Bounds for present value functions with stochastic interest rates and stochastic volatility

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Abstract

The distribution of the present value of a series of cash flows under stochastic interest rates has been investigated by many researchers. One of the main problems in this context is the fact that the calculation of exact analytical results for this type of distributions turns out to be rather complicated, and is known only for special cases. An interesting solution to this difficulty consists of determining computable upper bounds, as close as possible to the real distribution.

In the present contribution, we want to show how it is possible to compute such bounds for the present value of cash flows when not only the interest rates but also volatilities are stochastic. We derive results for the stop loss premium and distribution of these bounds.

1 Introduction

When investigating sums of dependent variables, one of the main problems that arise is the fact that due to the dependencies it is almost impossible to find the real distribution of such a sum. In some recent papers, we suggested to solve this problem by calculating upper bounds. Using the concept of

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comonotonicity, we are able to determine bounds in convexity order that are rather close to the original variable, and much easier to compute. For the meaning and consequences of this approach, we refer to section 2.

One of the applications of this kind of problems is the investigation of the present value of a series of non-negative payments at times 1 up to n

$$A = \sum_{t=1}^{n} \alpha_t e^{-Y_1 - Y_2 - ... - Y_t}, \tag{1}$$

where Y_t represents the stochastic continuous compounded rate of return over the period $[t-1, t]$ (see also [4]).

In the classical assumption, prices are log-normally distributed, and thus the variables Y_t are independent and normally distributed. In other words,

$$Y_t \sim N\left(\mu_t, \sigma_t^2\right) \tag{2}$$

where μ_t and σ_t are constants.

In the present contribution, we will generalize this classical assumption by replacing the constant σ_t by a random variable $\tilde{\sigma}_t$, where we assume that the volatilities $\tilde{\sigma}_t$ for the periods $[t-1, t]$ are mutually independent variables. For any realization σ_t we then have that

$$Y_t \mid \tilde{\sigma}_t = \sigma_t \sim N\left(\mu_t, \sigma_t^2\right). \tag{3}$$

This idea has been borrowed from [6].

In correspondence with the financial paradigma, in equation (1) we should correct the variables Y_t by means of their volatility, or

$$\begin{aligned} A &= \sum_{t=1}^{n} \alpha_t e^{-(Y_1 - \frac{1}{2}\tilde{\sigma}_1^2) - (Y_2 - \frac{1}{2}\tilde{\sigma}_2^2) - ... - (Y_t - \frac{1}{2}\tilde{\sigma}_t^2)} && (4) \\ &= \sum_{t=1}^{n} \alpha_t e^{-Y(t) + \frac{1}{2}\Sigma(t)}, && (5) \end{aligned}$$

where $Y(t) = Y_1 + Y_2 + ... + Y_t$ is used to denote the total compounded rate of return over the period $[0, t]$, and where $\Sigma(t)$ is defined as $\Sigma(t) = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + ... + \tilde{\sigma}_t^2$. The reason for this change by means of the volatility as suggested in equations (4) and (5) has to be found in the fact that with this adaptation, for the (new) accumulated values we then have the identity

$$E\left[e^{\left(Y_t - \frac{1}{2}\tilde{\sigma}_t^2\right)}\right] \cdot e^{-\mu_t} = 1. \tag{6}$$

Note that for the variable $Y(t)$ we have the obvious (conditional) moments

$$E[Y(t)|\tilde{\sigma}_1, ..., \tilde{\sigma}_t] = \mu_1 + ... + \mu_t \tag{7}$$

$$Var[Y(t)|\tilde{\sigma}_1, ..., \tilde{\sigma}_t] = \tilde{\sigma}_1^2 + ... + \tilde{\sigma}_t^2 = \Sigma(t). \tag{8}$$

For the distributions of the variables $Y(t)$ and $\Sigma(t)$, we will use the notations $F_t(x)$ and $G_t(x)$, or

$$F_t(x) = Prob[Y(t) \leq x] \qquad f_t(x) = \frac{d}{dx}F_t(x) \tag{9}$$

and

$$G_t(x) = Prob[\Sigma(t) \leq x] \qquad g_t(x) = \frac{d}{dx}G_t(x). \tag{10}$$

Since we already fixed the model for $Y(t)$, the function $F_t(x)$ is known. For the calculation of $G_t(x)$, we need to specify a model for the stochastic volatilities.

In order to study the distribution of the present value (5), we will use recent results concerning bounds for sums of stochastic variables. In the following section, we will explain the methodology we used for finding the desired answers. We will briefly repeat the most important results. Section 3 contains an expression for the function $G_t(x)$ for a few volatility models. The concrete boundary results for the quantity A of equation (5) are presented in section 4 and 5. Finally in section 6, we will give some numerical illustrations.

2 Methodology

2.1. Looking at the structure of the variable A in (5), we see that this quantity belongs to the class of variables

$$A = \sum_{t=1}^{n} \phi_t(Y(t), \Sigma(t)). \tag{11}$$

For the present problem the functions $\phi_t : \Re^2 \to \Re : (x, s) \mapsto \phi_t(x, s)$ are mainly exponential.

Even in case the distributions of the random variables $Y(t)$ and $\Sigma(t)$ are known, the calculation of the distribution function for random variables in this form is far from self-evident. The most important difficulty arises from the fact that neither the random variables $Y(t)$ nor the variables $\Sigma(t)$ are mutually independent. A “simple” convolution of the different individual distribution functions thus is not correct, since also the dependency structures of the random vectors $(Y(1), ..., Y(n))$ and $(\Sigma(1), ..., \Sigma(n))$ have to be taken into account. And this, unfortunately, is almost impossible to obtain in most cases.

Instead of calculating the exact distribution of the variable A, we therefore will look for bounds, in the sense of “less favourable / more dangerous” variables, with a simpler structure and as close as possible to the original variable. We briefly repeat the meaning and most important results of this technique. For proofs and more details, we refer to recent publications e.g. [1, 2, 4].

2.2. The notion “less favourable” or “more dangerous” variable can be formalized by means of the convex ordering, see [5], with the following definition :

Definition 2.1 *If two random variables V and W are such that for each convex function $u : \Re \to \Re : x \mapsto u(x)$ the expected values (provided they exist) are ordered as*

$$E[u(V)] \leq E[u(W)], \tag{12}$$

the variable V is said to be smaller in convex ordering than a variable W, which is denoted as

$$V \leq_{cx} W. \tag{13}$$

Since convex functions are functions that take on their largest values in the tails, this means that the variable W is more likely to take on extreme values than the variable V, and thus it can be considered to be more dangerous.

Condition (12) on the expectations can be rewritten as

$$E[u(-V)] \geq E[u(-W)] \tag{14}$$

for arbitrary concave utility functions $u : \Re \to \Re : x \mapsto u(x)$. Thus, for any risk averse decision maker, the expected utility of the loss W is smaller than the expected utility of the loss V. This means that replacing the unknown distribution function of the variable V by the distribution function of the variable W is a prudent stategy.

The functions $u(x) = x$, $u(x) = -x$ and $u(x) = x^2$ are all convex functions, and thus it follows immediately that $V \leq_{cx} W$ implies $E[V] = E[W]$ as well as $Var[V] \leq Var[W]$.

An equivalent characterisation of convex order is formulated in the following lemma, a proof of which can be found in [5] :

Lemma 2.1 *If two variables V and W are such that $E[V] = E[W]$, then*

$$V \leq_{cx} W \Leftrightarrow E[(V-k)_+] \leq E[(W-k)_+] \;\; \textit{for all} \;\; k, \tag{15}$$

with $(x)_+ = max(0, x)$.

Since more dangerous risks will correspond to higher (so-called) stop-loss premiums $E[(V-k)_+]$, again it can be seen that the notion of convex order is very adequate to describe an ordering in dangerousness. Indeed, $E[(V-k)_+]$ denotes the expected loss (in financial terms) of realizations exceeding k.

2.3. The notion of convex ordering can be extended from two single variables to two sums of variables, as is proved in [1, 2, 4]. In the following results, we use the notation

$$F_X(x) = Prob(X \leq x) \tag{16}$$

for the distribution of a random variable X, where $x \in \Re$, and

$$F_X^{-1}(p) = \inf\{x \in \Re : F_X(x) \geq p\} \tag{17}$$

for the inverse distribution of X, where $p \in [0, 1]$.

We will start by presenting bounds in convexity for 'ordinary' sums of variables, and continue with bounds for sums of functions of variables.

Proposition 2.1 *Consider an arbitrary sum of random variables*

$$V = X_1 + ... + X_n, \tag{18}$$

and define the related stochastic quantities

$$V_{upp} = F_{X_1}^{-1}(U) + ... + F_{X_n}^{-1}(U) \tag{19}$$

$$V_{upp*} = F_{X_1|Z}^{-1}(U) + ... + F_{X_n|Z}^{-1}(U) \ , \tag{20}$$

with U an arbitrary random variable that is uniformly distributed on $[0,1]$, and with Z an arbitrary random variable that is independent of U.

We then have

$$V \leq_{cx} V_{upp*} \leq_{cx} V_{upp} \tag{21}$$

and thus the stop-loss premiums satisfy the relation

$$E[(V-k)_+] \leq_{cx} E[(V_{upp*}-k)_+] \leq_{cx} E[(V_{upp}-k)_+]. \tag{22}$$

The corresponding terms in the original variable V and in the upper bounds V_{upp} and V_{upp*} are all mutually identically distributed, or

$$X_j \stackrel{d}{=} F_{X_j}^{-1}(U) \stackrel{d}{=} F_{X_j|Z}^{-1}(U) \ . \tag{23}$$

In fact, by construction the upper bound V_{upp} is the most dangerous combination of variables with the same marginal distributions as the original terms X_j in V. Indeed, the sum now consists of a sum of comonotonous variables all depending on the same stochastic U, and thus not usable as hedges against each other. The upper bound V_{upp*} is an improved bound, which is closer to V due to the extra information through conditioning.

The second proposition extends the previous results from ordinary sums of variables to sums of functions of variables.

Proposition 2.2 *Consider a sum of functions of random variables*

$$V = \phi_1(X_1) + ... + \phi_n(X_n). \tag{24}$$

For an arbitrary random variable U that is uniformly distributed on $[0,1]$, and an arbitrary random variable Z which is independent of U, define the related stochastic quantities

$$V_{upp} = \phi_1(F_{X_1}^{-1}(U)) + ... + \phi_n(F_{X_n}^{-1}(U)) \tag{25}$$

$$V_{upp*} = \phi_1(F_{X_1|Z}^{-1}(U)) + ... + \phi_n(F_{X_n|Z}^{-1}(U)) \tag{26}$$

in case each function $\phi_t : \Re \to \Re : x \mapsto \phi_t(x)$ *is* <u>*in*</u>*creasing, and*

$$V_{upp} = \phi_1(F_{X_1}^{-1}(1-U)) + ... + \phi_n(F_{X_n}^{-1}(1-U)) \tag{27}$$

$$V_{upp*} = \phi_1(F_{X_1|Z}^{-1}(1-U)) + ... + \phi_n(F_{X_n|Z}^{-1}(1-U)) \tag{28}$$

in case each function $\phi_t : \Re \to \Re : x \mapsto \phi_t(x)$ *is* <u>*de*</u>*creasing.*

We then have

$$V \leq_{cx} V_{upp*} \leq_{cx} V_{upp} \tag{29}$$

and thus also

$$E[(V-k)_+] \leq_{cx} E[(V_{upp*}-k)_+] \leq_{cx} E[(V_{upp}-k)_+]. \tag{30}$$

Both results are mainly based on the first proposition, combined with the property that for any increasing function ϕ and for any $p \in [0,1]$ it is true that

$$F_{\phi(X)}^{-1}(p) = \phi(F_X^{-1}(p)), \tag{31}$$

and that for any decreasing function ϕ and for any $p \in [0,1]$ we have the equality

$$F_{\phi(X)}^{-1}(p) = \phi(F_X^{-1}(1-p)). \tag{32}$$

Finally, once the boundary values for the investigated quantity and their stop-loss premiums are found, the distribution function follows immediately when use is made of lemma 2.2.

Lemma 2.2 *Consider an arbitrary variable A with distribution function*

$$F_A(k) = Prob[A \leq k] \; . \tag{33}$$

Provided the expectations exist, the relation between stop-loss premiums and distribution function is given by

$$\frac{d}{dk} E\left[(A-k)_+\right] = F_A(k) - 1 \; . \tag{34}$$

3 Distribution of $\Sigma(t)$

For the numerical illustration, we will need to know the concrete distribution of $\Sigma(t)$, which can be calculated if a model for the distribution of the stochastic volatilities $\tilde{\sigma}_t$ is specified.
As mentioned before, we assume the volatilities to be all independent and identically distributed. We suggest the following two models :

- an exponential distribution for $\tilde{\sigma}_t^2$, or

$$\tilde{\sigma}_t^2 \sim exp\,(\alpha)\,, \tag{35}$$

  where α is chosen large enough to minimize the chance of too large and unrealistic values for $\tilde{\sigma}_t^2$;

- a normal distribution for $\tilde{\sigma}_t$, or

$$\tilde{\sigma}_t \sim N\left(\sigma, \xi^2\right), \tag{36}$$

  where again ξ is chosen small enough to minimize the risk of negative values for $\tilde{\sigma}_t$.

The results for the distribution $G_t(x)$ are formulated in the following lemmas.

Lemma 3.1 *Define $\Sigma(t)$ as the sum $\Sigma(t) = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + ... + \tilde{\sigma}_t^2$, with the variables $\tilde{\sigma}_j^2$ independent and identically exponentially distributed,*

$$\tilde{\sigma}_j^2 \sim exp\,(\alpha)\,. \tag{37}$$

Then the distribution of $\Sigma(t)$ can be written as

$$G_t(x) = 1 - e^{-\alpha x} \sum_{k=0}^{t-1} \frac{(\alpha x)^k}{k!} = 1 - \frac{\Gamma(t, \alpha x)}{\Gamma(t)} \tag{38}$$

with $\Gamma(t,z) = \int_z^{+\infty} y^{t-1} e^{-y}\, dy$ the incomplete Gamma-function.

Proof. Trivial.

Lemma 3.2 *Define* $\Sigma(t)$ *as the sum* $\Sigma(t) = \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + ... + \tilde{\sigma}_t^2$ *, with the variables* $\tilde{\sigma}_j$ *independent and identically normally distributed,*

$$\tilde{\sigma}_j \sim N\left(\sigma, \xi^2\right) \ . \tag{39}$$

Then the distribution of $\Sigma(t)$ *is a convolution of*

(i) *a Gamma distribution with parameters* $\alpha = \frac{t}{2}$ *and* $\beta = 2\xi^2$*, and*

(ii) *a compound Poisson distribution with parameter* $\lambda = \frac{t\sigma^2}{2\xi^2}$ *and with claim size exponentially distributed with parameter* $2\xi^2$*.*

For the probability density, we have

$$g_t(x) = \frac{1}{2\xi^2} e^{-\frac{t\sigma^2}{2\xi^2} - \frac{x}{2\xi^2}} \left(\frac{x}{t\sigma^2}\right)^{t/4-1/2} I_{t/2-1}\left(\frac{\sqrt{xt\sigma^2}}{\xi^2}\right) \ . \tag{40}$$

Proof. We start by calculating the Laplace transform $L_t(u)$ of $g_t(x)$. A straightforward calculation gives

$$\begin{aligned} E\left[e^{-u\,\tilde{\sigma}_1^2}\right] &= \int_{-\infty}^{+\infty} e^{-u\,s^2}\, d\Phi\left(\frac{s-\sigma}{\xi}\right) \\ &= \frac{1}{\sqrt{1+2u\xi^2}} \exp\left\{\frac{\sigma^2}{2\xi^2}\left(\frac{1}{1+2u\xi^2} - 1\right)\right\}, \end{aligned} \tag{41}$$

and thus

$$L_t(u) = \left(\frac{1}{1+2u\xi^2}\right)^{t/2} \exp\left\{\frac{t\sigma^2}{2\xi^2}\left(\frac{1}{1+2u\xi^2} - 1\right)\right\}, \tag{42}$$

which proves the convolution.

Next, in order to find the denstiy function, we work out the Laplace inversion. At this stage, use can be made of the integral identity

$$\begin{aligned} &\int_0^{+\infty} e^{-ux} x^\beta I_{2\beta}\left(2\sqrt{\lambda x}\right) \\ &\qquad = \lambda^\beta u^{-1-2\beta} e^{\lambda/\beta}. \end{aligned} \tag{43}$$

A few transformations now lead to expression (40).

Q.E.D.

Combining the methods as described in section 2 with these distributional results, we will be able to calculate the bounds for the present value of a series of payments with stochastic interest rates and with stochastic volatility. Where needed, we will use the classical notation $\Phi(x)$ for the cumulative probabilities of the standard normal distribution.

4 Upper bound

We now return to the real problem of this contribution, the present value of a stochastic cash flow

$$A = \sum_{t=1}^{n} \alpha_t e^{-(Y_1 - \frac{1}{2}\tilde{\sigma}_1^2) - (Y_2 - \frac{1}{2}\tilde{\sigma}_2^2) - ... - (Y_t - \frac{1}{2}\tilde{\sigma}_t^2)} \quad (44)$$

$$= \sum_{t=1}^{n} \alpha_t e^{-Y(t)+\frac{1}{2}\Sigma(t)}, \quad (45)$$

where all payments α_j $(j = 1, ..., n)$ are non-negative, and with the variables modeled as specified in the introduction.

Since both interest rate and volatility are stochastic, we will need two successive applications of the results of the previous sections when calculating upper bounds. Indeed, in the first step we calculate an upper bound conditionally on all volatilities ; the second step is needed in order to eliminate this conditioning.
The results seem to be interesting even if the models of the volatility are not realistic in practical situations (see [6]), and they represent a first result on comonotonic bounds for scalar products of stochastic vectors.

4.1 General result

We will start by presenting the boundary variable for the present value A, and continue by calculating the stop-loss premiums and distribution.

Proposition 4.1 *Let U and V be independent variables which are uniformly distributed on* $[0, 1]$*, and define the variable*

$$W_{upp}(t) = \frac{1}{2}\Sigma(t) + \Phi^{-1}(U)\sqrt{\Sigma(t)} \quad (46)$$

with conditional distribution

$$H_{t,upp}(x|u) = Prob\left[W_{upp}(t) \leq x | U = u\right]. \tag{47}$$

We then have

$$A \leq_{cx} A_{upp} \overset{def}{=} \sum_{t=1}^{n} \alpha_t e^{-\left(\mu_1 + ... + \mu_t\right) + X_{t,upp}(U,V)} \tag{48}$$

with $X_{t,upp}(U,V)$ *defined by its realizations* $X_{t,upp}(u,v) = H^{-1}_{t,upp}(v|u)$.

Proof. We first apply proposition 2.2 (decreasing functions) to A, with respect to the variables $Y(t)$ and conditionally on the volatilities $\tilde{\sigma}_1, ..., \tilde{\sigma}_n$. This gives

$$A \leq_{cx} \tilde{A} \overset{def}{=} \sum_{t=1}^{n} \alpha_t e^{-F_t^{-1}(1-U) + \frac{1}{2}\Sigma(t)}, \tag{49}$$

where U is a uniformly distributed variable on $[0, 1]$, and where

$$F_t(x) = \Phi\left(\frac{x - (\mu_1 + ... + \mu_t)}{\sqrt{\Sigma(t)}}\right). \tag{50}$$

The sum $\tilde{A}$ can be rewritten as

$$\tilde{A} = \sum_{t=1}^{n} \alpha_t e^{-(\mu_1 + ... + \mu_t) + \sqrt{\Sigma(t)}\Phi^{-1}(U) + \frac{1}{2}\Sigma(t)}, \tag{51}$$

or

$$\tilde{A} = \sum_{t=1}^{n} \alpha_t e^{-(\mu_1 + ... + \mu_t) + W_{upp}(t)}, \tag{52}$$

where we defined $W_{upp}(t)$ as in equation (46).

A second application of proposition 2.2 (increasing functions), now for $\tilde{A}$ with respect to the variables $W_{upp}(t)$ gives the result displayed in (48).

Q.E.D.

Starting from the previous result, we arrive at the stop-loss premiums and distribution, as summarized in the following proposition.

Proposition 4.2 *Consider the quantity A_{upp} as mentioned in proposition 4.1. The stop-loss premium for this variable can be calculated as*

$$E\left[(A_{upp}-k)_+\right] \tag{53}$$
$$= \int_0^1 du \int_0^1 dv \left(\sum_{t=1}^{n} \alpha_t e^{-(\mu_1 + ... + \mu_t) + X_{t,upp}(u,v)} \quad - \quad k \right)_+ \; ;$$

the distribution follows as

$$F_{upp}(k) = Prob\left[A_{upp} \le k\right] = \text{area}(R(k)) \; , \tag{54}$$

where the region $R(k) \subset \{(u,v)|0 \le u \le 1, 0 \le v \le 1\}$ is the collection of all combinations of u and v for which

$$\sum_{t=1}^{n} \alpha_t e^{-(\mu_1 + ... + \mu_t) + X_{t,upp}(u,v)} \quad \le \quad k \; . \tag{55}$$

4.2 Calculation of the values $X_{t,upp}(u,v)$

In order to find an expression for the values $X_{t,upp}(u,v)$, we first have to determine the distribution function $H_{t,upp}(x|u)$ of the variable $W_{upp}(t)$ of equation (46). Since this variable $W_{upp}(t)$ is a specific transformation of the variable $\Sigma(t)$, the distribution $H_{t,upp}(x|u)$ of the first variable can be deduced by means of the distribution $G_t(x)$ of the second one (see section 3).

The following result can be applied :

Proposition 4.3 *Consider a non-negative variable X for which the distribution $F(x) = Prob[X \le x]$ is known. For positive constants a and b, define the variables*

$$\begin{cases} Z_1 &= aX + b\sqrt{X} \\ Z_2 &= aX - b\sqrt{X} \end{cases} \tag{56}$$

with distribution functions denoted by

$$\begin{cases} H_1(z) &= Prob[Z_1 \le z] \\ H_2(z) &= Prob[Z_2 \le z]. \end{cases} \tag{57}$$

Then

$$H_1(z) = \begin{cases} 0 & \textit{if } z \le 0 \\ F\left(\left(\sqrt{\frac{z}{a} + \frac{b^2}{4a^2}} - \frac{b}{2a}\right)^2\right) & \textit{if } z > 0\ , \end{cases} \tag{58}$$

and

$$H_2(z) = \begin{cases} 0 & \textit{if } z \le -\dfrac{b^2}{4a} \\ F\left(\left(\sqrt{\frac{z}{a} + \frac{b^2}{4a^2}} + \frac{b}{2a}\right)^2\right) & \\ \quad -F\left(\left(\sqrt{\frac{z}{a} + \frac{b^2}{4a^2}} - \frac{b}{2a}\right)^2\right) & \textit{if } -\frac{b^2}{4a} < z \le 0 \\ F\left(\left(\sqrt{\frac{z}{a} + \frac{b^2}{4a^2}} + \frac{b}{2a}\right)^2\right) & \textit{if } z > 0\ . \end{cases} \tag{59}$$

Proof. Both results can be found in a straightforward way, making use of the probability identity

$$Prob\left[aX \pm b\sqrt{X} \le z\right] = Prob\left[\left(\sqrt{X} \pm \frac{b}{2a}\right)^2 \le \frac{z}{a} + \frac{b^2}{4a^2}\right]. \tag{60}$$

Q.E.D.

Making use of the results of this proposition, with $a = \frac{1}{2}$, $b = \pm\Phi^{-1}(u)$, and $F(x) = G_t(x)$, the distribution $H_{t,upp}(x|u)$ can be written down immediately :

- if $u \ge 1/2$,

$$H_{t,upp}(x|u) = \begin{cases} 0 & \text{if } x \le 0 \\ G_t\left(\left(\sqrt{2x + \Phi^{-1}(u)^2} - \Phi^{-1}(u)\right)^2\right) & \text{if } x > 0\ ; \end{cases} \tag{61}$$

- if $u \leq 1/2$,

$$H_{t,upp}(x|u) = \begin{cases} 0 \\ \qquad \text{if } \; x \leq -\frac{1}{2}\Phi^{-1}(u)^2 \\ G_t\left(\left(\sqrt{2x+\Phi^{-1}(u)^2} - \Phi^{-1}(u)\right)^2\right) \\ \quad -G_t\left(\left(\sqrt{2x+\Phi^{-1}(u)^2} + \Phi^{-1}(u)\right)^2\right) \\ \qquad \text{if } \; -\frac{1}{2}\Phi^{-1}(u)^2 < x \leq 0 \\ G_t\left(\left(\sqrt{2x+\Phi^{-1}(u)^2} - \Phi^{-1}(u)\right)^2\right) \\ \qquad \text{if } \; x > 0 \; . \end{cases} \tag{62}$$

A few calculations lead to the inverse $X_{t,upp}(u,v)$:

- if $u \geq 1/2$,

$$X_{t,upp}(u,v) = \frac{1}{2}G_t^{-1}(v) + \Phi^{-1}(u)\sqrt{G_t^{-1}(v)}\; ; \tag{63}$$

- if $u \leq 1/2$ and $v \geq G_t\left(4\Phi^{-1}(u)^2\right)$,

$$X_{t,upp}(u,v) = \frac{1}{2}G_t^{-1}(v) + \Phi^{-1}(u)\sqrt{G_t^{-1}(v)}\; ; \tag{64}$$

- if $u \leq 1/2$ and $v < G_t\left(4\Phi^{-1}(u)^2\right)$,

$$X_{t,upp}(u,v) = C \tag{65}$$

  with $C \in \left[-\frac{1}{2}\Phi^{-1}(u)^2, 0\right[$ defined implicitely as the solution of

$$G_t\left(\left(\sqrt{\Phi^{-1}(u)^2+2C} - \Phi^{-1}(u)\right)^2\right) \\ -G_t\left(\left(\sqrt{\Phi^{-1}(u)^2+2C} + \Phi^{-1}(u)\right)^2\right) = v. \tag{66}$$

5 Improved upper bound

For the improved bound, we have to condition on a variable Z which has some resemblance to the investigated quantity. As in [4], we will choose linear combinations of the one-period compounded rates of return

$$Z = \sum_{t=1}^{n} \beta_t Y_t \tag{67}$$

and we use the notation ρ_t for the correlation between this variable Z and the compounded interest $Y(t) = Y_1 + ... + Y_t$. Note that conditionally on this variable Z, again the variable $Y(t)$ is normally distributed, with

$$E[Y(t)|Z, \tilde{\sigma}_1, ..., \tilde{\sigma}_t] = (\mu_1 + ... + \mu_t) + \rho_t \sqrt{\Sigma(t)} \left(\frac{Z - E[Z]}{\sqrt{Var[Z]}} \right) \tag{68}$$

$$Var[Y(t)|Z, \tilde{\sigma}_1, ..., \tilde{\sigma}_t] = (1 - \rho_t^2)\Sigma(t). \tag{69}$$

The correlation ρ_t can be calculated as

$$\rho_t = \frac{1}{\sqrt{\Sigma(t)}} \frac{\sum_{j=1}^{t} \beta_j \tilde{\sigma}_j^2}{\sqrt{\sum_{j=1}^{n} \beta_j^2 \tilde{\sigma}_j^2}} . \tag{70}$$

Due to the stochasticity of the volatilities, of course this correlation is also stochastic.

5.1 General result

We keep the same structure, starting by presenting the boundary variable for the present value A, and continuing by calculating the stop-loss premiums and distribution.

Proposition 5.1 *Let U_a, U_b and V be independent variables which are uniformly distributed on $[0, 1]$, and define the variable*

$$W_{upp*}(t) = \frac{1}{2}\Sigma(t) - \rho_t \Phi^{-1}(U_a)\sqrt{\Sigma(t)} + \sqrt{1 - \rho_t^2}\Phi^{-1}(U_b)\sqrt{\Sigma(t)} \tag{71}$$

with conditional distribution

$$H_{t,upp*}(x|u_a, u_b) = Prob\left[W_{upp*}(t) \leq x | U_a = u_a, U_b = u_b\right]. \tag{72}$$

We then have

$$A \leq_{cx} A_{upp*} \stackrel{def}{=} \sum_{t=1}^{n} \alpha_t e^{-\,(\mu_1 + ... + \mu_t) + X_{t,upp*}(U_a, U_b, V)} \tag{73}$$

with $X_{t,upp*}(u_a, u_b, v) = H^{-1}_{t,upp*}(v|u_a, u_b)$.

Proof. We first apply proposition 2.2 (decreasing functions) to A, with respect to the variables $Y(t)$ and conditionally on the volatilities $\tilde{\sigma}_1, ..., \tilde{\sigma}_n$. This gives

$$A \leq_{cx} \tilde{A}^* \stackrel{def}{=} \sum_{t=1}^{n} \alpha_t e^{-F^{-1}_{t|Z}(1-U)+\frac{1}{2}\Sigma(t)}, \tag{74}$$

where U is a uniformly distributed variable on $[0,1]$, and where due to equations (68) and (69)

$$F_{t|Z}(x) = \Phi\left(\frac{x - (\mu_1 + ... + \mu_t) - \rho_t\sqrt{\Sigma(t)}\Phi^{-1}(U_a)}{\sqrt{(1-\rho_t^2)\Sigma(t)}}\right). \tag{75}$$

The sum $\tilde{A}^*$ can be rewritten as

$$\tilde{A}^* = \sum_{t=1}^{n} \alpha_t e^{-(\mu_1 + ... + \mu_t) + \frac{1}{2}\Sigma(t)} \tag{76}$$

$$\cdot\; e^{-\rho_t\sqrt{\Sigma(t)}\Phi^{-1}(U_a) + \sqrt{1-\rho_t^2}\sqrt{\Sigma(t)}\Phi^{-1}(U_b)}, \tag{77}$$

or

$$\tilde{A}^* = \sum_{t=1}^{n} \alpha_t e^{-(\mu_1 + ... + \mu_t) + W_{upp*}(t)}, \tag{78}$$

where we defined $W_{upp*}(t)$ as in equation (71).

A second application of proposition 2.2 (increasing functions), now for $\tilde{A}^*$ with respect to the variables $W_{upp*}(t)$ now gives the result of (73).

Q.E.D.

Starting from the previous result, we arrive at the stop-loss premiums and distribution, as summarized in the following proposition.

Proposition 5.2 *Consider the quantity A_{upp*} as mentioned in proposition 5.1. The stop-loss premium for this variable can be calculated as*

$$E\left[(A_{upp*}-k)_+\right] \tag{79}$$
$$= \int_0^1 du_a \int_0^1 du_b \int_0^1 dv$$
$$\left(\sum_{t=1}^{n} \alpha_t e^{-(\mu_1+...+\mu_t)+X_{t,upp*}(u_a,u_b,v)} - k\right)_+ ;$$

the distribution follows as

$$F_{upp*}(k) = Prob\left[A_{upp*} \le k\right] = \text{volume}(R^*(k)) \ , \tag{80}$$

where the region $R^*(k) \subset \{(u_a,u_b,v)|0 \le u_a \le 1, 0 \le u_b \le 1, 0 \le v \le 1\}$ is the collection of all combinations of u_a, u_b and v for which

$$\sum_{t=1}^{n} \alpha_t e^{-(\mu_1+...+\mu_t)+X_{t,upp*}(u_a,u_b,v)} \le k \ . \tag{81}$$

5.2 Calculation of the values $X_{t,upp*}(u_a, u_b, v)$

As can be seen in equation (71), the variable $W_{upp*}(t)$ no longer depends on the variable $\Sigma(t)$ alone, but on a combination of the n variables $\Sigma(1),...,\Sigma(n)$ through the correlation ρ_t. As a consequence, the derivation of the distribution $H_{t,upp*}(x|u_a,u_b)$ and thus of $X_{t,upp*}(u_a,u_b,v)$ becomes more and more complicated as the linear combination for Z is more complete. This should not be surprising, since the improved upper bound becomes closer to the original variable A, the more the variables Z and A are alike. Under the present circumstances, this corresponds with a linear combination as complete as possible.

We will show the effect of a "small" conditioning by giving the results in case we take $\beta_2 = ... = \beta_n = 0$ and $\beta_1 = 1$, or $Z = Y(1)$. This choice for the conditioning is not unreasonable, since this means that we condition on the rate of return for the first period, for which a forecast seems to be more reliable than for periods later on.

When conditioning on $Z = Y(1)$, the correlation ρ_t (see equation (70)) can be simplified to

$$\rho_t = \sqrt{\frac{\Sigma(1)}{\Sigma(t)}} \,. \tag{82}$$

In this case, the variable $W_{upp*}(t)$ can be written as

$$\begin{aligned} W_{upp*}(t) &= \frac{1}{2}\Sigma(t) - \Phi^{-1}(U_a)\sqrt{\Sigma(1)} + \Phi^{-1}(U_b)\sqrt{\Sigma(t)-\Sigma(1)} \\ &= W_A(t) + W_B(t) \end{aligned} \tag{83}$$

where due to the assumptions about the volatilities the variables

$$W_A(t) = \frac{1}{2}\Sigma(1) - \Phi^{-1}(U_a)\sqrt{\Sigma(1)} \tag{84}$$

and
$$W_B(t) = \frac{1}{2}\left(\Sigma(t) - \Sigma(1)\right) + \Phi^{-1}(U_b)\sqrt{\Sigma(t)-\Sigma(1)} \tag{85}$$

are independent.

If we use the notations

$$H_{t,A}(x|u_a) = Prob\left[W_A(t) \leq x | U_a = u_a\right] \tag{86}$$

and

$$H_{t,B}(x|u_b) = Prob\left[W_B(t) \leq x | U_b = u_b\right], \tag{87}$$

it follows from (83) that the convolution of these two distributions results in the distribution $H_{t,upp*}(x|u_a, u_b)$ of $W_{upp*}(t)$.

In order to calculate the distributions of (86) and (87), proposition 4.3 can be used with $a = \frac{1}{2}$, $b = \pm\Phi^{-1}(u_a)$ and $F(x) = G_1(x)$ for $H_{t,A}(x|u_a)$, and with $a = \frac{1}{2}$, $b = \pm\Phi^{-1}(u_b)$ and $F(x) = G_{t-1}(x)$ for $H_{t,B}(x|u_b)$.

We find

- if $u_a \leq 1/2$,

$$H_{t,A}(x|u_a) = \begin{cases} 0 \\ \qquad\qquad \text{if } x \leq 0 \\ G_1\left(\left(\sqrt{2x + \Phi^{-1}(u_a)^2} + \Phi^{-1}(u_a)\right)^2\right) \\ \qquad\qquad \text{if } x > 0\,; \end{cases} \tag{88}$$

- if $u_a \geq 1/2$,

$$H_{t,A}(x|u_a) = \begin{cases} 0 \\ \qquad\qquad \text{if } \ x \leq -\frac{1}{2}\Phi^{-1}(u_a)^2 \\ G_1\left(\left(\sqrt{2x+\Phi^{-1}(u_a)^2}+\Phi^{-1}(u_a)\right)^2\right) \\ \quad -G_1\left(\left(\sqrt{2x+\Phi^{-1}(u_a)^2}-\Phi^{-1}(u_a)\right)^2\right) \\ \qquad\qquad \text{if } \ -\frac{1}{2}\Phi^{-1}(u_a)^2 < x \leq 0 \\ G_1\left(\left(\sqrt{2x+\Phi^{-1}(u_a)^2}+\Phi^{-1}(u_a)\right)^2\right) \\ \qquad\qquad \text{if } \ x > 0 \ ; \end{cases} \tag{89}$$

- if $u_b \leq 1/2$,

$$H_{t,B}(x|u_b) = \begin{cases} 0 \\ \qquad\qquad \text{if } \ x \leq -\frac{1}{2}\Phi^{-1}(u_b)^2 \\ G_{t-1}\left(\left(\sqrt{2x+\Phi^{-1}(u_b)^2}-\Phi^{-1}(u_b)\right)^2\right) \\ \quad -G_{t-1}\left(\left(\sqrt{2x+\Phi^{-1}(u_b)^2}+\Phi^{-1}(u_b)\right)^2\right) \\ \qquad\qquad \text{if } \ -\frac{1}{2}\Phi^{-1}(u_b)^2 < x \leq 0 \\ G_{t-1}\left(\left(\sqrt{2x+\Phi^{-1}(u_b)^2}-\Phi^{-1}(u_b)\right)^2\right) \\ \qquad\qquad \text{if } \ x > 0 \ ; \end{cases} \tag{90}$$

- if $u_b \geq 1/2$,

$$H_{t,B}(x|u_b) = \begin{cases} 0 \\ \qquad\qquad \text{if } \ x \leq 0 \\ G_{t-1}\left(\left(\sqrt{2x+\Phi^{-1}(u_b)^2}-\Phi^{-1}(u_b)\right)^2\right) \\ \qquad\qquad \text{if } \ x > 0 \ . \end{cases} \tag{91}$$

6 Numerical illustration

In this last section, we want to examine the accuracy of the upper bounds in comparison with the exact present value. In order to do so, we will investigate the first upper bound (the bound with the smallest precision) for three cash-flows with different structure :

- $\alpha_t = 10$ for $t = 1, ..., 10$;
- $\alpha_t = t$ for $t = 1, ..., 10$;
- $\alpha_t = 11 - t$ for $t = 1, ..., 10$.

For the normal distribution of the stochastic interest rate (see equation (3)), we choose $\mu_t = 0.07$ for each time point t ; the squared stochastic volatility (see equation (35)) is assumed to be exponentially distributed with parameter 20, i.e. with mean 0.05.

In figures 1, 3 and 5 (matching the three cases mentioned above) the distribution of the upper bound is depicted, together with an empirical distribution of the original present value obtained by Monte-Carlo simulation. In each of the three cases we see that the upper bound performs rather well and thus provide a good approximation of the exact distribution of the present value.

In order to show the calculation method of the distribution function as given in equation (54), figures 2, 4 and 6 give an idea of how the region $R(k)$ looks by graphing the surface

$$\text{sum}(u,v) = \sum_{t=1}^{n} \alpha_t e^{-(\mu_1 + ... + \mu_t) + X_{t,upp}(u,v)} \tag{92}$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 1$ for the same three cash-flows.

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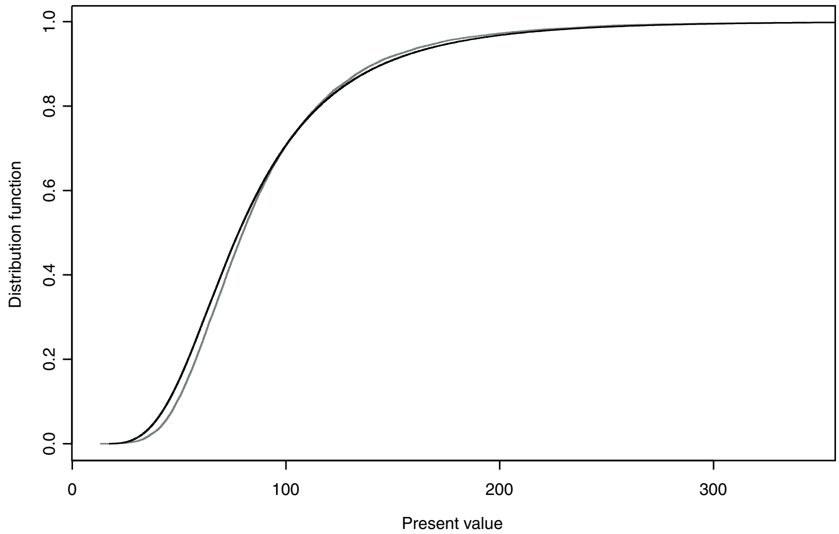

Figure 1: Distribution function of the upper bound A_{upp} (black) for $\alpha_t = 10$ $(t = 1, ..., 10)$, compared to a simulated version of the distribution of A (grey).

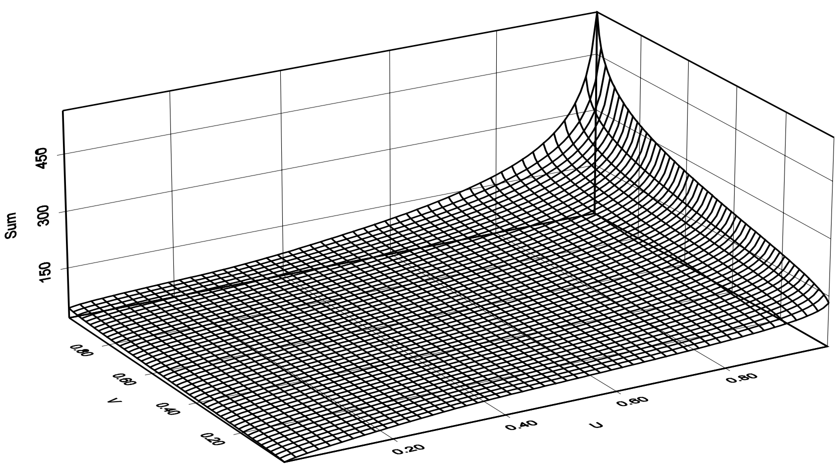

Figure 2: The surface sum(u, v) (see (92)) for $\alpha_t = 10$ $(t = 1, ..., 10)$.

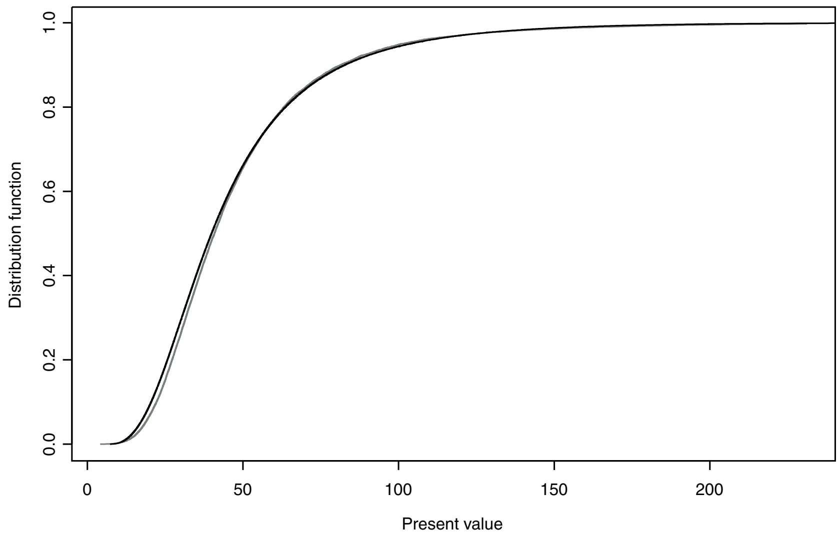

Figure 3: Distribution function of the upper bound A_{upp} (black) for $\alpha_t = t$ ($t = 1, ..., 10$), compared to a simulated version of the distribution of A (grey).

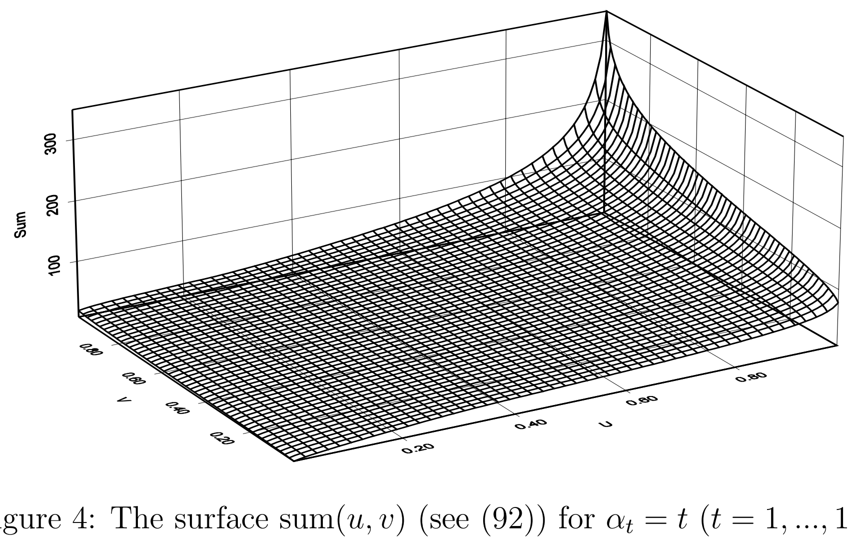

Figure 4: The surface sum(u, v) (see (92)) for $\alpha_t = t$ ($t = 1, ..., 10$).

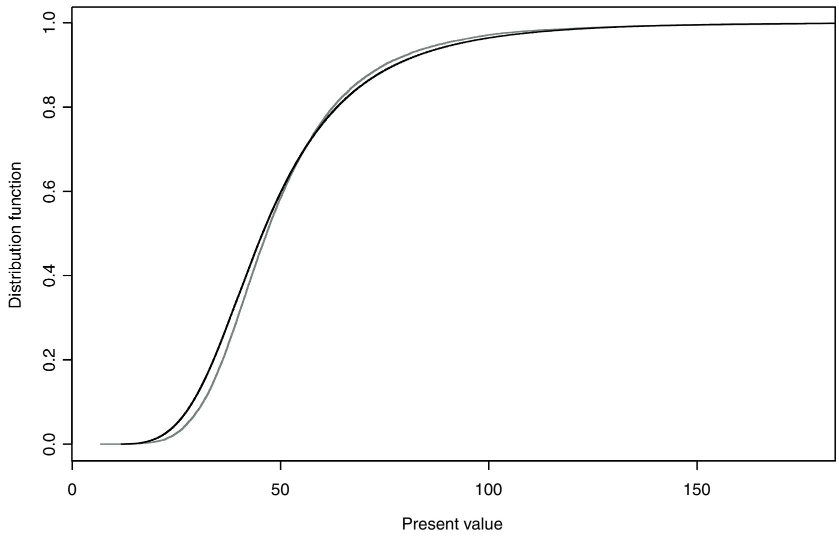

Figure 5: Distribution function of the upper bound A_{upp} (black) for $\alpha_t = 11 - t$ $(t = 1, ..., 10)$, compared to a simulated version of the distribution of A (grey).

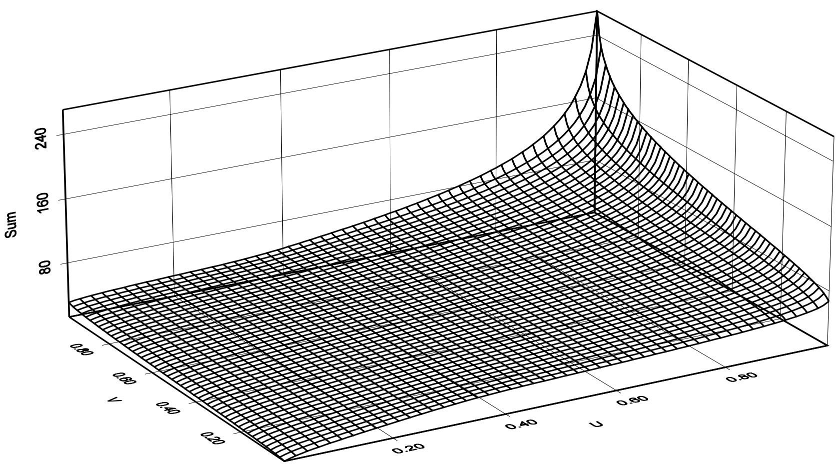

Figure 6: The surface sum(u, v) (see (92)) for $\alpha_t = 11 - t$ $(t = 1, ..., 10)$.