that we do not restrict to discrete valuations (with $\Gamma = \mathbb{Z}$) but it is harmless to assume that Γ is a subgroup of the additive group $\mathbb{R}^m_+$. Let w be the maximal ideal of D. By the definition of w, D/w is the residue field $k = k_v$ of v. For generalities on valuation rings we refer to Schilling's [7]. Let R be an arbitrary ring with unit element 1. A set of additive subgroups $\{F_\gamma R\colon\ \gamma \in \Gamma\}$ is said to be a *Γ-filtration* of R if the following conditions hold:

(1) $1 \in F_0 R$; for $\gamma \leqslant \tau$ in Γ we have $F_\gamma R \subset F_\tau R$.
(2) For $\gamma, \tau \in \Gamma$, $F_\gamma R F_\tau R \subset F_{\gamma+\tau} R$.
(2) $R = \bigcup_{\gamma \in \Gamma} F_\gamma R$, i.e., the filtration is always exhaustive.

A Γ-filtration FR is said to be *separated* if for every $x \neq 0$ in R there is a (unique) $\delta \in \Gamma$ such that $x \in F_\delta R$ and $x \notin F_{\delta'} R$ for $\delta' < \delta$. For a separated filtration FR we have:

1.1. Lemma. *If $\mathcal{A}$ is a subset of Γ such that $\gamma = \inf\{\alpha\colon\ \alpha \in \mathcal{A}\} \in \Gamma \cup \{-\infty\}$ then $F_\gamma R = \bigcap_{\alpha \in \mathcal{A}} F_\alpha R$.*

Proof. The inclusion $F_\gamma R \subset \bigcap_{\alpha \in \mathcal{A}} F_\alpha R$ is obvious. Look at $x \notin F_\gamma R$. Since FR is assumed to be separated, there is a unique $\delta \in \Gamma$ such that $x \in F_\delta R$ but $x \notin F_{\delta'} R$ for any $\delta' < \delta$. Then $\gamma < \delta$ (because otherwise $\delta \leqslant \gamma$ and $x \in F_\gamma R$), hence $\alpha_0 < \delta$ for some $\alpha_0 \in \mathcal{A}$ because $\gamma = \inf\{\alpha\colon\ \alpha \in \mathcal{A}\}$. Then $x \notin F_{\alpha_0} R$ and thus $x \notin \bigcap_{\alpha \in \mathcal{A}} F_\alpha R$. □

1.2. Example (Filtration of discrete support). Let $\Pi \subset \Gamma$ be any discrete subgroup (e.g., $\Pi \simeq \mathbb{Z}^n$). Consider $F_\pi R$, $\pi \in \Pi$, defining a separated Π-filtration on R, then put $F_\gamma R = F_\pi R$ for $\gamma \in \Gamma$ such that $\pi \leqslant \gamma < \pi'$, where π' is the smallest element in Π strictly bigger than π in Π. Note that, if $F_\gamma R \neq R$ then there exists a $\pi_0 \in \Pi$ such that $F_\gamma R \subset F_{\pi_0} R$ (take any π_0 such that $F_{\pi_0} R \not\subset F_\gamma R$); moreover since the Π-filtration is separated we have that $\bigcap_{\pi \in \Pi} F_\pi R = 0$ and therefore there is a $\pi_1 \in \Pi$ such that $F_{\pi_1} R \subset F_\gamma R$, $\pi_1 \leqslant \gamma$. Hence, the foregoing defines a Γ-filtration and it is obviously separated too. We say that the Γ-filtration $\{F_\gamma R\colon\ \gamma \in \Gamma\}$ has discrete support Π. For $\gamma \in \Gamma$ we have that $F_\gamma R / \bigcup_{\tau < \gamma} F_\tau R = 0$ except when $\gamma \in \Pi$, then the latter is $F_{\pi'} R / F_\pi R$ where $\pi \leqslant \gamma \leqslant \pi'$ is as above.

1.3. The associated Γ-graded ring

For $\gamma \in \Gamma$ and a Γ-filtration FR on R we define: $F^0_\gamma R = \bigcup_{\tau < \gamma} F_\tau R$, $G_\gamma(R) = F_\gamma R / F^0_\gamma R$ (as an additive group).

The additive group $\bigoplus_{\gamma\in T} G_\gamma(R)$ obviously becomes a Γ-graded ring if we define the multiplication as follows: for $\bar{a} \in G_\gamma(R)$, $\bar{b} \in G_\tau(R)$ let $a \in F_\gamma R$, respectively $b \in F_\tau R$, represent $\bar{a}$, respectively $\bar{b}$, and put $\bar{a}.\bar{b} = ab \bmod F^0_{\gamma+\tau} R$. The unit of $G(R)$ is $\bar{1} = 1 \bmod F^0_0 R$. We say that $G(R) = \bigoplus_{\gamma\in T} G_\gamma(R)$ is the associated graded ring for FR. In case FR is a Γ-filtration of discrete support Π, as in Example 1.2, then $G(R)$ may be considered as a Π-graded ring blown-up to a Γ-graded ring in the trivial way (i.e., parts of degree $\gamma \in \Gamma$ are zero unless possibly whenever $\gamma \in \Pi$).

1.4. Algebra filtrations extending a valuation filtration

Consider a K-algebra A and a filtration fK on K, say with respect to some totally ordered Abelian group Γ. A Γ-filtration FA of A is said to be an extension of fK if $f_\gamma K = F_\gamma A \cap K$ for all $\gamma \in \Gamma$.

Recall (cf. [3]) that a Γ-filtration FR on an arbitrary ring R is said to be a strong Γ-filtration if $F_\gamma R F_\tau R = F_{\gamma+\tau} R$ for all $\gamma, \tau \in \Gamma$, or, equivalently: $F_\gamma R F_{-\gamma} R = F_0 R$ for $\gamma \in \Gamma$. If fK is a strong filtration and FA extends fK then it is easily seen that FA is a strong filtration, too. To any valuation v of K, i.e., to the valuation ring D, there corresponds a valuation filtration $f_v K$, simply written fK once v has been fixed, given by $f_\gamma K = \{\lambda \in K\colon\ v(\lambda) \leqslant -\gamma\}$. Obviously fK is a separated Γ-filtration with $f_0 K = O_v = D$ and $G(K)_0 = k$. Furthermore, it is also clear that $f_\gamma K f_\tau K = f_{\gamma+\tau} K$ for $\gamma, \tau \in \Gamma$, i.e., the valuation filtration fK is a strong filtration. Any Γ-filtration FA extending the valuation filtration fK is therefore always a strong filtration, in particular, for $\gamma \in \Gamma$ we have $F_\gamma A = f_\gamma K F_0 A$. For such a filtration FA, $G(A)$ is strongly Γ-graded in the sense that $G_\gamma(A) G_\tau(A) = G_{\gamma+\tau}(A)$ for $\gamma, \tau \in \Gamma$.

1.5. Proposition. *Let A be a K-algebra with a filtration FA extending the valuation filtration fK then $G(A)$ is the group ring $G(A)_0\Gamma$ and $G(A)_0$ may be viewed as the residual algebra of A.*

Proof. Pick nonzero $\bar{a} \in G(K)_\gamma$, $\bar{b} \in G(K)_\tau$ and let $a \in F_\gamma K - F^0_\gamma K$, $b \in F_\tau K - F^0_\tau K$ represent $\bar{a}$, respectively $\bar{b}$. Then $ab \in F_{\gamma+\tau} K - F^0_{\gamma+\tau} K$ because $F_\gamma K = f_\gamma K$ and fK is the valuation filtration. Consequently $\bar{a}\bar{b} \neq 0$, i.e., it follows that $G(K)$ is a gr-domain (that is, there are no nontrivial homogeneous zero divisors). Since Γ is ordered, $G(K)$ is a domain if it is a gr-domain. Look again at $\bar{a} \in G(K)_\gamma$, $\bar{a} \neq 0$, represented by $a \in F_\gamma K - F^0_\gamma K$, then $a^{-1} \in F_{-\gamma} K$ is easily checked. Hence, $\bar{a}$ has inverse $a^{-1} \bmod F^0_{-\gamma} K_0$, thus $G(K)$ is a Γ-graded field. Note that $F^0_\gamma K = F^0_0 K F_\gamma K$, i.e., $F^0_\gamma K = w F_\gamma K$ for all $\gamma \in \Gamma$. The graded field $G(K)$ is necessarily of the form $k\Gamma$. □

The Rees ring associated to the filtration FA is defined to be $\widetilde{A} = \sum_{\gamma\in\Gamma} F_\gamma A.\gamma \subset A\Gamma$, the latter being the group ring of Γ over A. Because $1 \in F_\gamma A$ for every positive $\gamma \in \Gamma$, i.e., $0 \leqslant \gamma$, the set of strictly positive elements $\Gamma_+ = \{\gamma\colon\ 0 < \gamma\}$ is in $\widetilde{A}$ thus, as Γ is Abelian, it is a central set in $\widetilde{A}$. It is easy to verify that $\widetilde{A}/\widetilde{A}\Gamma_+ = G(A)$ as Γ-graded rings, where $\widetilde{A}\Gamma_+$ is the ideal of $\widetilde{A}$ generated by Γ_+.

It is well known, cf. [3], that by inverting the multiplicatively closed subset Γ_+ of $\widetilde{A}$ we obtain $A\Gamma$; moreover, if we write I for the ideal generated in $\widetilde{A}$ by the central elements $\gamma - 1$, $\gamma \in \Gamma_+$, then $\widetilde{A}/I \cong A$ and $F_\gamma A$ is the image of $\widetilde{A}_\gamma$ modulo I.

1.6. *Γ-filtered vectorspaces*

Consider a field K with a separated Γ-filtration fK; the category of filtered vectorspaces over K is denoted by K-filt. A Γ-filtration FV on a K-vectorspace V is said to be a *good filtration* if there exist sets $\{v_\alpha\colon \alpha \in \mathcal{A}\} \subset V$, $\{\gamma_\alpha\colon \alpha \in \widetilde{\mathcal{A}}\} \subset \Gamma$, such that for all $\gamma \in \Gamma$ we have $F_\gamma V = \sum_{\alpha \in \mathcal{A}} f_{\gamma - \gamma_\alpha} K v_\alpha$. Obviously $\{v_\alpha\colon \alpha \in \mathcal{A}\}$ is a set of K-generators for V but it needs not be a K-basis. The foregoing definition extends the definition given for $\mathbb{Z}$-filtrations, cf. [1], to the case of any totally ordered group Γ. Suppose fK is a strong filtration in the sequel. Then it follows from $f_{\gamma-\gamma_\alpha} K v_\alpha \subset F_\gamma V$ that $v_\alpha \in F_{\gamma_\alpha} V$ for all $\alpha \in \widetilde{\mathcal{A}}$. Moreover, we also have then that $F_\gamma V = f_\gamma K F_0 V$ for every $\gamma \in \Gamma$, hence if FV is a good filtration then without loss of generality we may assume that the generating set $\{v_\alpha\colon \alpha \in \mathcal{A}\}$ is taken in $F_0 V$ and $F_\gamma V = \sum_{\alpha \in \mathcal{A}} f_\gamma K v_\alpha$ for $\gamma \in \Gamma$. Now in case $F_0 V$ is free over $f_0 K$, say with basis $\{w_i\colon i \in J\}$, then the good filtration FV may also be described by the K-basis $\{w_i\colon i \in J\}$, $F_\gamma V = \sum_{i \in J} f_\gamma K w_i$. In fact, we shall be dealing with this situation frequently, because we will be looking at valuation filtrations fK, i.e., these are strong and $f_0 K = O_v$ is a valuation ring, so that torsion-free finitely generated O_v-modules will be free.

A *filtered K-Hopf algebra H* can be defined as being a Hopf algebra in the category k-filt. This definition is equivalent to H being a Hopf algebra over K having a Hopf filtration FH; let us write down explicitly the properties of the structure morphisms involved in this definition. For detail on Hopf algebras the reader may consult [6,8],.... We have a Hopf algebra H with an algebra filtration FH such that the counit ε and the comultiplication Δ are filtered morphisms, e.g.,

(1) $\varepsilon(F_\gamma H) \subset F_\gamma K = f_\gamma K$, for all $\gamma \in \Gamma$.
(2) $S(F_\gamma H) \subset F_\gamma H$, for all $\gamma \in \Gamma$.
(3) $\Delta(F_\gamma H) \subset \sum_{\sigma+\tau=\gamma} F_\sigma H \otimes F_\tau H$, for all $\gamma \in \Gamma$.

The condition (3) just expresses that Δ is a filtered morphism where $H \otimes_K H$ is equipped with the tensor algebra filtration defined by putting $F_\gamma(H \otimes_K H) = \sum_{\sigma+\tau=\gamma} F_\sigma H \otimes F_\tau H$, $\gamma \in \Gamma$.

1.7. Proposition. *Let H be a Hopf algebra over K equipped with a Hopf filtration FH, then $G(H)$ is a Γ-graded Hopf algebra. If FH extends fK then $G(H) = k\Gamma \otimes_k F_0 H$ with Hopf algebra structure derived from $F_0 H$ (in fact via $F_0 H / F_0^0 H$) making it into a graded Hopf algebra over the gr-field $k\Gamma$.*

Proof. The statements about the algebra structure of $G(H)$ are obvious. Now observe that $F_0 H$ is a sub-Hopf algebra of H (but over $f_0 K$), indeed $\Delta(F_0 H) \subset \sum_{\gamma \in \Gamma} F_\gamma H \otimes F_{-\gamma} H$, but since $F_\gamma H = f_\gamma K \otimes F_0 H$, $F_{-\gamma} H = f_{-\gamma} K \otimes F_0 H$, it follows that $\Delta(F_0 H) \subset F_0 H \otimes F_0 H$. The restriction of ε to $F_0 H$ defines first the $f_0 K$-linear $\varepsilon|_{F_0 H}$ and then the

$k\Gamma$-linear $\bar{\varepsilon}$ on $G(H)$ extending $\bar{\varepsilon}|_{F_0H}: F_0H/F_0^0H \to k$. Similarly, $S|_{F_0H}: F_0H \to F_0H$ defines the $k\Gamma$-linear $\overline{S}: G(H) \to G(H)$ extending the k-linear $\overline{S|_{F_0H}}: G(H)_0 \to G(H)_0$. All claims in the proposition are obvious from the arguments. □

Note that for a Hopf filtration FH, the inclusion $K \hookrightarrow H$, is a filtered morphism. For a strong Hopf filtration FH the condition of extending fK is equivalent to $F_0H \cap K = f_0K$.

2. Hopf valuation functions

In this section we consider a Hopf algebra H over the field K together with a Γ-valuation ring $O_v = D$ in K. The valuation associated to O_v is $v: K^* \twoheadrightarrow \Gamma$, we shall also write v for the function $K \twoheadrightarrow \Gamma \cup \{\infty\}$ extending v on K^* by putting $v(0) = \infty$. The residue field for v is k.

A *Hopf valuation function extending* v is a function $-\xi : H \twoheadrightarrow \Gamma \cup \{\infty\}$, usually viewed as a *Hopf valuation filtration function* $\xi : H \twoheadrightarrow \Gamma \cup \{-\infty\}$, satisfying:

HV1. $\xi(h) = -\infty$ if and only if $h = 0$.
HV2. $\xi(1) = 0$.
HV3. $\xi(\lambda h) = \xi(h) - v(\lambda)$ for $h \in H$, $\lambda \in K$.
HV4. $\xi(gh) \leqslant \xi(g) + \xi(h)$ for $g, h \in H$.
HV5. $\xi(g + h) \leqslant \max\{\xi(g), \xi(h)\}$ for $g, h \in H$.
HV6. $\xi(Sh) \leqslant \xi(h)$; $\xi(\varepsilon(h)) \leqslant \xi(h)$.
HV7. $\xi(h) \geqslant \inf\{\max_{\sum}\{\xi(h_1) + \xi(h_2)\}\}$,

where $\Delta(h) = \sum h_1 \otimes h_2$; here $\max_{\sum}$ is taken over the terms in a fixed expression of $\Delta(h)$ in Sweedler notation, while inf is over all possible decompositions of $\Delta(h)$.

2.1. Observation. Condition HV7 actually leads to an equality. Indeed from $\Delta(h) = \sum h_1 \otimes h_2$ we may derive:

$$h = \sum \varepsilon(h_1)h_2 = \sum h_1\varepsilon(h_2).$$

Applying HV3 we arrive at

$$\xi(h) \leqslant \max_{\sum}\bigl\{\xi\bigl(\varepsilon(h_1)\bigr)h_2\bigr\} = \max_{\sum}\bigl\{\xi\bigl(\varepsilon(h_1)\bigr) + \xi(h_2)\bigr\}, \tag{$*$}$$

where the latter equality is a consequence of HV3. Since the inequality $(*)$ holds for every decomposition of $\Delta(h)$ we obtain $\xi(h) \leqslant \inf\{\max_{\sum}\{\xi(\varepsilon(h_1)) + \xi(h_2)\}\}$. Now, from HV6 it follows that

$$\xi\bigl(\varepsilon(h_1)\bigr) + \xi(h_2) \leqslant \xi(h_1) + \xi(h_2),$$

hence $\xi(h) \leqslant \inf\{\max_{\sum}\{\xi(\varepsilon(h_1)) + \xi(h_2)\}\} \leqslant \inf\{\max_{\sum}\{\xi(\varepsilon(h_1)) + \xi(h_2)\}\}$. Therefore HV7 actually expresses an equality, even moreover:

$$\xi(h) = \inf\Big\{\max_{\sum}\big\{\xi(h_1) + \xi(h_2)\big\}\Big\} = \inf\Big\{\max_{\sum}\big\{\xi\big(\varepsilon(h_1)\big) + \xi(h_2)\big\}\Big\}$$
$$= \inf\Big\{\max_{\sum}\big\{\xi(h_1) + \xi\big(\varepsilon(h_2)\big)\big\}\Big\}.$$

2.2. Warning. When comparing our theory to the multiplicative theory of group valuations as these have been used in [2], note that those group valuations are really the associated absolute value functions, not the valuation order functions; the relation between these is classically given by "minus the natural logarithm." Hence if our ξ is restricted to the group $G(H)$ of group-like elements of H this does not immediately yield the group valuation values but there is the usual logarithmic relation via $-\ln$.

2.3. Theorem. *The Hopf valuation filtration functions $\xi : H \to \Gamma \cup \{-\infty\}$ satisfying* HV1, ..., HV7*, correspond bijectively to the separated Hopf filtrations FH extending the valuation filtration fK corresponding to the valuation v.*

Proof. Let us start from a Hopf valuation filtration function $\xi : H \twoheadrightarrow \Gamma \cup \{-\infty\}$ with properties HV1, ..., HV7. For $\gamma \in \Gamma$ put $F_\gamma H = \{h \in H\colon\ \xi(h) \leqslant \gamma\}$. Properties HV5 and HV3 entail that $F_\gamma H$ is an additive subgroup of H, containing 0 because of HV1. From HV2, HV4, HV5 it follows that FH is a filtration of the ring H. Putting $h = 1$ in HV3 entails $\xi(\lambda) = -v(\lambda)$ for $\lambda \in K$, hence FH does extend the valuation filtration fK corresponding to v. For $h \in H$ we have $h \in F_{\xi(h)}H$, therefore FH defines an exhaustive filtration of H. From HV6 we infer that S and ε are filtered maps of degree zero with respect to FH. Now suppose that FH were non-separated. Then there would exist a nonzero $x \in H$ such that for every $\gamma \in \Gamma$ such that $x \in F_\gamma H$ we automatically must have $x \in F_\gamma^0 H$. In any case we have $x \in F_{\xi(x)}H$, however if for some $\gamma < \xi(x)$ we would have $x \in F_\gamma H$ then by definition of $F_\gamma H$ it means that $\xi(x) < \gamma$, contradicting $\gamma < \xi(x)$. Therefore $x \in F_{\xi(z)}H - F^0_{\xi(z)}H$. Considering $h \in H$ and $\Delta(h) = \sum h_1 \otimes h_2$, we may apply Observation 2.1 and arrive at

$$\xi(h) = \inf\Big\{\max_{\sum}\big\{\xi(h_1) + \xi(h_2)\big\}\Big\}.$$

In order to prove $\Delta(F_\gamma H) \subset \sum_{\tau\in\Gamma} F_\tau H \otimes F_{\gamma-\tau} H$, it is enough to establish this for $\gamma = \xi(h)$, any $h \in H$; indeed, ξ is (like v) assumed to be surjective. Now if $\delta > \xi(h) = \gamma$ then for some decomposition $\Delta(h) = \sum h_1 \otimes h_2$ we must have that $\delta \geqslant \max_{\sum}\{\xi(h_1) + \xi(h_2)\}$.

The latter comes down to

$$\Delta(h) \in \sum_{\tau\in\Gamma} F_\tau H \otimes F_{\delta-\tau} H. \tag{$*$}$$

At this point recall that FH is a strong filtration and $F_\tau H = F_\tau K F_0 H = f_\tau K F_0 H$ for every $\tau \in \Gamma$. Hence for the tensor filtration on $H \otimes_K H$ we obtain:

$$F_\tau(H \otimes_K H) = \sum_{\sigma\in\Gamma} F_{\tau-\sigma} H \otimes F_\sigma H = \sum_{\sigma\in\Gamma} F_{\tau-\sigma} K F_\sigma K F_0 H \otimes F_0 H = F_\tau H \otimes F_0 H.$$

From $(*)$ we have $\Delta(h) \in F_\sigma H \otimes F_0 H$ for every $\delta > \xi(h)$. As a D-module, $F_0 H$ is flat; indeed, over a valuation domain every finitely generated torsion free module is projective and every projective is free, so $F_0 H$ is the direct limit of free modules of finite rank, thus flat. It follows that $\bigcap_\delta (F_\delta H \otimes_D F_0 H) = (\bigcap_\delta F_\delta H) \otimes_D F_0 H$. But either $\xi(h)$ is equal to some $\max_{\sum}\{\xi(h_1) + \xi(h_2)\}$ for a certain decomposition $\Delta(h) = \sum h_1 \otimes h_2$ in which case $(*)$ applies with $\delta = \xi(h)$ and then there is nothing else to prove, or else $\xi(h)$ appears as the inf of elements $\max_{\sum}\{\xi(h_1) + \xi(h_2)\} \in \Gamma$ (observe that the $\max_{\sum}$ is over a finite set) and by the separatedness (see Lemma 1.1) we have: $F_\gamma H = \bigcap_{\delta > \gamma} F_\delta H$ for $\gamma = \xi(h)$. Consequently, also in the latter case $\Delta(h) \in F_\gamma H \otimes F_0 H$ for $\gamma = \xi(h)$. We have proved that FH is a Hopf filtration.

Conversely, start from a given separated Hopf filtration FH extending fK. In view of the separatedness, any nonzero $x \in H$ is in $F_\gamma H - F_\gamma^0 H$ for some unique $\gamma \in \Gamma$. Therefore the function $\xi : H \to \Gamma \cup \{-\infty\}$ defined by $\xi(0) = -\infty$, $\xi(x) = \inf\{\tau\colon x \in F_\tau H\}$ is well-defined and it is surjective since FH extends fK and $v : K \twoheadrightarrow \Gamma \cup \{-\infty\}$ is surjective. We may think of $\xi(x)$ as the "filtration degree" with respect to FH. Verification of the properties HV1, ..., HV7 is straightforward. $\square$

Let us point out that it is possible to define a pseudovaluation associated to ξ, this is, a function defined on the lattice of D-submodules of H and takes values in the completion $\widehat{\Gamma}$ of Γ. Such pseudovaluations of algebras were considered first in [4]; although they seem to be useful in case Γ is discrete or FH has discrete support, we do not go into this theory here.

The advantage of Theorem 2.3 is that we know a Hopf valuation ξ if and only if we know the Hopf order $F_0 H = \{h \in H\colon \xi(h) \leqslant 0\}$ in H. It is of some importance to realize that in case of a group algebra $H = KG$ for a finite group G, the knowledge of a Zassenhaus valuation on G does not unambiguously determine a Hopf valuation on KG nor a Hopf order of KG. Indeed, the construction of Larson orders different from DG in KG exactly shows that different orders may be constructed from the same Zassenhaus valuation of G, cf. [2].

Theorem 2.3 applied to the case of a group algebra, entails that both DG and a nontrivial Larson order $DG \subsetneqq \mathcal{L} \subset KG$ correspond to different Hopf valuation functions on KG but these may take the same values on some K-basis of the Hopf algebra, e.g., on G in the case of $H = KG$. It turns out that some basis or generating set is better than others! This may be related to the notion of "good filtration" on H as defined in 1.6. In the following section we hope to clarify this dependence on the choice of sets of generators. In doing so we will obtain a constructive method for making (maximal) orders of Larson-type in any finite-dimensional Hopf algebra, plus a general explanation of their existence in terms of the above mentioned base change (see Section 4 where also several examples are given).

3. The derived valuation filtration function

Throughout this section H is as before a Hopf algebra over K with a separated Hopf filtration FH extending fK associated to a valuation v of K. The Hopf valuation function associated to FH is ξ. In this situation $F_0 H$ is a Hopf algebra over $D = f_0 K$ and it is an order of H in the sense that $KF_0H = K \otimes_D F_0H = H$. For $h \in H$ define $I_h \subset K$ by $I_h = \{x \in K\colon xh \in F_0H\}$. The next proposition establishes that ξ may be calculated from data in K.

3.1. Proposition. *For $h \in H$, $\xi(h) = v(I_h)$, in particular, for $h \in K$ we have $\xi(h) = -v(h)$. Let ξ_1, respectively ξ_2, correspond to Hopf filtrations F^1H, respectively F^2H; then $F_0^1H \subset F_0^2H$ is equivalent to $\xi_1 \geqslant \xi_2$.*

Proof. Take $h \neq 0$ in H, then for some $\gamma \in \Gamma$, $h \in F_\gamma H - F_\gamma^0 H$. Suppose $xh \in F_0H$ for $x \in K$; such an x exists because $KF_0H = H$. Then $x \in F_\delta H$ entails $h \in F_{\delta^{-1}}K = F_{\delta^{-1}}H \cap K$ and this forces $\delta = \gamma^{-1}$. Therefore $I_h \subset f_{\gamma^{-1}}K$.

On the other hand, $f_{\gamma^{-1}}K \subset I_h$, therefore we arrive at $I_h = f_{\gamma^{-1}}K$, hence $\xi(h) = \gamma = v(I_h)$. For the second statement, observe that F^1H and F^2H are strong filtrations, therefore an inclusion $F_0^1H \subset F_0^2H_1$ entails $F_\gamma^1 H \subset F_\gamma^2 H$ for every $\gamma \in \Gamma$ and the relation $\xi_2 \leqslant \xi_1$ follows. Conversely, $\xi_2 \leqslant \xi_1$ clearly implies $F_0^1H = \{h \in H\colon \xi_1(h) \leqslant 0\} \subset F_0^2H = \{h \in H\colon \xi_2(h) \leqslant 0\}$. □

When $h \in F_0H$ then $\varepsilon(h) \in D$ and if $v(\varepsilon(h)) = \gamma$ then we may divide h by $\lambda \in K$ with $v(\lambda) = \gamma$, we still have that $\varepsilon(\lambda^{-1}h) \in D$ but we do not know whether $\lambda^{-1}h \in F_0H$. If a suitable set of K-generators for H, B say, can be selected such that $\{\lambda_i^{-1}h_i\colon h_i \in B\}$ generates a ring over D, then we might obtain a method to construct D-orders in H. Elements of H that are obvious candidates for the best divisibility properties are those $h \in H$ with $\varepsilon(h) = 0$, i.e., elements of the augmentation ideal. This is at the basis of the definition of the derived valuation filtration function $d\varepsilon$ associated to ξ.

Consider the K-space H/K, K embedded in H in the canonical way, and define $d\xi : H/K \to \Gamma \cup \{-\infty\}$ by putting $d\xi(\bar{h}) = \xi(h - \varepsilon(h))$, where $\bar{h}$ is the class of h in H/K; we shall also write $d\xi$ for the function defined on H by $d\xi(h) = \xi(h - \varepsilon(h))$, taking into account that $d\xi(\lambda) = -\infty$ for every $\lambda \in K$.

3.2. Lemma. *Either $\xi(h) = \xi(\varepsilon(h))$ or $\xi(h) = d\xi(h)$, in other words, $d\xi(h) < \xi(h)$ is possible only when $\xi(h) = \xi(\varepsilon(h))$ and $\varepsilon(h) \neq 0$.*

Proof. Since $h = (h - \varepsilon(h)) + \varepsilon(h)$, in case $\varepsilon(h) \neq 0$ we have $\xi(h) \leqslant \max\{d\xi(h), \xi(\varepsilon(h))\}$; on the other hand, by definition of $d\xi$ we also have $d\xi(h) \leqslant \max\{\xi(h), \xi(\varepsilon(h))\}$. Combining these yields immediately that either $\xi(h) = \xi(\varepsilon(h))$ or $\xi(h) = d\xi(h)$, the second statement in the lemma is clear. □

The calculus with respect to $d\xi$ is close to the one for ξ, but there are some obvious modifications which we phrase in the following proposition.

3.3. Proposition. *With notation and conventions as before, the function $d\xi$ satisfies the following properties*:

DHV1. $d\xi(h) = -\infty$ *if and only if* $h \in K$.
DHV2. *For* $\lambda \in H$, $h \in H$, $d\xi(\lambda h) = d\xi(h) - v(\lambda)$.

DHV3. *For* $g,h\in H$, $d\xi(gh)\leqslant\max\{d\xi(g)+\xi(h),\ \xi(g)+d\xi(h)\}$. *In case* $\varepsilon(g)=\varepsilon(h)=0$, $d\xi(gh)\leqslant d\xi(g)+d\xi(h)$.

DHV4. *For* $g,h\in H$, $d\xi(g+h)\leqslant\max\{d\xi(g),d\xi(h)\}$.

DHV5. *For* $h\in H$, $d\xi(Sh)=d\xi(Sh)$.

DHV6. *For* $h\in H-K$ *with* $\varepsilon(h)=0$ *and* $\Delta(h)=\sum h_1\otimes h_2$,

$$d\xi(h)\geqslant\inf\Bigl\{\max_{\sum}\bigl\{d\xi(h_1)+d\xi(h_2)\bigr\}\Bigr\}.$$

Proof. The proof of DHV1, DHV2, DHV4 is straightforward.

DHV3. The first statement follows from

$$gh-\varepsilon(gh)=\bigl(g-\varepsilon(g)\bigr)h+\varepsilon(g)\bigl(h-\varepsilon(h)\bigr),$$

hence $d\xi(gh)\leqslant\max\{\xi((g-\varepsilon(g))h),\ \xi(\varepsilon(g)(h-\varepsilon(h)))\}$. The latter combined with HV4 and $\xi(\varepsilon(g))\leqslant\xi(g)$ yields the statement. In case $\varepsilon(g)=\varepsilon(h)=0$ we may assume $g\notin K$, $h\notin K$ because otherwise there is nothing to prove. In that case, $\xi(g)\neq\xi(\varepsilon(g))$ and $\xi(h)\neq\xi(\varepsilon(h))$, thus Lemma 3.2 applies and we obtain $d\xi(g)=\xi(g)$, $d\xi(h)=\xi(h)$, and also

$$d\xi(gh)\leqslant\xi(gh)\leqslant\xi(g)+\xi(h)=d\xi(g)+d\xi(h).$$

DHV5 follows from $d\xi(Sh)\leqslant\xi(Sh-\varepsilon(Sh))=\xi(h-\varepsilon(h))=d\xi(h)$.

DHV6. From $\varepsilon(h)=0$ it follows that $\xi(\varepsilon(h))\neq\xi(h)$ because $h\notin K$; thus in view of Lemma 2.2 $d\xi(h)=\xi(h)$. Therefore $d\xi(h)\geqslant\inf\{\max_{\sum}\{\xi(h_1)+\xi(h_2)\}\}\geqslant\inf\{\max_{\sum}\{d\xi(h_1)+d\xi(h_2)\}\}$, as desired. □

To any Hopf valuation filtration function ξ, there corresponds a Hopf order over D, $H(\xi)=\{h\in H:\ \xi(h)\leqslant 0\}$. To the derived function, we now associate $H(d\xi)$ defined by $H(d\xi)=D\otimes\{h\in\varepsilon^{-1}(D):\ -\infty\leqslant d\xi(h)\leqslant 0\}$. Since $\xi(h)\leqslant 0$ entails $\xi(\varepsilon(h))\leqslant 0$, we may apply Lemma 2.2 and conclude that $H(\xi)\subset H(d\xi)$.

3.4. Theorem. *With notation as above, in particular,* w *is the maximal ideal of* D, $H(d\xi)$ *is a Hopf order over* D *in* H, *it is the* D*-algebra generated by the elements of* $w^{d\xi(h)}(h-\varepsilon(h))$ *for* $h\in\varepsilon^{-1}(D)$. *In fact,* $H(\xi)=H(d\xi)$.

Proof. Following the line of proof of Theorem 2.3, we may define a Hopf filtration F^dH on H by

$$F^d_\gamma H=f_\gamma K\oplus\bigl\{h\in\varepsilon^{-1}(D):\ d\xi(h)\leqslant\gamma\bigr\},\quad \text{for } \gamma\in\Gamma.$$

Obviously, $F^d_\gamma H=f_\gamma K F^d_0 H=f_\gamma K H(d\xi)$. Note that, while following the lines of proof of Theorem 2.3, we now have to use DVH3, DVH5, ... It is clear that $d\xi(w^{d\xi(h)}(h-\varepsilon(h))\leqslant 0$. On the other hand, if $h\in\varepsilon^{-1}(D)$ is such that $-\infty<d\xi(h)\leqslant 0$, then $h-\varepsilon(h)\in w^{d\xi(h)}(h-\varepsilon(h))$ with $\varepsilon(h)\in D$; therefore such an h is in the D-algebra generated by the

$w^{d\xi(h)}(h - \varepsilon(h))$. Finally, consider $h \in H(d\xi)$. In case $\xi(h) = \xi(\varepsilon(h))$, $h \in H(\xi)$, but if $\xi(h) \neq d\xi(h)$ and $\varepsilon(h) \neq 0$ then $\xi(h) = \xi(\varepsilon(h))$ entails $\xi(h) \leqslant 0$ as $\varepsilon(h) \in D$, hence $H(d\xi) = H(\xi)$ follows. □

3.5. Remark.

(1) So far the term Hopf order only referred to the property $H = KH(\xi)$ or $H = KF_0H$. In case H is finite-dimensional, we want to consider D-orders that are finite over D, this is done in Section 4.

(2) The foregoing theorem may be disappointing if one hoped for a new order $H(\xi)$ to appear. Nevertheless, the description of $H(\xi)$ in terms of $d\xi$ or of elements of the form $h - \varepsilon(h)$ will be essential in the construction of new orders depending on the choice of some suitable K-basis for H, in case H is finite-dimensional.

4. The finite case. Examples

We consider a Hopf algebra H, finite-dimensional over K. We look at comparable Hopf valuation filtration functions $\xi \leqslant \xi_0$ with associated Hopf algebras over D, $H(\xi_0) \subset H(\xi)$. In view of concrete applications and examples we will consider, we may start from a Hopf order $H(\xi_0)$ *having a finite D-basis B*, and it is not restrictive to assume that $\varepsilon(b) = 0$ for $b \in B$. Define $H_B(\xi)$ to be the D-algebra generated by the $w^{d\xi(b)}b$ for b in B.

4.1. Theorem. *With notation as above, $H_B(\xi)$ is a Hopf order in H such that $H(\xi_0) \subset H_B(\xi) \subset H(\xi)$.*

Proof. Follow the lines of proof for Theorem 2.3 but using DHV5 and DHV3, taking into account that $\varepsilon(b) = 0$ with $b \neq 0$ yields $d_\xi(b) = \xi(b) \neq -\infty$. □

In case D is Noetherian, D is a discrete valuation ring, e.g., $\Gamma = \mathbb{Z}$, $w = (\pi)$, and π a uniformizing parameter. This is the case considered in [2]; we shall restrict to a discrete valuation ring case in the sequel!

4.2. Proposition. *Let H be a finite-dimensional Hopf algebra and assume it is semisimple, let D be a discrete valuation ring of K, then $H_B(\xi)$ as defined before has a finite D-basis.*

Proof. By definition $H_B(\xi)$ is generated by the $\pi^{\xi(b)}b$, $b \in B$, hence it is an affine algebra over a Noetherian ring D. Since H is finite-dimensional, $H_B(\xi)$ is also a PI-ring. Consequently, since H is semisimple, $H_B(\xi)$ is a semiprime ring, and by the foregoing a Noetherian PI ring. Hence we obtain that $H_B(\xi)$ is a finite D-module and, since it is torsion-free (contained in H a K-algebra), it is free of finite rank. Note that the minimal prime ideals of $H_B(\xi)$ correspond bijectively to the minimal prime ideals of H and $H_B(\xi)$ embeds in a finite product of prime Noetherian PI rings. □

4.3. Observation. The case considered by Larson in [2] is an example of the above situation because $H = KG$ over a field of characteristic zero; the Larson orders correspond to orders of type $H_B(\xi)$ with $H(\xi_0) = DG$ and $B = \{1,\ g-1\colon\ g \in G,\ g \neq 1\}$. The above proposition yields a short proof for the fact, proved in [2] by combinatorial methods, that the Larson orders have a finite basis. We can extend Proposition 3.2 for certain Hopf orders in case H is not semisimple. Let us call a D-order Λ of H a *moderate order* if its prime radical $\mathrm{rad}(\Lambda)$ is a finitely generated D-module. Since $K\Lambda = H$ and $\mathrm{rad}(H)$ is nilpotent, say $(\mathrm{rad}\,H)^d = 0$, it follows that $\mathrm{rad}(\Lambda) = \mathrm{rad}(H) \cap \Lambda$ and $\mathrm{rad}(\Lambda)^d = 0$.

4.4. Proposition. *Let D be a discrete valuation ring of K, H a finite-dimensional Hopf algebra. If $H_B(\xi)$ is a moderate order of H then $H_B(\xi)$ has a finite D-basis.*

Proof. $H_B(\xi)/\mathrm{rad}(H_B(\xi))$ is embedded in a product of prime Noetherian PI rings that are D-orders in some simple Artinian component of $H/\mathrm{rad}(H)$, hence this is a finitely generated D-module. Since $H_B(\xi)$ is assumed to be moderate, $\mathrm{rad}(H_B(\xi))$ is already a finitely generated D-module, hence so is $H_B(\xi)$. Again since $H_B(\xi)$ is a torsion-free D-module, it follows that it has a finite D-basis. □

Most often Hopf algebras will be given to us by specific generators and not too complicated multiplication and comultiplication rules. This usually allows immediately to find a rather trivial Hopf order by taking the specific generators over a subring of K containing the multiplication and comultiplication structure constants (very often this is just $\mathbb{Z}$ or $\mathbb{Z}[\rho]$ for some root of unity, ρ say). That order is chosen for $H(\xi_0)$. The nature of the generators given (e.g., group-like, skew derivations) turns out to be such that the radical part can be well controlled, e.g., all examples given later are moderate orders.

4.5. Remark.

(1) The construction of $H_B(\xi)$ shows that $H_B(\xi)$ is determined by $H_B(\xi_0)$ and the values of ξ on B. This is exactly the advantage of considering the basis B, e.g., in Observation 4.3 the chosen basis defines a Larson order. This may be done via a Zassenhaus valuation on G that can be translated to a Hopf valuation function ξ on KG that we need not describe on all elements of KG explicitly, because only the $\xi(b)$ have been used in the construction of $H_B(\xi)$. Whereas the elements of the basis $\{g\colon\ g \in G\}$ have valuation filtration degree zero in either DG, $H_B(\xi)$ or $H(\xi)$ in KG (because every $g \in G$ is a unit in each of these rings) the basis B is more "discerning."

(2) Is $H_B(\xi) = H(\xi)$? If not, then we can start from $H_B(\xi)$ as $H(\xi)$ and find a D-basis B_1 of $H_B(\xi)$, to repeat the construction of a Hopf order $H_{B_1}(\xi)$, $H_B(\xi) \subset H_{B_1}(\xi) \subset H(\xi)$. If $H(\xi)$ is moderate, i.e., Noetherian as a D-module, the process must terminate. However we can do slightly better:

4.6. Proposition. *Let D be a discrete valuation ring of K and H a finite-dimensional Hopf algebra over K; then for given valuation functions $\xi \leqslant \xi_0$ with corresponding moderate Hopf orders $H(\xi_0) \subset H(\xi)$, there exists a K-basis B of H contained in $H(\xi_0)$ such that $H(\xi) = H_B(\xi)$.*

Proof. Pick a D-basis B' for $H(\xi)$ and for $b_i' \in B'$ put $\beta_i = \xi_0(b_i')$, $i = 1, \ldots, n$. Since $\xi \leqslant \xi_0$, we have $\beta_i \geqslant 0$ and by definition $\pi^{\beta_i} b_i' \in H(\xi_0) - \pi H(\xi_0)$, where $w = (\pi)$.

Put $B = \{\pi^{\beta_i} b_i' \colon i = 1, \ldots, n\}$, then $H_B(\xi) \supset B'$ and thus $H_B(\xi) \supset D[B'] = H(\xi)$, consequently $H_B(\xi) = H(\xi)$. $\square$

In constructing examples we follow the following strategy. We start from a Hopf order $H(\xi_0)$ that can be chosen arbitrarily but it is usually the most trivial one we can find, as explained in the remarks preceding Remark 4.5. Let $\{x_1, \ldots, x_n\}$ be a D-basis for $H(\xi_0)$. We think of $H(\xi)$ as a hypothetical Hopf order containing $H(\xi_0)$ and calculate the conditions relating the $\xi(x_i)$, $i = 1, \ldots, n$, from the properties of ξ. Then we apply the $H_B(\xi)$ construction and we will obtain a Hopf order containing $H(\xi_0)$ but not necessarily $H_B(\xi) = H(\xi)$ (see Remark 4.5 and Proposition 4.6) because the ξ-values on B may not suffice to determine ξ. In any case, since $H_B(\xi)$ is a Hopf order, we have $H_B(\xi) = H(\xi')$ for some function ξ' such that $\xi \leqslant \xi' \leqslant \xi_0$ and ξ and ξ' agree on the basis B. Now, in constructing examples, we do not really care about this ambiguity because $H(\xi)$ was "virtual" to start with and constructing $H(\xi') \supsetneqq H(\xi_0)$ may be viewed as constructing just another $H(\xi)$ (we never specified which ξ we actually aim to arrive at). In practical situations there are two phenomena that seem to happen frequently (always, for the examples we consider hereafter): first, a D-basis for $\mathrm{rad}(H(\xi_0))$ is obvious from the way H is defined; secondly, when looking at $\pi^{\pi(b)} b$ for b in the selected D-basis (containing the D-basis for $\mathrm{rad}(H(\xi_0))$ in it) it turns out that these actually form a D-basis for the D-algebra generated by them! The latter fact is not always trivial to verify, e.g., for the Taft algebra H in Example 4.8 the technical Lemma 4.9 is necessary, Example 4.14 makes this even more clear.

4.7. Example. Consider the Sweedler Hopf algebra over the field of p-adic rationals Q_p, say $H = \mathbb{Q}_p[x, y]$ generated by x, y satisfying $x^2 = 1$, $y^2 = 0$, $xy + yx = 0$, such that:

$$\varepsilon(x) = 1, \qquad S(x) = x, \qquad \Delta(x) = x \otimes x,$$
$$\varepsilon(y) = 0, \qquad S(y) = xy, \qquad \Delta(y) = l \otimes y + y \otimes x.$$

Let $\mathbb{Z}_p$ denote the p-adic integers in $\mathbb{Q}_p$; then after putting $D = \mathbb{Z}_p$ with maximal ideal $(\pi) = (w)$;

$$H(m) = \mathbb{Z}_p + \mathbb{Z}_p(x - 1) + w^{-m} y + w^{-m} xy$$

is a Hopf order in H for every $m \geqslant 0$ in $\mathbb{N}$.

Proof. It is possible to verify this "by hand" because multiplication and comultiplication may be written down explicitly with respect to the basis $1, x - 1, y, xy$; nevertheless let us show how foregoing methods may be applied here.

Since $(x - 1)y = xy - y$ we have: $\xi(x - 1) + \xi(y) \leqslant \max\{\xi(xy), \xi(y)\}$ where ξ is supposed to be the valuation filtration function corresponding to some Hopf order $H(\xi)$ containing $\mathbb{Z}_p[x, y]$, the latter is also a Hopf order (trivial) we may consider as $H(\xi_0)$. From $(x - 1)xy = y - xy$ we obtain

$$\xi(x - 1) + \xi(xy) \leqslant \max\bigl\{\xi(xy), \xi(y)\bigr\}. \tag{$*$}$$

From $\xi(xy) \leqslant \xi(x) + \xi(y)$ HV4 and $\xi(Sh) \leqslant \xi(h)$, plus the fact that $S(y) = xy$ and $S(xy) = -y$, we obtain that $\xi(y) = \xi(xy)$. From $(*)$ above it then follows that $\xi(x-1) \leqslant 0$. So we put $\xi(y) = \xi(xy) = m \geqslant 0$ and $\xi(x-1) = 0$. Since

$$\Delta(x-1) = x \otimes x - 1 \otimes 1,$$
$$\Delta(y) = 1 \otimes y + y \otimes (x-1) + y \otimes 1,$$
$$\Delta(xy) = xy \otimes (x-1) + 1 \otimes y + xy \otimes 1,$$

the $\mathbb{Z}_p$-subalgebra $H(m)$ generated by the $x^{\xi(b)}(b - \varepsilon(b))$ (this is nothing but $H(m)$ as defined in the example) is a Hopf order, with $\mathbb{Z}_p$-basis $1, x-1, \pi^{-m}y, \pi^{-m}xy$. Note that for $m \neq 0$, $H(m) \neq H(\xi_0)$ and if $m < m'$ then $H(m) \subsetneqq H(m')$; consequently, there *cannot be a really maximal* Hopf order H. □

Recall that the two-generator Taft algebra $H_T(n)$ over K is defined by $H_T(n) = K[x, y]$ with $x^n = 1$, $y^n = 0$, $xy + yx = 0$, and $\Delta(x) = x \otimes x$, $\varepsilon(x) = 1$, $S(x) = x^{n-1}$, $\Delta(y) = 1 \otimes y + y \otimes x$, $\varepsilon(y) = 0$, $S(y) = -x^{n-1}y$.

4.8. Example. Let $H_T(n)$ be the Taft algebra as above over $K = \mathbb{Q}_p$. Then $H_T(-1) = \mathbb{Z}_p + \sum_{i=1}^{n-1} \mathbb{Z}_p(x^i - 1) + \sum_{i=0,j=1}^{h-1} \mathbb{Z}_p \pi^{-j} x^i y^j$ is a Hopf order in $H_T(n)$ with basis $\{x^i(\pi^{-1}y)^j\colon i, j = 0, \ldots, n-1\}$.

Proof. From $(x^{n-i} - 1)y^j = x^{n-i}y^j - y^j$ and $(x^i - 1)(x^{n-i} - y^j) = y^j - x^{n-i}y^j$, we obtain the inequalities:

$$\xi\bigl(x^{n-i} - 1\bigr) + \xi\bigl(y^j\bigr) \leqslant \max\bigl\{\xi\bigl(x^{n-i}y^j\bigr), \xi\bigl(y^j\bigr)\bigr\}, \tag{$*$}$$

$$\xi\bigl(x^i - 1\bigr) + \xi\bigl(x^{n-i}y^j\bigr) \leqslant \max\bigl\{\xi\bigl(x^{n-i}y^j\bigr), \xi\bigl(y^j\bigr)\bigr\}. \tag{$**$}$$

Moreover, HV4 yields $\xi(x^{n-i}y^j) \leqslant \xi(x^{n-i}) + \xi(y^j)$, and from HV6 combined with $S(y) = -x^{n-1}y$ we obtain $\xi(x^{n-1}y) \leqslant \xi(y)$. This leads to $\xi(y^j) = \max\{\xi(x^{n-i}y^j), \xi(y^j)\}$ and from $(*)$ we thus obtain $\xi(x^{n-i} - 1) \leqslant 0$. So taking $\xi(y^j) = 1$ for all $j \neq n$ and $\xi(x^{n-i} - 1) \leqslant 0$ for all $i \neq n$, we satisfy the foregoing conditions. Then $H_T(B)$ with respect to the basis $\{x^i y^j\colon i, j = 0, \ldots, n-1\}$ yields exactly the $H_T(-1)$ given in the example. To check that $\Delta(\pi^{-j}x^i y^j) \subset H_T(-1) \otimes H_T(-1)$, one can use the following combinatorial lemma. □

4.9. Lemma. *In a Taft algebra H_T we calculate for all $0 \leqslant i, j < n$ that*:

$$\Delta\bigl(x^i y^j\bigr) = x^i y^j \otimes \bigl(x^i - 1\bigr) + x^i y^j \otimes 1 + \sum_{r=1}^{j} \alpha_r \binom{j}{r} x^{i+r} y^{j-r} \otimes x^i y^r$$
$$= +1 \otimes x^i y^j + \bigl(x^{i+j} - 1\bigr) \otimes x^i y^j$$

where

$$\alpha_r = \begin{cases} 0 & \textit{when } r \textit{ is odd}, \\ (r-1)(r-3)\cdots/(j-1)(j-3)\cdots(j-(r-1)) & \textit{if } r \textit{ is even} \end{cases} \quad \textit{for } j \textit{ even};$$

and

$$\alpha_r = \begin{cases} r(r-2)\cdots/j(j-2)\cdots(j-(r-1)) & \textit{if } r \textit{ is odd}, \\ (r-1)(r-3)\cdots/j(j-2)\cdots(j-(r-2)) & \textit{if } r \textit{ is even} \end{cases} \quad \textit{for } j \textit{ odd}.$$

Proof. Since A is an algebra morphism:

$$\Delta\big(x^i y^j\big) = \big(\Delta(x)\big)^i \big(\Delta(y)\big)^j = \big(x^i \otimes x^i\big)(y \otimes 1 - x \otimes y)^j.$$

Now apply the binomial formula taking into account that $yx = -xy$. □

For group-like elements $g \in G(H) \subset H$ in a Hopf algebra H, the elements we have to consider are just the $g-1$ ($\varepsilon(g)=1$). Therefore the following numerical lemma will be useful.

4.10. Lemma. *Let g be any element in a Q-algebra A; then for any natural number n we have the equality*

$$\begin{aligned}(g-1)^n = g^n &+ \alpha_{n-1}(-1)(g-1)^{n-1} + \alpha_{n-2}(-1)^2(g-1)^{n-2} + \cdots \\ &+ \alpha_{n-i}(-1)^i(g-1)^{n-i} + \cdots + \alpha_2(-1)^{n-2}(g-1)^2 + \alpha_1(-1)^{n-1}(g-1) \\ &+ c_n(-1)^n,\end{aligned}$$

where

$$\alpha_{n-i} = (-1)^{i+1}\binom{n}{i} = (-1)^{i+1}\alpha_i.$$

Proof. Evaluate $T^n - (T+1)^n = -\sum_{k=1}^{n}\binom{n}{k}T^{n-k}$ at $T = g-1$. The author's original proof by induction is considerably shortened by this observation due to the referee. □

4.11. Remark.

(1) If $n = p_1^{i_1} p_2^{i_2} \cdots p_s^{i_s}$ where $p_1, \ldots, p_s$ are nonequal prime numbers, then $p_j^{i_j} \mid \alpha_{p_l^{i_l}}$, $0 \leqslant j, l \leqslant s$, $j \neq l$, and $p_j^{i_j} \nmid \alpha_{p_j^{i_j}}$.

(2) If $n = p^s$, $s \geqslant 1$, then $p^{s-i} \mid \alpha_{p^i}, \alpha_{p^i(p^{s-i}r-1)}$ and $p^{s-i+1} \nmid \alpha_{p_i}, \alpha_{p_i(p^{s-i}+1)}$.

Proof. (1) If $n = pq$ then $\alpha_p = (-1)^{n-p+1} pq(pq-1)\cdots(pq-p+1)/p! = qz$. In fact, no factor $pq - i$, $1 \leqslant i \leqslant p-1$, can be divided by p, otherwise $i = sp$, a contradiction.

(2) If $n = p^s$ then $\alpha_{p^i} = (-1)^{n-p^i+1} p^s(p^s-1)\cdots(p^s-p^i+1)/p^i! = p^{s-i}z$. In fact, no factor $p^i - t$, $1 \leqslant t \leqslant p^i - 1$, can be divided by p^l with $l \leqslant i$ without reducing p^l by a factor of $(p^i-1)!$ otherwise $i = kp^i$, a contradiction. □

4.12. Proposition. *Consider a number field $K/\mathbb{Q}$ and let D be a discrete valuation ring of K extending $\mathbb{Z}_p \subset Q$ (here $\mathbb{Z}_p$ is the localization of $\mathbb{Z}$ with respect to the prime p). Let $e = v(p)$ be the absolute ramification index of K. Consider a finite-dimensional Hopf algebra H over K and let $G = G(H)$ be its finite group of group-like elements. If ξ is any Hopf valuation filtration function on H extending v then we have*:

(1) $\xi(g-1) = 0$ *for $g \in G$ such that the order $0(g)$ of g in G is not a power of p.*
(2) $\xi(g-1) \leqslant e(p^s - p^{s-1})^{-1}$ *if the order of g is p^s.*

Proof. This follows from (3) and (4) in Remark 4.10. Indeed if $0(g) \neq p^s$ then it is a multiple of at least two different primes and none of these can divide all α's in the formula given in Lemma 4.9, hence in that case $\xi(g-1) = 0$. On the other hand, if $n = p^s$ then p will be a divisor of all α's in the formula in Lemma 4.9 but p^2 will not divide $\alpha_{p^{s-1}}$. From Lemma 4.9 we may obtain an expression for $(\pi^{\xi(g)}(g-1))^n$, i.e.: with $n = p^s$:

$$\left(\pi^{\xi(g-1)^n}\right) = \alpha_{n-1}(-1)\left(\pi^{\xi(g)}(g-1)\right)^{n-1} + \cdots$$
$$+ \alpha_{p^{s-i}}\pi^{(n-p^{s-i})\xi(g)}(-1)^{p^{s-i}}\left(\pi^{\xi(g)}(g-1)\right)^{p^{s-i}} + \cdots.$$

It follows from this that $p = v\pi^{(n-p^{s-1})\xi(g)}$ for some $v \in D$, therefore $e \geqslant (p^s - p^{s-1}) \times \xi(g-1)$. □

4.13. Remark.

(1) For $H = KG$, G a finite group, and ξ corresponding to a Larson order (i.e., a Hopf order $H_B(\xi)$ corresponding to the basis $\{1,\ 1-g\colon\ g \in G\}$), the conditions in Proposition 4.12 do reduce to the conditions also found by Larson in [2].
(2) The conditions (2) in Proposition 4.12 make it clear that the realization of a certain ξ forces rather demanding ramification properties of v, e.g., for $d\xi(g) = 1$ one needs $e \geqslant p^s - p^{s-1}$, $p^s = 0(g)$.
Recall the definition of the generalized Taft algebra with respect to a root of unity ρ, say $\rho^n = 1$. Put $H_T(n)$ equal to the K-algebra generated by x and y satisfying $x^n = 1$, $y^n = 0$ and $xy = \rho yx$, with Hopf algebra structure given by

$$\varepsilon(x) = 1, \qquad S(x) = x^{n-1}, \qquad \Delta(x) = x \otimes x,$$
$$\varepsilon(y) = 1, \qquad S(y) = -\rho^{-1}x^{n-1}y, \qquad \Delta(y) = 1 \otimes y + y \otimes x.$$

4.14. Example. Let D be a discrete valuation ring of K, $\rho \in D$.

In case $n \neq p^s$, for $x \geqslant 1$

$$H_T(-n) = D + \sum_{i=1}^{n-1} D(x-1)^i + \sum_{i=0,\,j=1}^{n-1} D\pi^{-jn}(x-1)^i y^j$$

is a Hopf order.

In case $n = p^s$, $\pi^{m(p^s - p^{s-1})} \mid p$ and $\pi^m \mid (\rho - 1)$ in D, then

$$H_T\left(-p^s\right) = D + \sum_{i=1}^{n-1} D\pi^{-im}(x-1)^i + \sum_{i=0,\,j=1}^{n-1} D\pi^{-im-jn}(x-1)^i y^j$$

is a Hopf order.

Observe that $(\rho-1)^{p^s} = p\sum_{i=1}^{p^s-1}(\alpha_i/p)(\rho-1)^i$ (Lemma 3.9) and then $(\rho-1)^{p^s} \in (p) \subset \pi^{(p^s p^{s-1})m}$, where $p^s m < e + p^{s-1}m$, $e = v(p)$.

A full proof of the claims can be obtained via the quantum binomial formula (see [5] applied to $f = \pi^{-m}(g-1)$, and via a careful coefficient calculation in an expression for $X^t f^s$, $t, s \in \mathbb{N}$. We omit these technical details here. Let us just provide a concrete case where all of the above phenomena are clear.

4.15. Example. Take $p = 2$ and look at the localization of $\mathbb{Z}_z[\rho]$ at $\rho - 1$ where $\rho = \sqrt[4]{-1} = \sqrt{\mathrm{i}}$. For D we take $\mathbb{Z}[(\rho-1)^{1/5}]_{((\rho-1)^{1/5})}$. In this case $\pi = (\rho-1)^{1/5}$, $(2) = (\pi^5)$, $v(\pi) = 1$, $e = v(2) = 5$, $v(\rho - 1) = 5$, $\rho^8 = 1$. So we have $(\rho-1)^8 \in (2)$ and $(2) \subset (\rho - 1) \subset (\pi) \subset D$. The constructions in Remark 4.13 apply in this case.

Let us conclude with an example showing the effect of base change.

4.16. Example. We start with the situation of Example 4.7 but with K a number field such that $\pi^m \mid 2$ in D. Put $H(m,n) = D[f, \chi]$, $f = \pi^{-m}(x-1)$, $\chi = \pi^{-n}y$. Then $H(m,n)$ is a Hopf algebra of rank 4 over D with

$$\begin{aligned}
&\Delta(f) = f \otimes x + 1 \otimes f,\\
&f\chi + \chi f = \nu\chi, \quad \text{where } 2 = \nu\chi^m \text{ with } \nu \in D,\\
&\Delta(f\chi) = f\chi \otimes 1 + 1 \otimes f\chi + f \otimes x\chi + \chi \otimes fx,\\
&\Delta(\chi) = 1 \otimes \chi + \chi \otimes x,\\
&f\chi - \chi = \nu x\chi.
\end{aligned}$$

We may also define $H(n)$ as in Proposition 4.6, it is of rank 4 over D with basis $\{1, x-1, \pi^{-n}y, \pi^{-n}xy\}$. Both $H(m,n)$ and $H(n)$ contain the Hopf order $D[x, y]$ (viewed as $H(\xi_0)$ in Proof of 3.6)

Now $H(n)$ is of the form $H_B|\xi|$ with respect to $B = \{1, x-1, y, xy\}$ and $H(m,n)$ with respect to $B' = \{1, x-1, y, (x-1)y\}$, and these orders are obviously different.

Acknowledgment

The authors thank the referee for careful reading of the manuscript, leading to serious enhancement of the presentation.

References

[1] J.E. Bjork, Rings of Differential Operators, North-Holland, Amsterdam.
[2] R.G. Larson, Hopf algebra orders determined by group valuations, J. Algebra 38 (1976) 414–452.
[3] C. Năstăsescu, F. Van Oystaeyen, Graded Ring Theory, in: Library of Math., Vol. 28, North-Holland, Amsterdam, 1982.
[4] F. Van Oystaeyen, Pseudo-places of algebras and the symmetric part of the Brauer group, PhD thesis, Amsterdam, 1972.
[5] C. Kassel, Quantum Groups, Springer, Berlin, 1995.
[6] S. Montgomery, Hopf Algebras and their Actions on Rings, in: CMBS, Vol. 82, American Mathematical Society, Providence, RI, 1993.
[7] O. Schilling, The Theory of Valuations, in: Amer. Math. Soc. Math. Surveys, Vol. IV, American Mathematical Society, Providence, RI, 1950.
[8] S. Dascalescu, C. Năstăsescu, S. Raianu, Hopf Algebras, Dekker, New York, 2001.