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The Divergence on Submanifolds of the Wiener Space

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Using Malliavin's calculus, the divergence, the covariant derivative, and the Riemann and Ricci curvatures of a submanifold of the Wiener space are defined. It is shown that the Ricci and Riemann curvatures appear in the commutator of the divergence operator and covariant derivative operator. Capacities are used to restrict the commutator formula to a submanifold and to deduce that the curvature belongs to all $L^p(da^\xi)$-spaces where a^ξ is the area measure associated to the submanifold. A version of Weitzenböck's formula, a Bochner type formula, and a criterion for the existence of the divergence are proved. A generalization to higher dimensional tensor fields is stated, and a connection with stochastic integration on the submanifold of the Wiener space is worked out.

INTRODUCTION

In many recent papers [3, 6–8, 10, 17–19] the loop space of a manifold M is looked upon as a subspace of the Wiener space in the following way: The Ito map Φ, which is the solution of the stochastic differential equation for Brownian motion on a manifold, associates to each path of the Wiener space $t \mapsto \omega(t)$ a path based in a fixed point m of the manifold $t \mapsto \Phi_t(\omega)$. The space of loops based in m on the manifold is the image under the Ito map of the subspace of the Wiener space which consists of the paths ω in the Wiener space for which the Ito map takes at time $t = 1$ the value m, i.e., paths for which $\Phi_1(\omega) = m$. Such a subspace is a submanifold defined implicitly by the nondegenerate functional Φ_1. Getzler and Airault introduced in [10, 4] an expression for the Ricci curvature of such submanifolds of the Wiener space. They used this expression to establish Weitzenböck type formulas for restrictions to the submanifold of exact differential one forms. The aim of this paper is to generalize their results in the following way: The Riemann curvature is introduced and in Theorem 30 it is shown

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that the Ricci curvature can be found as a limit of contractions of the Riemann curvature using a renormalization technique. It is shown that the Weitzenböck formula for differential forms in [4, 10] is valid even without the restrictive exactness condition. In fact it is shown that the formulae in [4, 10] (cf. Theorem 12) are contained in the more general result stated in Theorem 11, which calculates the commutator of the divergence operator and the covariant derivation on the submanifold. This theorem is a generalization of the results of Stroock [24], Shigekawa [22], and Malliavin [16]. By these improvements it is possible to prove the existence of the divergence for differentiable forms in $L^{2+\varepsilon}(da^{\xi})$ (cf. [5]) and to compare the divergence on the submanifold to a stochastic integral on the submanifold (Theorem 25).

1. Notation

The Wiener space $\mathscr{W}$ consists of continuous paths ω $(0 \leqslant t \leqslant 1)$, with values in $\mathbb{R}$ and starting in 0. The subset of an absolutely continuous path based in 0 and with derivative in $L^2[0, 1]$ is called the Cameron–Martin subspace H. The dual $\mathscr{W}^*$ of the Wiener space $\mathscr{W}$ can be considered as a subspace of $L^2[0, 1]$: An element λ in $\mathscr{W}^*$ is associated to the unique l in $L^2[0, 1]$ for which the identity $\lambda(h) = \int_{[0,1]} h'(s)\, l(s)\, ds$ holds for all the functions h in the Cameron–Martin space H. The Wiener measure on $\mathscr{W}$ is denoted by μ. A measurable map determined on the Wiener space up to a μ-negligible set and with values in $L^2([0, 1]^k) = L^2[0, 1]^k$ is called a k-parameter process. A square integrable one parameter process X_ω is a map $L^2([0, 1] \times \mathscr{W}, dt \otimes \mu, \mathbb{R})$. A square integrable two parameter process is a map in $L^2([0, 1]^2 \times \mathscr{W}, dt \otimes dt \otimes \mu, \mathbb{R})$, or equivalently this is a square integrable one parameter process on $\mathscr{W}$ with values in $L^2[0, 1]$. Multiparameter processes can be interpreted in the same way. This notation is close to the notation adopted in [21].

By definition the class $\mathscr{S}_k$ consists of k-parameter processes $X_\omega(t_1, ..., t_k)$ such that for some n and m there exists a representation as

$$X_\omega(t_1, ..., t_k) = \sum_{i=1}^{m} \phi_i(\lambda_1(\omega), ..., \lambda_n(\omega))\, v_i^1(t_1) \cdots v_i^k(t_k). \tag{1.1}$$

Here $\phi_1, ..., \phi_m$ are rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, $\lambda_1, ..., \lambda_n$ belong to $\mathscr{W}^*$, and the v_i^j are functions in $L^2[0, 1]$. The classes $\mathscr{S}_k$ $(k = 0, 1, ...)$ are used to introduce the divergence and derivative operators. Since $L^2[0, 1]^k$ is the closure of the subspace spanned by functions of the form $v_i^1(t_1) \cdots v_i^k(t_k)$ and since the functions of the form $\phi_i(\lambda_1(\omega), ..., \lambda_n(\omega))$ span $L^2(\mathscr{W}, \mu)$ this class is dense in $L^2_{L^2[0,1]^k}(\mathscr{W}, \mu)$.

The derivative of the process $X_\omega(t_1, ..., t_k)$ is a $k+1$-parameter process in $\mathscr{S}_{k+1}$ denoted by $\nabla_{t_0} X_\omega(t_1, ..., t_k)$ (or by ∇X when the parameters are not mentioned explicitly) and it is given by the formula

$$\nabla_{t_0} X_\omega(t_1, ..., t_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x^j}(\lambda_1(\omega), ..., \lambda_n(\omega))\, l_j(t_0)\, v_i^1(t_1) \cdots v_i^k(t_k). \quad (1.2)$$

The L^2-function l_j in this formula is associated to the functional λ_j in $\mathscr{W}^*$. $\nabla_{t_0} X_\omega$ is denoted $(DX)_{t_0}$ in [21]. This definition of the derivation operator is equivalent with the definition used by Kree [12] and Malliavin [15]. They introduce derivations of maps on the Wiener space with values in a real separable Hilbert space G. The derivation of such a map takes values in the space of Hilbert–Schmidt operators from the Cameron–Martin space H to G which is denoted $\mathscr{H}.\mathscr{S}.(H, G)$. The equivalence of definition is obtained when $L^2[0, 1]^k$ is taken for G and H is identified with $L^2[0, 1]$ because then $\mathscr{H}.\mathscr{S}.(H, K)$ equals $L^2[0, 1]^{k+1}$.

We now introduce the divergence operator. Suppose that the functions $l_1(t), ..., l_m(t)$ in the following representation are orthogonal in $L^2[0, 1]$:

$$Y_\omega(t_0, ..., t_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} \phi_{ij}(\lambda_1(\omega), ..., \lambda_n(\omega))\, l_j(t_0)\, v_{ij}^1(t_1) \cdots v_{ij}^k(t_k). \quad (1.3)$$

This representation is slightly more involved than the representation of X in (1.1) because it is necessary to pick functions l_j out of a fixed orthogonal basis in the expression $l_j(t_0)\, v_{ij}^1(t_1) \cdots v_{ij}^k(t_k)$ to define the divergence of the $k+1$ parameter process $Y_\omega(t_0, ..., t_k)$. The divergence is a k-parameter process denoted by $\delta_{s_0} Y(s_0, ..., s_k)$ (or by δY) and defined by the expression

$$(\delta_{s_0} Y_{s_0})_\omega (s_1, ..., s_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(-\frac{\partial \phi_{ij}}{\partial x^j}(\lambda_1(\omega), ..., \lambda_n(\omega)) + \lambda_j(\omega)\, \phi_{ij}(\lambda_1(\omega), ..., \lambda_n(\omega)) \right) v_{ij}^1(s_1) \cdots v_{ij}^k(s_k). \quad (1.4)$$

The differential operators ∇ and δ are adjoint, i.e., for all processes X in $\mathscr{S}_k$ and Y in $\mathscr{S}_{k+1}$ the equality

$$\mathbf{E}^\mu \left[\int_{[0,1]^{k+1}} \nabla_{t_0} X_\omega(t_1, ..., t_k)\, Y_\omega(t_0, ..., t_k)\, dt_0 \cdots dt_k \right] = \mathbf{E}^\mu \left[\int_{[0,1]^k} X_\omega(t_1, ..., t_k)(\delta_{t_0} Y_{t_0})_\omega (t_1, ..., t_k)\, dt_1 \cdots dt_k \right] \quad (1.5)$$

holds.

By this property and since $\mathcal{S}_k$ is dense in $L^p_{L^2[0,1]^k}(\mathcal{W}, \mu)$ for all p $(1 < p < \infty)$ it follows that the operator ∇ from $\mathcal{S}_k \subset L^p_{L^2[0,1]^k}(\mathcal{W}, \mu)$ to $L^p_{L^2[0,1]^{k+1}}(\mathcal{W}, \mu)$ and the operator δ from $\mathcal{S}_{k+1} \subset L^p_{L^2[0,1]^{k+1}}(\mathcal{W}, \mu)$ to $L^p_{L^2[0,1]^k}(\mathcal{W}, \mu)$ are closable. The closure of the domain of the operator ∇ on $\mathcal{S}_k$ provided with the norm $\|X\|_{p,1} = \|X\|_p + \|\nabla X\|_p$ is a Banach space denoted by $W_{p,1}(L^2[0,1]^k)$ and it can be considered as a subspace of $L^p_{L^2[0,1]}(\mathcal{W}, \mu)$. The Banach spaces $W_{p,2}(L^2[0,1]^k), \ldots, W_{p,n}(L^2[0,1]^k), \ldots,$ are constructed by closing the family of functions $\mathcal{S}_k$ with respect to the inductively defined sequence of norms $\|X\|_{p,n} = \|X\|_p + \|\nabla X\|_{p,n-1}$ for $n \geqslant 2$.

The family $(W_{p,n}(L^2[0,1]^k))_{1<p<\infty, n\in\mathbb{N}}$ is a decreasing sequence of subspaces contained in $L^1_{L^2[0,1]^k}(\mathcal{W}, \mu)$ and the intersection $\bigcap W_{p,n}(L^2[0,1]^k)$ is denoted $W_\infty(L^2[0,1]^k)$. The closure of the operator δ contains $W_{p,1}(L^2[0,1]^{k+1})$ and δ is a continuous operator from $W_{p,n+1}(L^2[0,1]^{k+1})$ to $W_{p,n}(L^2[0,1]^k)$. These results are proved by Meyer, Kree, and Sugita in [12, 20, 23].

We will need the following properties for the spaces W_∞

(1) The operators δ and ∇ are continuous from W_∞ to W_∞.

(2) If A is a infinitely differentiable mapping from $L^2[0,1]^k$ to $L^2[0,1]^l$ with slowly increasing derivatives of all order, then the induced mapping which sends X_ω to $A \circ X_\omega$ is continuous from $W_\infty(L^2[0,1]^k)$ to $W_\infty(L^2[0,1]^l)$.

In [11, 13] these results are proved in a more general setting, with the L^2 spaces replaced by arbitrary real separable Hilbert spaces, and also the proof of the following derivation formulas can be found there. To state the first formula the derivative of A in $v \in L^2[0,1]^k$ is denoted DA_v and is considered as a continuous linear functional from $L^2[0,1]^k$ to $L^2[0,1]^l$.

$$\nabla(A \circ X)_\omega = DA_{X_\omega} \circ (\nabla X_\omega). \tag{1.6}$$

For a bilinear continuous map b from $L^2[0,1]^k \times L^2[0,1]^l$ to $L^2[0,1]^m$ the indentities

$$\nabla_t b(X^1, X^2) = b(\nabla_t X^1, X^2) + b(X^1, \nabla_t X^2) \tag{1.7}$$

and,

$$\delta_t b(X^1_t, X^2) = b(\delta_t X^1_t, X^2) - \int_{[0,1]} b(X^1_t, \nabla_t X^2)\, dt \tag{1.8}$$

hold true.

To verify that the last integral makes sense, observe that for all continuous bilinear maps b from the product of two Hilbert spaces $H_1 \times H_2$

to a Hilbert space H_0 and for all Hilbert spaces K there exists a unique continuous bilinear map from $\mathcal{H.S.}(K, H_1) \times \mathcal{H.S.}(K, H_2)$ to H_0 such that $(k \otimes h_1, k' \otimes h_2)$ is mapped to $\langle k, k' \rangle_K b(h_1, h_2)$. This result is applied with $K = L^2[0, 1]$, $H_1 = L^2[0, 1]^l$, $H_2 = L^2[0, 1]^k$, and $H_0 = L^2[0, 1]^m$.

2. Decomposition of the Wiener Space by a Nondegenerate Functional

We fix a nondegenerate smooth functional Φ with values in $\mathbb{R}^d$; i.e., we fix a collection of d real-valued stochastic variables $(\Phi^1, ..., \Phi^d)$ defined on the Wiener space with the following properties:

(1) The functional Φ^i belongs to $W_\infty(\mathbb{R})$ for all $i = 1, ..., d$.

(2) The covariance matrix σ with components $\sigma^{ij} = \int_{[0,1]} \nabla_t \Phi^i \nabla_t \Phi^j \, dt$ is μ-almost surely invertible and the components σ_{ij}^{-1} of the inverse matrix σ^{-1} also belongs to $W_\infty(\mathbb{R})$. This covariance matrix is the same as the matrix introduced by Malliavin in [19].

These properties imply that the functions $\rho = \frac{1}{2} \log \det \sigma$ and ρ^{-1} are in W_∞, and that for almost all ω the subspace of $L^2[0, 1]$ spanned by the gradient vectors $\{\nabla\Phi^1, ..., \nabla\Phi^d\}$ is d-dimensional, where $\nabla\Phi^i$ denotes the derivative of Φ, i.e., a map from $\mathcal{W}$ in $L^2[0, 1]$ as defined in the previous section. The orthogonal complement of this normal subspace is the tangent space given by

$$\{\nabla\Phi^1, ..., \nabla\Phi^d\}^\perp = \left\{ l \in L^2[0, 1] \text{ such that } \int_{[0,1]} l(t) \nabla_t \Phi^i \, dt = 0 \text{ for all } i \right\}. \tag{2.1}$$

The orthogonal projection of $L^2[0, 1]$ onto the normal subspace is a finite rank operator $N_\omega \in \mathcal{H.S.}(H, H; \mathbb{R})$ with a kernel given by the expression

$$N(s, t) = \sum_{i,j} \nabla_s \Phi^i \sigma_{ij}^{-1} \nabla_t \Phi^j.$$

A one parameter process $X_t \in W_\infty(L^2[0, 1])$ can be decomposed as a sum of a normal and tangent component as

$$X_t - \int_{[0,1]} X_s N(s, t) \, ds = P^t X_t. \tag{2.2}$$

The path in ω of the process $P^t X_t$ is the projection on the tangent space in ω of the path $X_\omega(t)$. The normal projection N, the tangent projection P, and their sum Id gives rise to 3^k projection operators on $L^2[0,1]^k$ generally denoted by $S_1 \otimes \cdots \otimes S_k$ where S_i is N, P, or Id. The projections are defined by their action on the generators of $L^2[0,1]^k$:

$$(S_1 \otimes \cdots \otimes S_k)(h_1(t_1) \cdots h_k(t_k)) = (S_1 h_1)(t_1) \otimes \cdots \otimes (S_k h_k)(t_k).$$

$P^{t_{i_1}, \ldots, t_{i_r}} A(t_1, \ldots, t_k)$, denotes $(S_1 \otimes \cdots \otimes S_k)(A(t_1, \ldots, t_k))$ where S_i is equal to P if i belongs to $\{i_1, \ldots, i_r\}$ and else equal to Id. For example, suppose that Y is a 2 parameter process, i.e., a map from the Wiener space with values in $L^2[0,1]^2$, and denote the function value of Y in ω by $Y_\omega(s,t)$. Also denote the tangent space of the decomposition Φ in ω by T_ω. The product $T_\omega \times T_\omega$ is a subspace of $L^2[0,1]^2$ and the orthogonal projection of the L^2 function $Y_\omega(s,t)$ on $T_\omega \times T_\omega$ is denoted by $P^{s,t} Y_\omega(s,t)$. In particular $P^{s,t} Y(s,t)$ is an example of the following class of processes:

Definition. A process $A(t_1, \ldots, t_k)$ is said to be tangent if $P^{t_1, \ldots, t_k} A(t_1, \ldots, t_k) = A(t_1, \ldots, t_k)$ for almost all ω.

In [2, 4] the construction of a family of measures $(a^\xi)_{\xi \in \mathbb{R}^d}$ with the following properties is carried out:

(1) For all continuous stochastic variables $Y \in W_\infty$, the map $\xi \in \mathbb{R}^d$ to $\int_{\mathscr{W}} Y \, da^\xi$ has a continuous modification;

(2) For all continuous functions ϕ with compact support on $\mathbb{R}^d$ the following equality is valid:

$$\int_{\mathbb{R}^d} \phi(\xi) \int_{\mathscr{W}} Y \, da^\xi \, d\xi = \int_{\mathscr{W}} (\phi \circ \Phi) \, Y e^\nu \, d\mu.$$

The measures a^ξ on $\mathscr{W}$ are called the area measures associated to the functional Φ. The area measure is normalized to p^ξ for values of ξ for which $a^\xi(\mathscr{W})$ doesn't vanish.

$$p^\xi = \frac{a^\xi}{a^\xi(X)}.$$

The expectation operator with respect to this propability p^ξ is denoted by $\mathbf{E}^{p^\xi}$.

3. Divergence and Derivative on Submanifolds

The definition of a derivative and a divergence operator with respect to the decomposition of the Wiener space by Φ is motivated by the following proposition.

PROPOSITION 1 (see also [2, 1, 10]). *For all one parameter processes X_t and two parameter processes $Y(s, t)$ in W_∞, and for all continuous functions with compact support on $\mathbb{R}^d$ the identity*

$$\int_{\mathbb{R}^d} \phi(\xi)\, \mathbf{E}^{P^\xi}\left[\int_{[0,1]^2} \nabla_s(P^t X_t)\, P^{s,t} Y(s,t)\, ds\, dt\right] d\xi$$

$$= \int_{\mathbb{R}^d} \phi(\xi)\, \mathbf{E}^{P^\xi}\left[\int_{[0,1]} P^t X_t\, e^{-\rho}\, \delta_s(e^\rho P^{s,t} Y(s,t))\, dt\right] d\xi$$

holds true.

Proof. It is sufficient to prove the formula for smooth functions with compact support on $\mathbb{R}^d$ and it is equivalent to use integration with respect to a^ξ instead of taking the expectation $\mathbf{E}^{P^\xi}$. Hence we will show the equality

$$\int_{\mathbb{R}^d} \phi(\xi) \int_{\mathscr{W}} \int_{[0,1]^2} \nabla_s(P^t X_t)\, P^{s,t} Y(s,t)\, ds\, dt\, da^\xi\, d\xi$$

$$= \int_{\mathbb{R}^d} \phi(\xi) \int_{\mathscr{W}} \int_{[0,1]} P^t X_t\, e^{-\rho}\, \delta_s(e^\rho P^{s,t} Y(s,t))\, dt\, da^\xi\, d\xi$$

which is equivalent to

$$\int_{\mathscr{W}} (\phi \circ \Phi)\, e^\rho \int_{[0,1]^2} \nabla_s(P^t X_t)\, P^{s,t} Y(s,t)\, ds\, dt\, d\mu$$

$$= \int_{\mathscr{W}} (\phi \circ \Phi) \int_{[0,1]} P^t X_t\, \delta_s(e^\rho P^{s,t} Y(s,t))\, ds\, dt\, d\mu \tag{3.1}$$

by the characterization of the area measure. This last equality is valid by virtue of the adjointness of ∇ and δ. Indeed the divergence $\delta_s(e^\rho(\phi \circ \Phi)\, P^{s,t} Y(s,t))$ in Eq. (3.1) can be calculated by rule (1.8). This yields

$$\delta_s(e^\rho(\phi \circ \Phi)\, P^{s,t} Y(s,t))$$

$$= (\phi \circ \Phi)\, \delta_s(e^\rho P^{s,t} Y(s,t)) - e^\rho \int_{[0,1]} \nabla_s(\phi \circ \Phi)\, P^{s,t} Y(s,t)\, ds.$$

Here $\int_{[0,1]} \nabla_s(\phi \circ \Phi)\, P^{s,t} Y(s,t)\, ds = 0$ because $P^{s,t} Y(s,t)$ is a tangent process and $\nabla_s(\phi \circ \Phi)$ is normal. ∎

DEFINITION. The derivative with respect to the decomposition of a k-parameter process X in W_∞ is denoted $\nabla^\Phi X$ and it is given by the expression

$$\nabla^\Phi_{t_0} X(t_1, ..., t_k) = P^{t_0, ..., t_k}\, \nabla_{t_0} P^{t_1, ..., t_k} X(t_1, ..., t_k).$$

The divergence of a $k+1$-parameter process Y in W_∞ is denoted by $\delta^\Phi Y$ and it is defined as

$$\delta^\Phi_{t_0} Y(t_0, ..., t_k) = e^{-\rho} P^{t_1, ..., t_k} \delta_{t_0}(e^\rho P^{t_0, ..., t_k} Y(t_0, ..., t_k)).$$

Remark. By the same techniques as are used in the proof of Proposition 1 the following generalization can be shown. For processes X in $W_\infty(L^2[0, 1]^k)$ and Y in $W_\infty(L^2[0, 1]^{k+1})$ the equality

$$\int_{\mathbb{R}^d} \phi(\xi)\, \mathbf{E}^{P^\xi}\left[\int_{[0,1]^{k+1}} \nabla^\Phi_{t_0} X(t_1, ..., t_k)\, Y(t_0, ..., t_k)\, dt_0 \cdots dt_k\right] d\xi$$
$$= \int_{\mathbb{R}^d} \phi(\xi)\, \mathbf{E}^{P^\xi}\left[\int_{[0,1]^k} X(t_1, ..., t_k)\, \delta^\Phi_{t_0} Y(t_0, ..., t_k)\, dt_1 \cdots dt_k\right] d\xi \qquad (3.2)$$

is valid.

4. Relation of δ to δ^Φ, and of ∇ to ∇^Φ

Lemma 2. *The identity*

$$P^t \nabla_t \tfrac{1}{2} \log \det \sigma = -P^t \delta_u N(u, t)$$

holds true in W_∞.

Proof. To find $\nabla_t \rho$ the chain rule (1.6) is applied with $A(\sigma) = \frac{1}{2} \log \det \sigma$. The result reads as

$$P^t \nabla_t \rho = \sum_{i,j} \sigma_{ij}^{-1} \int_{[0,1]} \nabla_t \nabla_s \Phi^i \nabla_s \Phi^j\, ds.$$

By the derivation rules (1.7) and (1.8) the divergence $P^t \delta_u (\sum_{i,j} \sigma_{ij}^{-1} \nabla_u \Phi^i \nabla_t \Phi^j)$ is calculated. Three terms appear and two of them cancel by composition with the projection P^t because they are normal. Only the following term remains:

$$P^t \delta_u \left(\sum_{i,j} \sigma_{ij}^{-1} \nabla_u \Phi^i \nabla_t \Phi^j\right) = -P^t \sum_{i,j} \sigma_{ij}^{-1} \int_{[0,1]} \nabla_u \Phi^i \nabla_u \nabla_t \Phi^j\, du.$$

By symmetry of $\nabla_u \nabla_s \Phi^j$ in (u, s) this equals $P^t \nabla_t \rho$. ∎

Proposition 3. *The divergence of a* $k+1$ *parameter tangent process* Y *in* W_∞ *can be read from the formulas*

$$\delta_{t_0}^{\Phi} Y(t_0, ..., t_k) = P^{t_1, ..., t_k}\left(\delta_{t_0} Y(t_0, ..., t_k) + \int_{[0,1]} \delta_{t_0} N(t_0, t)\, Y(t, t_1, ..., t_k)\, dt\right) \tag{4.1}$$

$$= P^{t_1, ..., t_k}\left(\delta_{t_0} Y(t_0, ..., t_k) + \int_{[0,1]^2} \nabla_s Y(t, t_1, ..., t_k)\, N(s, t)\, ds\, dt\right). \tag{4.2}$$

Proof. The expansion

$$\delta_{t_0}^{\Phi} Y_{t_0 \cdots t_k} = P^{t_1, ..., t_k}\left(\delta_{t_0} Y(t_0, ..., t_k) - \int_{[0,1]} Y(t, t_1, ..., t_k)\, \nabla_t \rho\, dt\right)$$

is carried out by application of the derivation rules (1.8) and (1.7). Using Lemma 2 this expression reduces to (4.1) and from this expression the formula (4.2) is obtained by the identity

$$\int_{[0,1]} \delta_{t_0} N(t, t_0)\, Y(t, t_1, ..., t_k)\, dt - \int_{[0,1]^2} \nabla_s Y(t, t_1, ..., t_k)\, N(s, t)\, ds\, dt = 0.$$

This identity is found by expanding the divergence

$$\delta_{t_0}\left(\int_{[0,1]} Y(t, t_1, ..., t_k)\, N(t, t_0)\, dt\right) = 0.$$

Indeed, $\int_{[0,1]} Y(t, t_1, ..., t_k)\, N(t, t_0)\, dt$ vanishes because Y is a tangent process. ∎

LEMMA 4. *The identity*

$$P^s \nabla_u N(r, s) = \int_{[0,1]} N(r, t)\, \nabla_u N(t, s)\, dt \tag{4.3}$$

is verified in $W_\infty(L^2[0, 1]^3)$.

Proof. Since $N(r, s)$ is the kernel of a projection operator it fulfills the relation

$$N(r, s) = \int_{[0,1]} N(r, t)\, N(t, s)\, dt.$$

The derivative of this equality is calculated,

$$\nabla_u N(r, s) = \int_{[0,1]} \nabla_u N(r, t)\, N(t, s)\, dt + \int_{[0,1]} N(r, t)\, \nabla_u N(t, s)\, dt$$

and this expression is transformed to

$$\int_{[0,1]} N(r, t)\, \nabla_u N(t, s)\, dt = \nabla_u N(r, s) - \int_{[0,1]} \nabla_u N(r, t)\, N(t, s)\, dt.$$

Using (2.2) it is readily verified that this equation is equivalent to Eq. (4.3). ▮

DEFINITION. By the previous lemma, the following three expressions define the same 3-parameter process,

$$\begin{aligned} S_r(s, t) &= P^{s,t}\, \nabla_s N(r, t) \\ &= \int_{[0,1]} N(r, u)\, P^s\, \nabla_s N(u, t)\, du \\ &= \int_{[0,1]} N(r, u)\, P^{s,t}\, \nabla_s N(u, t)\, du. \end{aligned} \tag{4.4}$$

This process is called the second fundamental form.

PROPOSITION 5. *The derivative* ∇^{Φ} *of a tangent process* X *in* W_{∞} *can be calculated by the formula*

$$\nabla_s^{\Phi} X_t = P^s\, \nabla_s X_t - \int_{[0,1]} X_r S_t(s, r)\, dr. \tag{4.5}$$

Proof. Equation (2.2) is used to calculate the difference $\nabla_s^{\Phi} X_t - P^s\, \nabla_s X_t$:

$$\nabla_s^{\Phi} X_t - P^s\, \nabla_s X_t = P^s \int_{[0,1]} \nabla_s X_r\, N(r, t)\, dr.$$

Since the process X is tangent, its normal component $\int_{[0,1]} X_r N(r, t)\, dr$ vanishes and the derivative of this expression,

$$\int_{[0,1]} \nabla_s X_r\, N(r, t)\, dr + \int_{[0,1]} X_r\, \nabla_s N(r, t)\, dr = 0, \tag{4.6}$$

is used to reduce Eq. (4.6) to Eq. (4.5). ▮

PROPOSITION 6. *The second fundamental form $S_r(s, t)$ is symmetric in s and t.*

Proof. Since the operator N projects on the subspace spanned by $\{\nabla_s \Phi^1, ..., \nabla_s \Phi^d\}$ it leaves the gradients spanning the subspace invariant: $\nabla_s \Phi^i = \int_{[0,1]} \nabla_t \Phi^i N(s, t)\, dt$. In this way a more involved expression

$$\nabla_u \nabla_s \Phi^i = \int_{[0,1]} \nabla_u \nabla_t \Phi^i\, N(s, t)\, dt + \int_{[0,1]} \nabla_t \Phi^i\, \nabla_u N(s, t)\, dt$$

for the second derivative of Φ is found. The tangent projection $P^{u,s}$ of the first term in this expansion vanishes, and the other term is rewritten using the definition of the second fundamental form. Hence we obtain

$$P^{u,s} \nabla_u \nabla_s \Phi^i = \int_{[0,1]} \nabla_t \Phi^i\, P^{u,s} \nabla_u N(s, t)\, dt = \int_{[0,1]} \nabla_t \Phi^i\, S_t(u, s)\, dt.$$

Since the 2-parameter process $P^{u,s} \nabla_u \nabla_s \Phi^i$ is symmetric in u and s, the previous formula shows that the normal projection in the r-variable of $S_r(s, t)$ is symmetric in s and t. From the definition (4.4) it can be read that $S_r(s, t)$ has no tangent component in the r-variable, and this proves the proposition. ∎

5. THE STROOCK–SHIGEKAWA COMMUTATOR FORMULA FOR SUBMANIFOLDS OF THE WIENER SPACE

Suppose $X_t \colon \mathscr{W} \times [0, 1] \mapsto \mathbb{R}$ is a predictable process with the following properties:

(1) X_t belongs to W_∞;

(2) $\nabla_s X_t$ belongs to W_∞;

(3) The following integral is finite for all p:

$$\int_{\mathscr{W}} \int_{[0,1]} |X_t|^p + \left(\int_{[0,1]} |\nabla_s X_t|^2\, ds \right)^{p/2} dt\, d\mu.$$

Stroock and Kusuoka have proved the formula stated in the following lemma about the derivative of the stochastic integral of such a process.

LEMMA 7. *For all h in the Cameron–Martin space the equality*

$$\int_{[0,1]} \nabla_s \left(\int_0^1 X_t\, dB_t \right) h'(s)\, ds$$
$$= \int_0^1 \int_{[0,1]} \nabla_s X_t\, h'(s)\, ds\, dB_t + \int_{[0,1]} X_t h'(t)\, dt \tag{5.1}$$

is valid.

Since the Ito integral and the divergence coincide for predictable processes, this formula is equivalent to a commutator formula for the divergence and the derivation. To a process X_t on the Wiener space can be associated a differential form by the formula $X(h)=\int_{[0,1]} X_t h'(t)\,dt$ for all $h\in H$. Using this identification the following result of Shigekawa in [22] is a generalization of the formula (5.1) to differential forms of arbitrary order. We state the theorem in our notation.

THEOREM 8 (Shigekawa). *For all $k+1$ parameter processes $X_{t_0}(t_1, ..., t_k)$ in $W_\infty(L^2[0,1]^{k+1})$ the following commutator relation holds:*

$$\delta_{t_0}\nabla_s X_{t_0}(t_1, ..., t_k)-\nabla_s\delta_{t_0}X_{t_0}(t_1, ..., t_k)=X(s, t_1, ..., t_k).$$

Proof. See Proposition 3.1 in [22]. ∎

Showing an analogous commutator formula for derivation with respect to the decomposition will be the aim of this section. The result will be fully proved for one parameter processes. The result in the multiparameter case is stated in the final section.

LEMMA 9. *For all one parameter tangent processes X_t in $W_\infty(L^2[0,1])$ the identity*

$$\nabla^\Phi_s(\delta^\Phi_t X_t)-\nabla^\Phi_s(\delta_t X_t)$$

$$=P^s\int_{[0,1]^2}\nabla_s\nabla_u X_t\, N(u,t)\,du\,dt \tag{5.2}$$

$$+\int_{[0,1]^2}\nabla_r X_t\, S_r(s,t)\,dr\,dt \tag{5.3}$$

$$-P^s\int_{[0,1]} X_t\left(\int_{[0,1]^2} S_u(t,r)\,S_u(s,r)\,dr\,du\right)dt \tag{5.4}$$

holds true.

Proof. The divergence δ^Φ_t of X_t is given by formula (4.2). In fact we have

$$\delta^\Phi_t X_t=\delta_t X_t+\int_{[0,1]^2}\nabla_u X_t\, N(u,t)\,du\,dt.$$

Hence the following formula gives the difference in the left hand side of (5.2):

$$\nabla^\Phi_s(\delta^\Phi_t X_t-\delta_t X_t)=P^s\nabla_s\left(\int_{[0,1]^2}\nabla_u X_t\, N(u,t)\,du\,dt\right).$$

By the product rule (1.7) this is expanded as

$$\nabla_s^{\Phi}\left(\int_{[0,1]^2}\nabla_u X_t\, N(u,t)\,du\,dt\right)=P^s\left(\int_{[0,1]^2}\nabla_s\nabla_u X_t\, N(u,t)\,du\,dt\right)$$

$$+P^s\left(\int_{[0,1]^2}\nabla_u X_t\,\nabla_s N(u,t)\,du\,dt\right).$$

The first term appears in the right hand side of (5.2). The second is decomposed as a sum of a normal and a tangent component by relation (2.2):

$$P^s\int_{[0,1]^2}\nabla_r X_t\,\nabla_s N(r,t)\,dr\,dt$$

$$=P^s\int_{[0,1]^2}\nabla_r X_t P^t\,\nabla_s N(r,t)\,dr\,dt$$

$$+P^s\int_{[0,1]^3}\nabla_r X_t\, N(t,u)\,\nabla_s N(r,u)\,dr\,du\,dt. \tag{5.5}$$

Using the definition of the second fundamental form $P^{s,t}\nabla_s N(r,t)=S_r(s,t)$, the tangent component is identified with term (5.3). The following two identities are applied to transform the second component. The first is equivalent with the vanishing of the derivative ∇ of $\int_{[0,1]} X_t N(r,t)\,dt$:

$$\int_{[0,1]}\nabla_r X_t\, N(t,u)\,dt=-\int_{[0,1]} X_t\,\nabla_r N(t,u)\,dt. \tag{5.6}$$

The second can be found by applying Lemma 4 twice:

$$P^t\int_{[0,1]}\nabla_v N(t,u)\,\nabla_s N(r,u)\,du=P^{t,r}\int_{[0,1]}\nabla_v N(t,u)\,\nabla_s N(r,u)\,du.$$

Hence the second component in formula (5.5) transforms to

$$P^s\int_{[0,1]^3}\nabla_r X_t\, N(t,u)\,\nabla_s N(r,u)\,dr\,du\,dt$$

$$=-P^s\int_{[0,1]} X_t\int_{[0,1]^2}\nabla_r N(t,u)\,\nabla_s N(r,u)\,dr\,du\,dt$$

$$=-\int_{[0,1]} X_t P^{s,t,r}\int_{[0,1]^2}\nabla_r N(t,u)\,\nabla_s N(r,u)\,dr\,du\,dt.$$

By using twice the definition of the second fundamental form $P^{s,r}\nabla_s N(r,u) = S_u(s,r)$ the expression

$$P^{s,t,r}\int_{[0,1]^2} \nabla_r N(t,u)\,\nabla_s N(r,u)\,dr\,du$$

reduces to

$$P^t\int_{[0,1]^2} S_u(t,r)\,S_u(s,r)\,du\,dr$$

and hence term (5.5) equals term (5.4). ∎

LEMMA 10. *For all* 1-*parameter tangent processes* X_t *in* $W_\infty(L^2[0,1])$ *the expansion*

$$\delta_t^\Phi \nabla_s^\Phi X_t - P^s\,\delta_t \nabla_s X_t = P^s\int_{[0,1]^2} \nabla_u \nabla_s X_t\, N(u,t)\,dt\,du \tag{5.7}$$

$$+\int_{[0,1]^2} \nabla_r X_t\, S_r(t,s)\,dr\,dt \tag{5.8}$$

$$+P^s\int_{[0,1]} X_t \int_{[0,1]} S_r(s,t)\,\delta_u N(u,r)\,dr\,dt$$

is valid.

Proof. As X_t is tangent, the derivative $\nabla_s^\Phi X_t$ is given by $\nabla_s^\Phi X_t = P^{s,t}\nabla_s X_t$. The divergence is calculated by formula (4.1):

$$\delta_t^\Phi \nabla_s^\Phi X_t - P^s\,\delta_t \nabla_s X_t$$
$$= P^s\left(\delta_t(P^{s,t}\nabla_s X_t) + \int_{[0,1]} P^{s,t}\nabla_s X_t\,\delta_u N(u,t)\,dt - \delta_t \nabla_s X_t\right).$$

In this expression the double projection $P^{s,t} = P\otimes P$ is written as $(I-N)\otimes(I-N) = I\otimes I - N\otimes I + N\otimes N - I\otimes N$ and we obtain

$$= P^s\left(\delta_t(\nabla_s X_t) - \delta_t\left(\int_{[0,1]} \nabla_r X_t\, N(r,s)\,dr\right)\right.$$
$$+\delta_u\left(\int_{[0,1]^2} N(u,t)\,\nabla_r X_t\, N(r,s)\,dr\,dt\right) - \delta_u\left(\int_{[0,1]} N(u,t)\,\nabla_s X_t\,dt\right)$$
$$\left.+\int_{[0,1]} \nabla_s X_t P^t\,\delta_u N(u,t)\,dt - \delta_t(\nabla_s X_t)\right).$$

The first and the last terms cancel. The product rule (1.8) for the divergence is applied to expand this expression further. In the expansion

$$\begin{aligned}\delta_t^{\Phi}\nabla_s^{\Phi}X_t - P^s\,\delta_t\nabla_s X_t = P^s &\int_{[0,1]^2} \nabla_r X_t \nabla_t N(r,s)\,dr\,dt \\ &- P^s\int_{[0,1]^3} \nabla_r X_t\, N(u,t)\,\nabla_u N(r,s)\,dr\,dt\,du \\ &+ P^s\int_{[0,1]^2} \nabla_u\nabla_s X_t\, N(u,t)\,dt\,du \\ &- P^s\int_{[0,1]} \nabla_s X_t\,\delta_u N(u,t)\,dt \\ &+ P^s\int_{[0,1]} \nabla_s X_t\, P^t(\delta_u N(u,t))\,dt \end{aligned} \tag{5.9}$$

the terms which vanish by composition with P^s are omitted. The third term is the right hand side of formula (5.7). The sum of the first and the second term reduces to (5.8). Indeed, we have

$$\begin{aligned} &P^s\int_{[0,1]^2} \nabla_r X_t \nabla_t N(r,s)\,dr\,dt \\ &\quad - P^s\int_{[0,1]^2} \nabla_r X_t \int_{[0,1]} N(u,t)\,\nabla_u N(r,s)\,dr\,dt\,du = \\ &P^s\int_{[0,1]^2} \nabla_r X_t P^t \nabla_t N(r,s)\,dr\,dt = \int_{[0,1]^2} \nabla_r X_t\, S_r(s,t)\,dr\,dt. \end{aligned}$$

The final equality holds true by the definition of the second fundamental form. To the sum of the last two terms in (5.9), the identity (5.6) is applied, and the result is also rewritten by use of the second fundamental form:

$$\begin{aligned} &- P^s\int_{[0,1]} \nabla_s X_t\,\delta_u N(u,t)\,dt + P^s\int_{[0,1]} \nabla_s X_t\, P^t(\delta_u N(u,t))\,dt \\ &= -P^s\int_{[0,1]^2} \nabla_s X_t\, N(t,r)\,\delta_u N(u,r)\,dr\,dt \\ &= P^s\int_{[0,1]^2} X_t \nabla_s N(t,r)\,\delta_u N(u,r)\,dr\,dt \\ &= P^s\int_{[0,1]^2} X_t S_r(s,t)\,\delta_u N(u,r)\,dr\,dt. \quad \blacksquare \end{aligned}$$

THEOREM 11. *For all 1-parameter processes X_t in $W_\infty(L^2[0,1])$ the identity*

$$\nabla_s^\Phi \delta_t^\Phi X_t - \delta_t^\Phi \nabla_s^\Phi X_t = P^s X_s + \int_{[0,1]} X_t \operatorname{Ricci}(s,t)\, ds \tag{5.10}$$

is valid. Here Ricci *denotes the 2-parameter process defined by the expression*

$$\begin{aligned}\operatorname{Ricci}(s,t) = & -P^{s,t} \int_{[0,1]} \delta_u N(u,r)\, S_r(s,t)\, dr \\ & - P^{s,t} \int_{[0,1]^2} S_r(u,t)\, S_r(u,s)\, dr\, du.\end{aligned} \tag{5.11}$$

Proof. By the symmetry of the second fundamental form $S_r(s,t) = S_r(t,s)$ and of the normal projection $N(u,t) = N(t,u)$ the terms (5.2) and (5.7), and (5.3) and (5.8) cancel out in the expression

$$\begin{aligned}&(\nabla_s^\Phi \delta_t^\Phi X_t - \delta_t^\Phi \nabla_s^\Phi X_t) \\ &\quad = +P^s \nabla_s \delta_t X_t - P^s \int_{[0,1]} X_t \left(\int_{[0,1]^2} S_u(t,r)\, S_u(s,r)\, dr\, du \right) dt \\ &\qquad - P^s \delta_t \nabla_s X_t - P^s \int_{[0,1]} X_t \left(\int_{[0,1]} S_r(s,t)\, \delta_u N(u,r)\, dr \right) dt \\ &\quad = P^s(\nabla_s \delta_t X_t - \delta_t \nabla_s X_t) + \int_{[0,1]} X_t \operatorname{Ricci}(s,t)\, dt.\end{aligned}$$

Theorem 8 is used to identify $P^s(\nabla_s \delta_t X_t - \delta_t \nabla_s X_t)$ with $P^s X_s$. ∎

We denote by $\mathscr{L}^v u$ the operation $\mathscr{L}^v u = -\frac{1}{2} \delta_s^\Phi \nabla_s^\Phi u$ defined for functions u belonging to $W_\infty(\mathbb{R})$.

THEOREM 12 (Weitzenböck's Formula for Submanifolds; see also [10, 4]). *For all functions f and g in $W_\infty(\mathbb{R})$ the identity*

$$\begin{aligned}&\mathscr{L}^v \left(\int_{[0,1]} \nabla_s^\Phi f \nabla_s^\Phi g\, ds \right) - \int_{[0,1]} \nabla_s^\Phi (\mathscr{L}^v f) \nabla_s^\Phi g\, ds - \int_{[0,1]} \nabla_s^\Phi f \nabla_s^\Phi (\mathscr{L}^v g)\, ds \\ &\quad = \int_{[0,1]^2} \nabla_s^\Phi \nabla_t^\Phi f \nabla_s^\Phi \nabla_t^\Phi g\, dt\, dt + \int_{[0,1]} \nabla_s^\Phi f \nabla_s^\Phi g\, ds \\ &\qquad + \int_{[0,1]^2} (\nabla_s^\Phi f)(\nabla_t^\Phi g) \operatorname{Ricci}(s,t)\, ds\, dt\end{aligned} \tag{5.12}$$

is valid.

Proof. The following equality is easily verified by using the definition of $\mathscr{L}^v$ and the derivation rules (1.7) and (1.8):

$$\mathscr{L}^v\left(\int_{[0,1]} \nabla_s^\Phi f \nabla_s^\Phi g\, ds\right) + \tfrac{1}{2}\int_{[0,1]} \delta_t^\Phi(\nabla_t^\Phi \nabla_s^\Phi f)\, \nabla_s^\Phi g\, ds$$

$$+ \tfrac{1}{2}\int_{[0,1]} \nabla_s^\Phi f\, \delta_t^\Phi(\nabla_t^\Phi \nabla_s^\Phi g)\, ds = \int_{[0,1]^2} \nabla_s^\Phi \nabla_t^\Phi f \nabla_s^\Phi \nabla_t^\Phi g\, ds\, dt. \qquad (5.13)$$

We prove the theorem by calculating the difference of the second terms of the left hand sides of formula (5.12) and formula (5.13). The following identity is valid by the equality $\nabla_s^\Phi \nabla_t^\Phi f = \nabla_t^\Phi \nabla_s^\Phi f$:

$$-\int_{[0,1]} \nabla_s^\Phi \mathscr{L}^v f \nabla_s^\Phi g\, ds + \tfrac{1}{2}\int_{[0,1]} \delta_t^\Phi \nabla_t^\Phi \nabla_s^\Phi f \nabla_s^\Phi g\, ds$$

$$= -\tfrac{1}{2}\int_{[0,1]} (\nabla_s^\Phi \delta_t^\Phi \nabla_t^\Phi f - \delta_t^\Phi \nabla_s^\Phi \nabla_t^\Phi f)\, \nabla_s^\Phi g\, ds.$$

Now we apply Theorem 11 to the expression $\nabla_s^\Phi \delta_t^\Phi \nabla_t^\Phi f - \delta_t^\Phi \nabla_s^\Phi \nabla_t^\Phi f$ and we obtain

$$\tfrac{1}{2}\int_{[0,1]} \left(P^s \nabla_s^\Phi f + \int_{[0,1]} \nabla_t^\Phi f \operatorname{Ricci}(s,t)\, dt\right) \nabla_s^\Phi g\, ds.$$

This expression equals half the difference of the right hand sides of formulas (5.12) and (5.13). By symmetry the difference of the remaining terms in the left hand sides of the formulas (5.12) and (5.13) yields the other half. ∎

6. Capacities and Redefinitions

All the equalities of Wiener functionals stated in the previous sections are μ-almost sure equalities. Since the Wiener measure is singular to all area measures, there is no immediate relation with equalities which are a^ξ-almost sure. By using the capacities introduced by Malliavin [14] it is possible to exploit the smoothness of the functionals to obtain better results.

Definition. The Sobolev norm $\| \ \|_{p,r}$ is used to define the capacity $c_{p,r}$ of an open set O of the Wiener space in the following way:

$c_{p,r}(O) = \inf\{ \|Y\|_{p,r} \mid$ where Y is a positive stochastic variable on the Wiener space belonging to $W_{p,r}$ and μ-almost surely greater than 1 on $O\}$.

The $c_{p,r}$-capacity $c_{p,r}(A)$ of an arbitrary subset $A \subset \mathscr{W}$ is the infimum of the $c_{p,r}$-capacities of the open sets containing A.

It is clear that the capacities are finer for higher p and r, i.e., $c_{p,r}$ is increasing with increasing p and r. A set $A \subset \mathscr{W}$ is called slim iff the capacity $c_{p,r}(A)$ vanishes for all p and r. Capacities are used to construct nice versions of smooth functionals on the Wiener space.

DEFINITION. A functional Y^* is called a redefinition (up to a slim set) of a functional Y in W_∞, if there exists a sequence of continuous functionals $(Y_n)_n \in W_\infty$ and a sequence of open sets $(O_k)_k$ such that the following three properties hold:

(1) Y_n converges to Y in W_∞;

(2) Y_n converges to Y^*, uniformly on the complement O_k^c of each open set O_k;

(3) The capacities $c_{k,k}(O_k)$ converge to zero.

When these conditions are fulfilled $((Y_n)_n, (O_k)_k, Y^*)$ is called a redefinition triple for Y.

By the technique of redefinition the class of μ-almost sure modifications of a smooth functional can be reduced to a class of functionals which are pairwise equal up to a slim set.

The basic tool for manipulations involving redefinitions is provided by the following theorem.

THEOREM 13 (Malliavin). *From each sequence of continuous smooth functionals $(Y_n)_n$ converging to Y in W_∞, a subsequence $(Y_{n_m})_m$ can be extracted such that the triple $((Y_{n_m})_m, (O_k)_k, \lim_{m\to\infty} Y_{n_m} = Y^*)$ is a redefinition triple for an appropriately chosen sequence of decreasing open sets $(O_k)_k$.*

By this theorem it is simple to prove that redefinitions behave well with respect to simple transformations.

PROPOSITION 14. *Suppose that A is a continuous map from a Hilbert space H_1 to H_2 and such that the induced map from $Y \in W_\infty(H_1)$ to $A \circ Y$ in $W_\infty(H_2)$ is also continuous. Then $A \circ Y^*$ is a redefinition of $A \circ Y$ for all functionals Y in $W_\infty(H_1)$.*

Proof. Choose a sequence of continuous smooth functionals $(Y_n)_n$ converging to Y in $W_\infty(H_1)$ and denote by $((Y_{n_m}, A \circ Y_{n_m})_m, (O_k)_k, (Y^*, A \circ Y^*))$ the redefinition triple for $(Y, A \circ Y) \in H_1 \times H_2$ provided by the previous theorem. For all elements ω not belonging to the intersection $\bigcap_k O_k$ the equalities $A \circ Y^*(\omega) = \lim_{k\to\infty} A \circ Y_{n_m}(\omega) = (A \circ Y)^*(\omega)$ are valid. ∎

The usefulness of redefinitions to prove results with respect to the decomposition of the Wiener space is shown by the following refinement of the definition of the area measures.

THEOREM 15 [4, 2]. *For all ξ in $\mathbb{R}^d$ there exists a unique measure a^ξ on the Wiener space such that*

> *for all functionals $Y \in W_\infty$ the map $\xi \mapsto \int Y^* \, da^\xi$ is a continuous version of the conditional expectation of Ye^P with respect to Φ;* (6.1)
>
> *slim sets are negligible for all the measures a^ξ.* (6.2)

PROPOSITION 16. *The map from $Y \in W_\infty$ to Y^* in $L^p(a^\xi)$ is continuous for all p, $1 \leq p < \infty$.*

Proof. See [4]. ∎

7. DIVERGENCE AND COVARIANT DERIVATIVE WITH RESPECT TO THE AREA MEASURE

We fix an element ξ in $\mathbb{R}^d$ such that the area measure a^ξ doesn't vanish. The Banach space of p-integrable processes defined up to p^ξ-modification and with values in $L^2[0, 1]^k$ is denoted $L^p_{L^2[0,1]^k}(da^\xi)$. Such a process $A(s_1, ..., s_k)$ is called tangent iff $(P^{s_1, ..., s_k})^* A(s_1, ..., s_k)$ equals $A(s_1, ..., s_k)$ p^ξ-almost surely. The class of tangent processes constitutes a closed subspace of $L^p_{L^2[0,1]^k}(da^\xi)$ denoted by $W^\xi_{p,0}(L^2[0, 1]^k)$.

LEMMA 17. *The set of redefinitions of arbitrary X^* (resp. tangent) k-parameter processes in W_∞ constitutes a dense subspace of $L^p_{L^2[0,1]^k}(da^\xi)$ (resp. of $W^\xi_{p,0}(L^2[0, 1]^k)$).*

Proof. It is sufficient to prove the density in $L^p_{L^2[0,1]^k}(\mathscr{W}, \mu)(da^\xi)$ of the set of redefinitions of k-parameter processes. The other density result stated is an easy consequence. For 0-parameter processes, i.e., for stochastic variables with values in $\mathbb{R}$, the result is proved in [4] and the same argument works in general. ∎

DEFINITION. We introduce two unbounded operators on $W^\xi_{p,0}(L^2[0, 1]^k)$. The domain of the operator $\nabla^{r\xi}$ on $W^\xi_{p,0}(L^2[0, 1]^k)$ $(1 < p < \omega)$ consists of the redefinitions $X^*(s_1, ..., s_k)$ of tangent processes X in W_∞. The operation is defined as

$$(\nabla^{r\xi}_{s_0} X^*)(s_1, ..., s_k) = (\nabla^\Phi X)^*_{s_0} (s_1, ..., s_k), \qquad da^\xi\text{-almost surely.}$$

In a analogous way the domain of $\delta^{v\xi}$ on $W^{\xi}_{p,0}(L^2[0,1]^{k+1})$ consists of the redefinitions $Y^*_{s_0,\dots,s_k}$ of $(k+1)$-parameter processes Y in W_∞ and $\delta^{v\xi}$ operates as

$$(\delta^{v\xi}_{s_0} Y^*)(s_1, \dots, s_k) = (\delta^{\Phi}_{s_0} Y_{s_0})^* (s_1, \dots, s_k), \qquad da^{\xi}\text{-almost surely.}$$

These definitions are justified by the following proposition.

PROPOSITION 18. *The operators $\nabla^{v\xi}$ and $\delta^{v\xi}$ are well defined and adjoint and consequently closable operators on $W^{\xi}_{p,0}(L^2[0,1]^k)$.*

Proof. Since the maps

$$\xi \mapsto \int_{\mathscr{W}} \int_{[0,1]^{k+1}} (\nabla^{\Phi}_{s_0} X)^* (s_1, \dots, s_k)\, Y^*(s_0, \dots, s_k)\, ds_0 \cdots ds_k\, da^{\xi}$$

and

$$\xi \mapsto \int_{\mathscr{W}} \int_{[0,1]^k} X^*(s_1, \dots, s_k)(\delta^{\Phi}_{s_0} Y_{s_0})^* (s_1, \dots, s_k)\, ds_1 \cdots ds_k\, da^{\xi}$$

are continuous and since they have the same integral with respect to all smooth functions with compact support by formula (3.2) they coincide pointwise. Hence, using redefinitions the following improved version of (3.2) is obtained. For all ξ and for all tangent processes X and Y in W_∞ the identity

$$\int_{\mathscr{W}} \int_{[0,1]^{k+1}} (\nabla^{\Phi}_{s_0} X)^* (s_1, \dots, s_k)\, Y^*(s_0, \dots, s_k)\, ds_0 \cdots ds_k\, da^{\xi}$$

$$= \int_{\mathscr{W}} \int_{[0,1]^k} X^*(s_1, \dots, s_k)(\delta^{\Phi}_{s_0} Y_{s_0})^* (s_1, \dots, s_k)\, ds_1 \cdots ds_k\, da^{\xi}$$

is valid. All the results stated are readily verified from this formula. Indeed suppose that the redefinition $X^*(s_1, \dots, s_k)$ of a tangent process X in W_∞ vanishes in $W^{\xi}_{p,0}(L^2[0,1]^k)$. Then the covariant derivative has vanishing integral with respect to all redefinitions of functions in W_∞. By the density of such redefinitions showed in the previous lemma the covariant derivative $\nabla^{v\xi}_{s_0} X_{s_1}$ vanishes, and this proves that $\nabla^{v\xi}$ is well-defined. The same argument works for $\delta^{v\xi}$. The adjointness and closability are also immediate. ∎

The closure of the domain of the operator $\nabla^{v\xi}$ on $L^p_{L^2[0,1]^k}(da^{\xi})$ with respect to the norm $\|X\|_p + \|\nabla^{v\xi} X\|_p = \|X\|^{\xi}_{p,1}$ is denoted by $W^{\xi}_{p,1}(L^2[0,1]^k)$. The main result of this section will be a relation between the space $W^{\xi}_{p,1}(L^2[0,1])$ and the closure of the domain of $\delta^{v\xi}$. We need the following theorem.

THEOREM 19. *For all tangent processes Y_t in $W_\infty(L^2[0, 1])$ the following identity holds true*:

$$\int_{\mathscr{W}} (\delta_t^\Phi Y_t)^{*2}\, da^\xi = \int_{\mathscr{W}} \int_{[0,1]} (Y_t^*)^2\, dt\, da^\xi$$
$$+ \int_{\mathscr{W}} \int_{[0,1]^2} (\nabla_t^\Phi Y_s)^* (\nabla_s^\Phi Y_t)^*\, ds\, dt\, da^\xi$$
$$+ \int_{\mathscr{W}} \int_{[0,1]^2} \mathrm{Ricci}(s, t)^*\, Y_t^* Y_s^*\, ds\, dt\, da^\xi. \qquad (7.1)$$

Proof. By adjointness of ∇^Φ and δ^Φ and by the commutator formula 11 the following manipulations are valid:

$$\int_{\mathscr{W}} (\delta_t^\Phi Y_t)^{*2}\, da^\xi = \int_{\mathscr{W}} \int_{[0,1]} (\nabla_s^\Phi \delta_t^\Phi Y_t)^*\, Y_s^*\, ds\, da^\xi$$
$$= \int_{\mathscr{W}} \int_{[0,1]} (\delta_t^\Phi \nabla_s^\Phi Y_t)^*\, Y_s^*\, ds\, da^\xi + \int_{\mathscr{W}} \int_{[0,1]} (Y_t^*)^2\, dt\, da^\xi$$
$$+ \int_{\mathscr{W}} \int_{[0,1]^2} \mathrm{Ricci}(s, t)^*\, Y_s^* Y_t^*\, ds\, dt\, da^\xi.$$

After using again the adjointness of ∇^Φ and δ^Φ the last formula is equivalent to formula (7.1). ∎

MAIN THEOREM 20. *The union*

$$\bigcup_{\varepsilon > 0} W^\xi_{2(1+\varepsilon)}(L^2[0, 1])$$

is continuously contained in the closure of the domain of δ^{v_ξ} on $W^\xi_{2,0}(L^2[0, 1])$.

Proof. Fix $\varepsilon > 0$ and suppose Y is a tangent process in W_∞. The right hand side of (7.1) can be estimated as

$$\int_{\mathscr{W}} \int_{[0,1]^2} (\nabla_t^\Phi Y_s)^* (\nabla_s^\Phi Y_t)^*\, ds\, dt\, da^\xi \leqslant \int_{\mathscr{W}} \int_{[0,1]^2} (\nabla_t^\Phi Y_s^*)^2\, ds\, dt\, da^\xi$$

and

$$\int_{[0,1]^2} \mathrm{Ricci}(s, t)^*\, Y_t^* Y_s^*\, ds\, dt$$
$$\leqslant \left(\int_{[0,1]^2} (\mathrm{Ricci}(s, t)^*)^2\, ds\, dt \right)^{1/2} \int_{[0,1]} (Y_t^*)^2\, dt.$$

To the last expression Hölder's inequality is applied with $1+\varepsilon$ and $1+1/\varepsilon$ as conjugate exponents:

$$\int_{\mathscr{W}} \int_{[0,1]^2} \text{Ricci}(s,t)^* \, Y_t^* Y_s^* \, ds \, dt \, da^{\xi}$$
$$\leqslant \left(\int_{\mathscr{W}} \left(\int_{[0,1]} (Y_t^*)^2 \, dt \right)^{1+\varepsilon} da^{\xi} \right)^{1+1/\varepsilon}$$
$$\times \left(\int_{\mathscr{W}} \left(\int_{[0,1]^2} (\text{Ricci}(s,t)^*)^2 \, ds \, dt \right)^{1/2(1+1/\varepsilon)} \right)^{\varepsilon/(1+\varepsilon)}.$$

This yields the estimate

$$(\|(\delta_s^{\nu_\xi} Y_s)^*\|_2^{\xi})^2 \leqslant (\|Y^*\|_{2(1+\varepsilon),1}^{\xi})^2 \, (1+\|\text{Ricci}^*\|_{1+1/\varepsilon,0}^{\xi}).$$

Since Ricci belongs to $W_{\infty}(L^2[0,1]^2)$, Lemma 16 shows that the norm $\|\text{Ricci}^*\|_{1+1/\varepsilon,0}^{\xi}$ is finite. Hence it is proved that the divergence is continuous as an operator from $W_{\infty}(L^2[0,1])$ to $W_{2,0}^{\xi}(L^2[0,1])$ with respect to the norm $\| \; \|_{2(1+\varepsilon),1}^{\xi}$. As the value of $\varepsilon > 0$ is arbitrary, the theorem is proved. ∎

8. Connection with Stochastic Integration

Suppose that it is possible to define a stochastic integral with respect to the process $X_t(\omega) = \omega(t)$ defined on $(\mathscr{W}, p^{\xi}, \mathscr{F}_t = \sigma(X_s : 0 \leqslant s \leqslant t))$. We will compare in this section the stochastic integral $\int_{[0,1]} Y_t \, dX_t$ with the divergence $\delta_t^{\nu_\xi} Y_t$ operator introduced in the previous section.

Lemma 21. *For all processes Y_t in $W_{\infty}(L^2[0,1])$ the identity*

$$(\delta Y_t) - \delta^{\Phi} Y_t = \int_{[0,1]^2} Y_t(\delta_s N(s,r)) \, N(r,t) \, dr \, dt$$
$$- \int_{[0,1]^2} (\nabla_s Y_t) \, N(s,t) \, ds \, dt \tag{8.1}$$

is valid.

Proof. In the case of tangent processes this difference is calculated in Proposition 3. When also the normal component of the process is taken into account the following expression is obtained:

$$(\delta Y_t) - \delta^{\Phi} Y_t = \delta_s \left(\int_{[0,1]} Y_t N(s,t) \, dt \right) - \int_{[0,1]} P' Y_t \, \delta_s N(s,t) \, dt.$$

By the chain rule (1.8), the following expression is deduced from the last equation:

$$= \int_{[0,1]} Y_t\, \delta_s N(s,t) - \int_{[0,1]} \nabla_s Y_t N(s,t)\, ds\, dt - \int_{[0,1]} P^t X_t\, \delta_s N(s,t)\, dt.$$

Taking the first and the last term of this expression together leads to (8.1). ▌

Notation. In this section $K_1, ..., K_k$ denote families of functions in $L^2[0, 1]$. Denote by $\mathscr{S}^{K_1, ..., K_k}$ the class of k-parameter processes $Y_\omega(t_1, ..., t_k)$ which admit a representation as

$$Y_\omega(t_1, ..., t_k) = \sum_i \phi_i(\lambda_1^i(\omega), ..., \lambda_m^i(\omega))\, v_i^1(t_1) \cdots v_i^k(t_k).$$

Here the functions ϕ_i are rapidly decreasing on $\mathbb{R}^m$, the linear functionals λ_j^i belong to $\mathscr{W}^*$, and the functions v_i^j belong to K_i. The subfamily $\mathscr{P}^K$ of $\mathscr{S}^K$ consists of 1-parameter processes which admit a similar representation with the extra condition that for all numbers i the interval $[0, 1]$ can be split in two parts $[0, t_i]$ and $[t_i, 1]$, such that the function v_i is supported by $[t_i, 1]$ and such that for all j, the function l_j^i which is associated to λ_j^i, is supported by $[0, t_i]$. Processes in $\mathscr{P}^K$ are predictable. If the families $K_1, ..., K_k$ are dense in $L^2[0, 1]$ then the family $\mathscr{S}^{K_1, ..., K_k}$ is dense in $L^p_{L^2[0,1]^k}(p^\xi)$. If the family $K|_{[t,1]} = \{f|_{[t,1]} \mid f \in K$ and f supported by $[t, 1]\}$ is dense in $L^2[t, 1]$ for all t, then $\mathscr{P}^K$ is dense in the class of square-integrable predictable processes on $\mathscr{W} \times [0, 1]$.

We fix an element in $\xi \in \mathbb{R}^d$. We introduce the following hypotheses (H.1) to (H.5).

(H.1) The area measure a^ξ doesn't vanish.

(H.2) The process $(X_t, \mathscr{F}_t, \mathscr{F}, \mathscr{W}, p^\xi(d\omega))$ is L^2-primitive. This means that there exists a finite measure α on the predictable σ-algebra of $\mathscr{W} \times [0, 1]$ with the following dominating property. For all simple processes Y with a representation $Y_\omega(t) = \sum_i a_i 1_{[s_i, t_i]}(t)\, 1_{F_i}(\omega)$ where F_i belongs to $\mathscr{F}_{t_i}$ the stochastic integral of Y with respect to X can be estimated as

$$\mathbf{E}^{p^\xi}\left[\left(\int_{\mathscr{W}} Y_t\, dX_t\right)^2\right] \leqslant \int_{\mathscr{W} \times [0,1]} Y_t^2(\omega)\, d\alpha(t, \omega).$$

(H.3) The measure α is absolutely continuous with respect to $dt \otimes dp^\xi$.

Denote by K^α the subfamily of $L^2[0,1]$ consisting of functions (deterministic processes) x for which the integral $\int_{\mathscr{W}\times[0,1]} x^2(t)\, d\alpha(t,\omega)$ is finite.

LEMMA 22. *For all predictable processes Y_t in $\mathscr{P}^{K^\alpha}$ the equality*

$$\int Y_t\, dX_t = (\delta_t Y_t)^*$$

is valid p^ξ-almost surely.

Proof. For a process $Y_t = \phi(\omega)\, 1_{[s_i, t_i]}(t)$ where ϕ is $\mathscr{F}_{s_i}$-measurable in $\mathscr{S}$ the stochastic integral equals

$$\int Y_t\, dX_t = \phi(\omega)\cdot(X_{t_1} - X_{s_1})$$

and the divergence is given by

$$(\delta_t Y_t)^* = \left(\int_{[0,1]} Y_t\, dB_t\right)^* = \phi(\omega)(B_{t_1} - B_{s_1}),$$

because the functional $\lambda(\omega) = \omega(t_1) - \omega(s_1)$ is the element of $\mathscr{W}^*$ associated to $1_{[s_1, t_1]} \in L^2[0,1]$. The formula holds true for finite sums of such processes. For all processes Y_t in $\mathscr{P}^{K^\alpha}$ there exists a sequence Y_t^n of processes which are sums of such simple processes and which converge in $L^2([0,1]\times\mathscr{W}, \mathscr{P}, \alpha)$ and in $W_\infty(L^2[0,1])$. There exists a subsequence $(Y^{n_m})_m$ such that $(\delta_t Y_t^{n_m})^*$ converges to $(\delta_t Y_t)^*$ on the complement of a slim set and such that $\int Y_t^{n_m}\, dX_t$ converges to $\int Y_t\, dX_t$ p^ξ-almost surely. This proves that the equality holds for all processes in $\mathscr{P}^{K^\alpha}$. ∎

For predictable processes in $\mathscr{P}^K$ the divergence δ coincides with the stochastic integration with respect to the process X_t and hence δ can be considered as an unbounded operator on $L^2([0,1]\times\mathscr{W}, \mathscr{P}, dt\otimes dp^\xi)$. The divergence with respect to the decomposition $\delta^{v\xi}$ and the first correction term in (8.1)

$$\int_{[0,1]^2} Y_t(\delta_s N(s,r))\, N(r,t)\, dr\, dt$$

can also be considered as unbounded operators on $L^2([0,1]\times\mathscr{W}, \mathscr{P}, dt\otimes dp^\xi)$. For the last correction term in (8.1), $\int_{[0,1]^2} \nabla_s Y_s\, N(s,t)\, ds\, dt$ an interpretation as an unbounded operator on $L^2([0,1]\times\mathscr{W}, \mathscr{P}, dt\otimes dp^\xi)$ is less obvious.

LEMMA 23. *There exists a process $A(s, t)$ tangent in the s-variable (i.e., such that $P^s A(s, t) = A(s, t)$) such that for all predictable processes X_t in W_∞ the following equality is valid*:

$$\int_{[0,1]^2} \nabla_s Y_t N(s, t)\, ds\, dt = \int_{[0,1]} \left(\int_{[0,1]} \nabla_s Y_t A(s, t)\, ds \right) dt. \tag{8.2}$$

Proof. Denote by $1_\nabla(s, t) = 1_{[t,1]}(s)$ the characteristic function of $(t \leqslant s)$ on $[0, 1]^2$. For all predictable processes Y_t in W_∞ the product $1_\nabla(s, t) \nabla_s Y_t = 0$ vanishes. Consequently for all 2-parameter processes which can be written as $A(s, t) = N(s, t) + 1_\nabla(s, t)\, B(s, t)$ and for all predictable processes identity (8.2) is fulfilled. Such a process A is tangent in s iff for all i the equality

$$\int_{[0,1]} (N(s, t) + 1_\nabla(s, t)\, B(s, t))\, \nabla_s \Phi^i\, ds = 0$$

is valid. This equality is equivalent with

$$\nabla_t \Phi^i + \int_t^1 B(s, t)\, \nabla_s \Phi^i\, ds = 0.$$

The function $B(s, t)$ which fulfills this equality and which minimalizes for almost all t the integral

$$\int_{[0,1]} B^2(s, t)\, ds$$

is determined by the map $t \mapsto l_t(s) \in L^2[0, 1]$, where for all t the function l_t is the element associated in $L^2[0, 1]$ to the linear map in $L^2[0, 1]^*$ which sends $(\nabla_s \Phi^i)\, 1_{[t,1]}(s)$ to $\nabla_t \Phi^i$ and which sends the orthogonal complement of the space spanned by the set $\{(\nabla_s \Phi^1)\, 1_{[t,1]}(s), ..., (\nabla_s \Phi^n)\, 1_{[t,1]}(s)\}$ to 0. ∎

Remark. In general $B(s, t)$ does not belong to $L^2[0, 1]^2$.

We introduce a partial covariance matrix with components σ_t^{ij} given by

$$\sigma_t^{ij} = \int_t^1 \nabla_s \Phi^i\, \nabla_s \Phi^j\, ds.$$

We make the following hypothesis.

(H.4) The determinant $\det(\sigma_t^{ij})$ is strictly positive $dt \otimes p^\xi(d\omega)$-almost surely.

In this case the function $B(s, t)$ in the previous lemma has the explicit expression

$$\begin{aligned} B(s, t) &= -\sum_{i,j=1}^{n} (\sigma_t^{-1})_{ij} \nabla_t \Phi^i \nabla_s \Phi^j \\ &= -(\det \sigma_t)^{-1} \left(\sum_{i,j=1}^{n} M_{ij}^t \nabla_t \Phi^i \nabla_t \Phi^j \right) \\ &= -(\det \sigma_t)^{-1} \Xi(s, t). \end{aligned}$$

Here, $(\sigma_{ij}^{-1})^t$ denotes the inverse matrix of σ_t^{ij}, and the components M_{ij}^t are the minors of the matrix σ_t^{ij}. In particular the function $\Xi(s, t) = \sum M_{ij}^t \nabla_s \Phi^i \nabla_t \Phi^j$ belongs to W_∞. We make a final assumption:

(H.5) There exists a family K of functions in $L^2[0, 1]$ such that the following properties hold true.

(1) $s(t)(\det \sigma_t)^{-1}$ belongs to $W_\infty(L^2[0, 1])$ for all s in K;

(2) K is dense in $L^2([0, 1], (\det \sigma_t)^{-1}\, dt)$ for p^ξ-almost all ω;

(3) $K|_{[t,1]} = \{f|_{[t,1]} \mid f \in K \text{ and } f \text{ supported by } [t, 1]\}$ is dense in $L^2([t, 1], dt)$ for all t.

Definition. $\nabla^{v_\xi, \sigma}$ is an unbounded operator on the space $L^p_{L^2([0,1], dt)}(\mathscr{W}, p^\xi(d\omega))$ and with values in the space $L^p_{L^2([0,1]^2, \rho_\omega(t)\, dt \otimes ds)}(\mathscr{W}, p^\xi(d\omega))$. It is defined on $\mathscr{S}^K$ by the identity

$$\begin{aligned} &\nabla_s^{v_\xi, \sigma} \left(\sum_i \phi_i(\lambda_1^i(\omega), ..., \lambda_m^i(\omega))\, v^i(t) \right) \\ &\quad = \sum_{i,j} \frac{\partial \phi_i}{\partial x^j} (\lambda_1^i(\omega), ..., \lambda_m^i(\omega))\, P^s l_j^i(s)\, v_i(t). \end{aligned}$$

Proposition 24. *$\nabla^{v_\xi, \sigma}$ is closable.*

Proof. The family $\mathscr{S}^{L^2, K}$ is dense in $L^p_{L^2([0,1]^2, \rho_\omega(t)\, dt \otimes ds)}$ and the product $Z_\omega(s, t)\, \rho_\omega(t)$ belongs to the domain of δ_s for all $Z_\omega(s, t)$ in $\mathscr{S}^{L^2, K}$. Hence the equality

$$\begin{aligned} &\int_{\mathscr{W}} \int_{[0,1]^2} \nabla_s^{v_\xi, \sigma}(Y_\omega(t))\, Z(s, t)\, \rho_\omega(t)\, ds\, dt\, dp^\xi \\ &\quad = \int_{\mathscr{W}} \int_{[0,1]} Y_\omega(t)\, \delta_s(P^s Z(s, t)\, \rho_\omega(t))\, dt\, dp^\xi \end{aligned}$$

is valid. By the density and the adjointness stated in the last identity the proposition is readily verified. ∎

THEOREM 25. *For a process Y in $\mathcal{S}^K$ and for $p>2$ we denote by $\| Y_t \|_p^\sigma$ the norm*

$$\| Y_t \|_p^\sigma = \left(\int_{\mathscr{W}} \left(\int_{[0,1]^2} (\nabla_s^{v_\xi, \sigma} Y_t)^2 \, ((\det \sigma_t)^{-2} + 1) \, ds \, dt \right)^{p/2} dp^\xi(\omega) \right)^{1/p} + \left(\int (Y_t(\omega))^2 \, d\alpha(\omega, t) \right)^{1/2} + \left(\int \left(\int_{[0,1]} (X_t(\omega))^2 \, dt \right)^{p/2} dp^\xi \right)^{1/p}.$$

The following operators are continuous from $\mathcal{S}^K$ to $L^2[0, 1](p^\xi)$ with respect to this norm.

(1) *Stochastic integration $I(Y) = \int Y_t \, dX_t$ with domain $\mathcal{P}^K$.*

(2) *The divergence with respect to the decomposition $\delta_t^{v_\xi}$ with domain $\mathcal{S}^K$.*

(3) *The operator $A(Y) = \int_{[0,1]^2} A(s, t) \nabla_s^{v_\xi, \sigma} Y_t \, ds \, dt$ with domain $\mathcal{S}^K$.*

(4) *The operator $S(Y) = \int_{[0,1]^2} Y_t(\delta_u N(u, s)) \, N(s, t) \, ds \, dt$ with domain $\mathcal{S}^K$.*

Moreover for all processes in the $\| Y_t \|_p^\sigma$-closure of $\mathcal{P}^K$ the identity

$$I(Y) = \delta_t^{v_\xi} Y - A(Y) + S(Y)$$

is valid.

Proof. The combination of Lemma 21, Lemma 22, and Lemma 23 yields the following identity for all processes Y in $\mathcal{P}^K$:

$$\int Y_t \, dX_t = (\delta_t^{v_\xi} Y_t) + \int_{[0,1]^2} Y_t N(r, t) \, \delta_s N(s, r) \, dt \, dr - \int_{[0,1]^2} \nabla_s Y_y \, A(s, t) \, ds \, dt.$$

Since Y_t is predictable the last term coincides with $A(Y)$ and this proves the theorem for processes in $\mathcal{P}^K$. Showing the continuity of the four operators is sufficient for finishing the proof. The stochastic integration I is continuous with respect to $\int Y_t^2(\omega) \, d\alpha(t, \omega)$ by definition of the dominating measure. The continuity of δ^{v_ξ} is proved in Theorem 20. Since $\int_{[0,1]} \delta_s N(s, r) \, N(r, t) \, dr$ belongs to $\bigcap_q W_{q,0}^\xi(L^2[0, 1])$ the operator S is continuous with respect to the norm $X \mapsto (\int (\int_{[0,1]} (X_t(\omega))^2\, dt)^{p/2}\, dp^\xi)^{1/p}$ for all $p > 2$. Finally $A(Y)$ can be estimated as

$$\begin{aligned}
&\int_{\mathscr{W}} \int_{[0,1]^2} (N(s,t) + (\det \sigma_t)^{-1}\, \Xi(s,t))\, \nabla_s^{v_\xi, \sigma} Y_t\, ds\, dt\, dp^\xi \\
&\quad \leqslant \int_{\mathscr{W}} \Bigg(\int_{[0,1]^2} N(s,t)^2\, ds\, dt \int_{[0,1]^2} (\nabla_s^{v_\xi, \sigma} Y_t)^2\, ds\, dt \\
&\qquad + \int_{[0,1]^2} \Xi(s,t)^2\, ds\, dt \left(\int_{[0,1]^2} (\nabla_s^{v_\xi, \sigma} Y_t)^2 (\det \sigma_t^{-2} + 1)\, ds\, dt \right)^{1/2} \Bigg)\, dp^\xi \\
&\quad \leqslant \int_{\mathscr{W}} \left(\int_{[0,1]^2} N(s,t)^2 + \Xi(s,t)^2\, ds\, dt \right)^{1/2} \\
&\qquad \times \left(\int_{[0,1]^2} (\nabla_s^{v_\xi, \sigma} Y)^2 (\det \sigma_t^{-2} + 1)\, ds\, dt \right)^{1/2} dp^\xi \\
&\quad \leqslant \mathrm{Cte} \left(\int_{\mathscr{W}} \left(\int_{[0,1]^2} (\nabla_s^{v_\xi, \sigma} Y_t)^2 (\det \sigma_t^{-2} + 1)\, ds\, dt \right)^{p/2} dp^\xi \right)^{1/p}.
\end{aligned}$$

Since $\int_{[0,1]^2} (N(s,t)^2 + \Xi(s,t)^2)\, ds\, dt$ belongs to $\bigcap_p L^p(p^\xi)$ the continuity of the operator S is also verified. ∎

9. Connection with Riemannian Geometry and Generalization to Higher Dimensional Tensor Fields

Definition. By $A(s, t, u, v)$ we denote a four-parameter process in $W_\infty(L^2[0,1]^4)$ defined by the identity

$$A(s, t, u, v) = \int_{[0,1]} \nabla_s N(v, r)\, \nabla_t N(r, u)\, dr.$$

The process *Riemann* is defined as the tangent component of the antisymmetric part of A:

$$\begin{aligned}
\mathrm{Riemann}(s, t, u, v) &= P^{s,t,u,v}(A(s,t,u,v) - A(t,s,u,v)) \\
&= P^{s,t,u}(A(s,t,u,v) - A(t,s,u,v)) \\
&= P^{s,t,v}(A(s,t,u,v) - A(t,s,u,v)).
\end{aligned}$$

The equivalence of these definitions, i.e., the fact that one of both projections P^u and P^v is redundant, is verified by applying Lemma 4 twice.

PROPOSITION 26. *The antisymmetric part of the second covariant derivative of a process X_t in $W_\infty(L^2[0, 1])$ is calculated by Riemann,*

$$\nabla_s^\Phi \nabla_t^\Phi X_v - \nabla_t^\Phi \nabla_s^\Phi X_v = \int_{[0,1]} X_u \,\text{Riemann}(s, t, u, v)\, du. \tag{9.1}$$

Proof. It is sufficient to prove formula (9.1) for tangent processes. In this case the derivative $\nabla_t^\Phi X_v$ equals $P^{t,v}\nabla_t X_v$, and this derivative is expanded after replacing $P^{t,v} = P\otimes P = (I-N)\otimes(I-N)$ by $I\otimes I - N\otimes I - I\otimes N + N\otimes N$. This yields

$$\begin{aligned}\nabla_t^\Phi X_v &= P^{t,v}\nabla_t X_v \\ &= \nabla_t X_v - \int_{[0,1]} N(t, r)\nabla_r X_v\, dr - \int_{[0,1]} \nabla_t X_r\, N(r, v)\, dv \\ &\quad + \int_{[0,1]^2} N(t, r)\nabla_r X_x\, N(x, v)\, dx\, dr.\end{aligned}$$

The expansion of the second derivative $\nabla_s^\Phi \nabla_t^\Phi X_v$ is carried out by the product rule and the terms which vanish by composition with $P^{s,t,v}$ are omitted:

$$\begin{aligned}\nabla_s^\Phi \nabla_t^\Phi X_v &= P^{s,t,v}\nabla_s P^{t,v}\nabla_t X_v \\ &= P^{s,t,v}\nabla_s\nabla_t X_v - P^{s,t,v}\int_{[0,1]} \nabla_s N(t, r)\nabla_r X_v\, dr \\ &\quad - P^{s,t,v}\int_{[0,1]} \nabla_t X_r \nabla_s N(r, v)\, dr.\end{aligned}$$

The second term is rewritten using the second fundamental form:

$$-P^v\int_{[0,1]} \nabla_r X_v\, S_r(s, t)\, dr = -P^{s,t,v}\int_{[0,1]} \nabla_s N(t, r)\nabla_r X_v\, dr.$$

By (5.6) and Lemma 4 the third term is transformed to

$$\begin{aligned}&-P^{s,t}\int_{[0,1]} \nabla_t X_r P^v \nabla_s N(r, v)\, dr \\ &\quad = -P^{s,t}\int_{[0,1]^2} \nabla_t X_u\, N(u, r)\nabla_s N(r, v)\, dr\, du \\ &\quad = P^{s,t}\int_{[0,1]} X_u \int_{[0,1]} \nabla_t N(u, r)\nabla_s N(r, v)\, dr\, du.\end{aligned}$$

Finally this yields the expression

$$\nabla_s^{\Phi}\nabla_t^{\Phi}X_v = P^{s,t,v}\nabla_s\nabla_t X_v - P^{s,t,v}\int_{[0,1]}\nabla_r X_v\, S_r(s,t)\,dr$$
$$+ P^{s,t}\int_{[0,1]} X_u\, A(s,t,u,v)\,dr.$$

The terms involving $\nabla_s\nabla_t X_v$ and $S_r(s,t)$ vanish in the antisymmetric component, and this proves (9.1). ∎

PROPOSITION 27. *The process Riemann fulfills the following symmetries.*

$$\text{Riemann}(s,t,u,v) = -\text{Riemann}(t,s,u,v) = -\text{Riemann}(s,t,v,u)$$
$$= \text{Riemann}(t,s,v,u); \tag{9.2}$$

$$\text{Riemann}(s,t,u,v) + \text{Riemann}(t,u,s,v) + \text{Riemann}(u,s,t,v) = 0; \tag{9.3}$$

$$\nabla_w \text{Riemann}(s,t,u,v) + \nabla_s \text{Riemann}(t,w,u,v)$$
$$+ \nabla_t \text{Riemann}(w,s,u,v) = 0. \tag{9.4}$$

Proof. The equalities in (9.2) follow immediately from the definition. The sum in (9.3) equals the antisymmetric component of the map $(s,t,u) \mapsto P^{s,t,u,v}A(s,t,u,v)$. Since the equality

$$P^{s,t,u,v}A(s,t,u,v) = P^s\int_{[0,1]} S_r(s,v)\,S_r(u,t)\,dr$$

is valid, it is seen that this map is symmetric in u and t, and hence the antisymmetric component vanishes. Along the same lines (9.4) is proved. The sum is the antisymmetric component of the map $(w,s,t) \mapsto \nabla_w^{\Phi}P^{s,t,u}A(s,t,u,v)$, and the projection $P^{s,t,u}$ is written as $(I-N)^{\otimes 3}$ to find the following expansion for the derivative:

$$\nabla_w^{\Phi}P^{s,t,u}\int_{[0,1]}\nabla_s N(v,r)\,\nabla_t N(r,u)\,dr$$
$$= P^{s,t,u,v,w}\int_{[0,1]}\nabla_w\nabla_s N(v,r)\,\nabla_t N(r,u)\,dr$$
$$+ P^{s,t,u,v,w}\int_{[0,1]}\nabla_s N(v,r)\,\nabla_w\nabla_t N(r,u)\,dr$$
$$- P^{s,t,u,v,w}\int_{[0,1]}\int_{[0,1]}\nabla_w N(s,x)\,\nabla_x N(v,r)\,dx\,\nabla_t N(r,u)\,dr$$
$$- P^{s,t,u,v,w}\int_{[0,1]}\nabla_s N(v,r)\int_{[0,1]}\nabla_x N(r,u)\,\nabla_w N(x,t)\,dx\,dr$$
$$- P^{s,t,u,v,w}\int_{[0,1]}\nabla_s N(v,r)\int_{[0,1]}\nabla_t N(r,x)\,\nabla_w N(x,u)\,dx\,dr.$$

The antisymmetric component of the first and the third term vanish by symmetry in w and s, and the same is true for the second and the fourth term by symmetry in w and t. The last term is identically zero, as is found by repeated use of Lemma 4:

$$P^{s,t,u,v,w}\int_{[0,1]^2}\nabla_s N(v,r)\,\nabla_t N(r,x)\,\nabla_w N(x,u)\,dx\,dr$$

$$=P^{s,t,u,w}\int_{[0,1]^3}\nabla_s N(v,y)\,N(y,r)\,\nabla_t N(r,x)\,\nabla_w N(x,u)\,dx\,dr\,dy$$

$$=P^{s,t,u,w}\int_{[0,1]^3}\nabla_s N(v,r)\,\nabla_t N(r,x)\,P^x\,\nabla_w N(x,u)\,dx\,dr\,dy$$

$$=P^{s,t,u,w}\int_{[0,1]^3}\nabla_s N(v,r)\,\nabla_t N(r,x)\,\nabla_w N(x,y)\,N(y,u)\,dx\,dr=0. \qquad\blacksquare$$

Suppose that Z^k is an increasing sequence of finite dimensional subspaces of $L^2[0,1]$ spanned by the functions $l_1(s), ..., l_{n_k}(s)$ and such that the associated linear functionals $\lambda_1(\omega), ..., \lambda_{n_k}(\omega)$ are continuous on $\mathscr{W}$. Then the orthogonal projection in $L^2[0,1]$ with kernel $P_k(s,t)=\sum_{i=1}^{n_k} l_i(s)\,l_i(t)$ has a continuous extension to an operator from $\mathscr{W}$ to Z^k given by the formula

$$P_{Z^k}(\omega)=\sum_{i=1}^{n_k}\lambda(\omega)\,l_i(s).$$

We denote by η_k the logarithm of the density of the Gaussian measure with respect to the Lebesgue measure on Z^k: $\eta_k=\frac{1}{2}\sum_{i=1}^{n_k}\lambda^2(\omega)=\frac{1}{2}\,\|P_{Z^k}(\omega)\|^2_{L^2[0,1]}$.

LEMMA 28. *A process $X(t_1, ..., t_k)$ in $W_\infty(L^2[0,1]^k)$ can be approximated as*

$$\lim_{k\to\infty}\int_{[0,1]}P_k(s,t_1)\,X(s,t_2,...,t_n)\,ds=X(t_1,...,t_n).$$

LEMMA 29. *If a process has a representation as a finite sum $Y=\sum_i Y^i l_i(t)$ where the Y^i are stochastic variables in $W_{2,1}(\mathbb{R})$ and the l_i are functions in $L^2[0,1]$ associated to λ_i in $\mathscr{W}^*$, then the divergence is given by the formula*

$$\delta_t\Big(\sum_i Y^i l_i(t)\Big)=-\sum_i\Big(\int_{[0,1]}\nabla_t Y^i\,l_i(t)\,dt+Y^i\lambda_i\Big).$$

Proof. This formula is the definition of the divergence if the functions Y^i belong to $\mathscr{S}_0$. The extension to functions in $W_{2,1}(\mathbb{R})$ is made by a limiting argument. ∎

THEOREM 30. *The 2-parameter process Ricci is a limit in W_∞ in the following way*:

$$\lim_{k\to\infty}\left(\int_{[0,1]^2} P_k(t,v)\,\text{Riemann}(s,t,u,v)\,dt\,dv - \nabla_s^\Phi\nabla_u^\Phi\eta_k + P^{s,u}P_k(s,u)\right) = -\text{Ricci}(s,u). \tag{9.5}$$

Proof. The operator Riemann is the difference of two processes. The first one is treated by applying Lemma 28:

$$\begin{aligned}
&\lim_{k\to\infty}\int_{[0,1]^2} P_k(t,v)\,P^{s,t,u,v}A(s,t,u,v)\,dt\,dv\\
&\quad=\int_{[0,1]^2}\lim_{k\to\infty}\int_{[0,1]} P_k(t,v)\,P^{s,v}\nabla_s N(v,r)\,dv\,P^{t,u}\nabla_t N(r,u)\,dt\,dr\\
&\quad=\int_{[0,1]^2} P^{s,t}\nabla_s N(t,r)\,P^{t,u}\nabla_t N(r,u)\,dt\,dr\\
&\quad=P^{s,u}\int_{[0,1]^2} S_r(s,t)\,S_r(u,t)\,dt\,dr.
\end{aligned}$$

This last expression appears in the definition (5.11) of the process Ricci. By definition (1.2) the first derivative of η_k is equal to

$$\nabla_u\eta_k=\sum_{i=1}^{n_k}\lambda_i(\omega)\,l_i(u)$$

and the second derivative is given by

$$\nabla_s\nabla_u\eta_k=\sum_{i=1}^{n_k} l_i(s)\,l_i(u).$$

Since $\nabla_s^\Phi\nabla_u^\Phi\eta_k$ can be found from $\nabla_s\nabla_u\eta_k$ from the relation

$$\nabla_s^\Phi\nabla_u^\Phi\eta_k=P^{s,u}\left(\nabla_s\nabla_u\eta_k-\int_{[0,1]}\nabla_r\eta_k\,S_r(s,u)\,dr\right),$$

the remaining terms appearing in the left hand side of formula (9.5) can be rewritten as

$$\begin{aligned}
&\nabla_s^{\Phi}\nabla_u^{\Phi}\eta_k - P^{s,u}P_k(s,u)\\
&\quad= -\sum_{i=1}^{n_k}\int_{[0,1]} l_i(r)\, S_r(s,u)\, dr\, \lambda_i(\omega)\\
&\quad= -\int_{[0,1]}\sum_{i=1}^{n_k}\int_{[0,1]} l_i(v)\, N(v,r)\, dv\, \lambda_i(\omega)\, S_r(s,u)\, dr. \qquad (9.6)
\end{aligned}$$

This expression will be combined with the term in (9.5) due to the second component of the process Riemann. Firstly, this second component is replaced by a more useful expression. This replacement operation is valid, because the limit of the difference vanishes, as is shown by the calculation

$$\begin{aligned}
&\lim_{k\to\infty}\Bigg(-\int_{[0,1]^2} P_k(t,v)\, P^{t,s,u}A(s,t,u,v)\, dt\, dv\\
&\quad+\int_{[0,1]}\sum_{i=1}^{n_k}\int_{[0,1]}\nabla_t\left(\int_{[0,1]} l_i(v)\, N(v,r)\, dv\right) l_i(t)\, dt\, S_r(s,u)\, dr\Bigg)\\
&=\lim_{k\to\infty}\Bigg(-\int_{[0,1]^2} P_k(t,v)\int_{[0,1]} P^t\,\nabla_t N(v,r)\, P^{s,u}\,\nabla_s N(r,u)\, dr\, dt\, dv\\
&\quad+\int_{[0,1]}\int_{[0,1]}\sum_{i=1}^{n_k} l_i(t)\, l_i(v)\,\nabla_t N(v,r)\, dt\, dv\, S_r(s,u)\, dr\Bigg)\\
&=\lim_{k\to\infty}\int_{[0,1]}\int_{[0,1]^2} P_k(t,v)(-P^t\,\nabla_t N(v,r)+\nabla_t N(v,r))\, dt\, dv\, S_r(s,u)\, dr\\
&=\int_{[0,1]}\int_{[0,1]^2}\lim_{k\to\infty}\int_{[0,1]} P_k(x,v)\, N(t,x)\, dx\,\nabla_t N(v,r)\, dt\, dv\, S_r(s,u)\, dr.
\end{aligned}$$

(9.7)

By lemma 28 the limit $\lim_{k\to\infty}\int_{[0,1]} P_k(x,v)\, N(t,x)\, dx$ is equal to the projection $N(t,v)$:

$$\int_{[0,1]}\int_{[0,1]^2} N(t,v)\,\nabla_t N(v,r)\, dt\, dv\, S_r(s,u)\, dr.$$

A redundant projection P^r can be introduced by Lemma 4:

$$\int_{[0,1]} P^r \int_{[0,1]^2} N(t,v)\,\nabla_t N(v,r)\, dt\, dv\, S_r(s,u)\, dr = 0.$$

P^r projects the second fundamental form to zero, indeed $P^r S_r(s, u) = 0$. The results (9.6) and (9.7) are added and this yields

$$\lim_{k \to \infty} \left(- \int_{[0,1]} \sum_{i=1}^{n_k} \int_{[0,1]} \nabla_t \left(\int_{[0,1]^2} l_i(v)\, N(v, r)\, dv \right) l_i(t)\, dt\, S_r(s, u)\, dr \right.$$
$$\left. + \int_{[0,1]} \sum_{i=1}^{n_k} \left(\int_{[0,1]} l_i(v)\, N(v, r)\, dv \right) \lambda_i(\omega)\, S_r(s, u)\, dr \right)$$
$$= \lim_{k \to \infty} \int_{[0,1]} \sum_{i=1}^{n_k} \left(- \int_{[0,1]} \nabla_t \left(\int_{[0,1]} l_i(v)\, N(v, r)\, dv \right) l_i(v)\, dt \right.$$
$$\left. + \int_{[0,1]} l_i(v)\, N(v, r)\, dv\, \lambda_i \right) S_r(s, u)\, dr.$$

The sum indexed from 1 to n_k in this expression equals a divergence by Lemma 29

$$\int_{[0,1]} - \lim_{k \to \infty} \delta_t \left(\sum_{i=1}^{n_k} \int_{[0,1]} l_i(v)\, N(v, r)\, dv\, l_i(t) \right) S_r(s, u)\, dr.$$

By continuity the limit and the divergence can be switched. This yields

$$\int_{[0,1]} \delta_t \lim_{k \to \infty} \left(\int_{[0,1]} P_k(t, v)\, N(v, r) \right) S_r(s, u)\, dr.$$

The limit is given by Lemma 29:

$$\int_{[0,1]} \delta_t N(t, r)\, S_r(s, u)\, dr.$$

This expression is the second term appearing in definition (5.11). ∎

In the expression $P_k(t, v)$ Riemann$(s, t, u, v)\, dt\, dv - \nabla_s^{\Phi} \nabla_u^{\Phi} \eta_k + P^{s,u} P_k(s, u)$ the term $-\nabla_s^{\Phi} \nabla_u^{\Phi} \eta_k$ is a renormalization needed to make the limit convergent. By this theorem the generalized Stroock–Shigekawa commutator formula is related to the following well-known formula for finite dimensional Riemannian manifolds (cf. §4.36 in [9]):

$$\nabla \delta X - \delta_2 \nabla X = \mathrm{Ricci}(X) - \langle \nabla\nabla \eta, X \rangle_{TM}. \tag{9.8}$$

Here ∇ denotes the Levi-Civita derivative on a finite dimensional Riemannian manifold and δ is the adjoint of the operator ∇ with respect to a measure with density e^{η} with respect to the natural measure on the Riemannian manifold. The correction

$$\int_{[0,1]} \mathrm{Ricci}(s, t)\, X_s\, ds + P^s X_s$$

in formula (5.10) corresponds to the right hand side of (9.8). In the infinite dimensional case it is impossible to dissociate the correction involving the Ricci curvature from the other term. Indeed the previous theorem shows that Ricci also contains a part corresponding to $\langle \nabla\nabla\eta, X\rangle_{TM}$.

For higher dimensional tensors (multi-parameter processes) a generalization which is analogous to the generalized Bochner formula can be proved. In our notation this reads as follows:

THEOREM 31. *For all k-parameter processes $X(t_1, ..., t_k)$ in $W_\infty(L^2[0, 1]^k)$ the equality*

$$\nabla_s^\Phi \delta_{t_1}^\Phi X(t_1, ..., t_k) - \delta_{t_1}^\Phi \nabla_s^\Phi X(t_1, ..., t_k)$$
$$= \int_{[0,1]} \text{Ricci}(s, t_1)\, X(t_1, ..., t_k)\, dt_1$$
$$+ \sum_{i=2}^{k} \int_{[0,1]^2} \text{Riemann}(t_1, s, t_2, u)\, X(s, t_1, ..., t_i, u, t_{i+1}, ..., t_k)\, ds\, du$$

is valid.

Proof. The proof is essentially similar to that of Theorem 11. It is carried out in full detail for 3-parameter processes in [25]. ∎

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REFERENCES

1. H. AIRAULT, Differential calculus on finite codimensional submanifolds of the Wiener space, *J. Funct. Anal.* **100** (1991), 291–316.
2. H. AIRAULT AND P. MALLIAVIN, Intégration géométric sur l'espace de Wiener, *Bull. Sci. Math. (2)* **112** (1988), 3–52.
3. H. AIRAULT AND P. MALLIAVIN, Integration on loop group. 2. Heat equation for the Wiener measure, *J. Funct. Anal.* **104** (1992), 71–109.
4. H. AIRAULT AND J. VAN BIESEN, Le processus d'Ornstein Ühlenbeck sur une sous-variété de l'espace de Wiener, *Bull. Sci. Math. (2)* **115** (1991), 185–210.
5. H. AIRAULT AND J. VAN BIESEN, Géométrie riemannienne en codimension finie sur l'espace de Wiener, *C.R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 125–130.
6. B. K. DRIVER, A Cameron–Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold, *J. Funct. Anal.* **110** (1992), 272–376.
7. B. K. DRIVER, A Cameron–Martin type quasi-invariance theorem for pinned Brownian motion on a compact Riemannian manifold, *Trans. Amer. Math. Soc.*, in press.

8. B. K Driver and M. Röckner, Construction of diffusions on path and loop spaces of compact Riemannian manifolds, *C.R. Acad. Sci. Paris Serie I Math.* **315**, No. 5 (1992), 603–608.
9. S. Gallot, D. Hulin, and J. Lafontaine, "Riemannian Geometry," Springer-Verlag, Berlin/Heidelberg, 1987.
10. E. Getzler, Dirichlet forms on loop space, *Bull. Sci. Math. (2)* **113** (1989), 151–174.
11. N. Ikeda and S. Watanabe, An introduction to Malliavin's calculus, *in* "Stochastic Analysis" (K. Ito, Ed.), pp. 1–52, Proceedings of the Taniguchi Symposium (Katata and Kyoto, 1982), Kinokuniya, Tokyo, 1984.
12. M. Kree and P. Kree, Continuité de la divergence dans les espaces de Sobolev relatif à l'espace de Wiener, *C.R. Acad. Sci. Paris Sér. I Math.* **296** (1983), 833–836.
13. S. Kusuoka and D. W. Stroock, Applications of the Malliavin calculus, Part I, *in* "Stochastic Analysis" (K. Ito, Ed.), pp. 271–306, Proceedings of the Taniguchi Symposium (Katata, Kyoto 1982), Kinokuniya, Kyoto, 1984.
14. P. Malliavin, Implicit functions in finite corank on the Wiener space, *in* "Stochastic Analysis" (K. Ito, Ed.), pp. 369–386, Proceedings of the Taniguchi Symposium (Katata and Kyoto, 1982), Kinokuniya, Tokyo, 1984.
15. P. Malliavin, Differential analysis in stochastic calculus, course, MIT.
16. P. Malliavin, Calcul des variations, intégrales stochastiques et complexe de Rham sur l'espace de Wiener, *C.R. Acad. Sci. Paris Sér. I Math.* **299** (1984), 347–350.
17. P. Malliavin, Integration on loop groups. I. Quasi invariant measures, *J. Funct. Anal.* **93** (1990), 207–237.
18. P. Malliavin, Integration on loop group. 2. Asymptotic Peter–Weyl orthogonality, *J. Funct. Anal.* **108** (1992), 13–46.
19. P. Malliavin, Stochastic calculus of variations and hypoelliptic operators, *in* "Proceedings, Intern. Conf. on Stochastic Differential Equations at Kyoto, 1976," pp. 195–263, Kinokuniya/Wiley, Tokyo/New York, 1978.
20. P. A. Meyer, Quelques résultats analytiques sur le semigroupe d'Ornstein–Ühlenbeck en dimension infinie, *in* "Theory and Applications of Random Fields" (G. Kallianpur, Ed.), Proc. IFIP–WG 7/1, Working Conference at Bangalore, Lecture Notes in Control and Inform. Sci., Vol. 49, Springer-Verlag, Berlin, 1983.
21. D. Nualart and E. Pardoux, Stochastic calculus with anticipating integrands, *Probab. Theory Related Fields* **78** (1988), 535–581.
22. I. Shigekawa, De Rham–Hodge–Kodaira's decomposition on an abstract Wiener space, *J. Math. Kyoto Univ.* **26**, No. 2 (1986), 191–202.
23. H. Sugita, Sobolev spaces of Wiener functionals and Malliavin calculus, *J. Math. Kyoto Univ.* **25** (1985), 31–48.
24. D. W. Stroock, The Malliavin's calculus, *Math. Systems Theory* **14** (1981), 25–65.
25. J. Van Biesen, Intrinsic conditions for the existence of the divergence on submanifolds of the Wiener space, research report.