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Grothendieck topology, coherent sheaves and Serre's theorem for schematic algebras

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Dedicated to the memory of Albert Collins

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Abstract

We define schematic algebras to be algebras which have "enough" Ore-sets. Many graded algebras studied nowadays are schematic. We construct a generalised Grothendieck topology for the free monoid on all Ore-sets of a schematic algebra R. This allows us to develop a sheaf theory which is similar to the scheme theory for commutative algebras. In particular, we obtain an equivalence between the category of all coherent sheaves and the category *Proj* R as it is defined in (Artin; 1992).

1. Introduction

For a positively graded commutative algebra C, Serre's theorem (cf. [4]) states that the category of coherent sheaves on the scheme *Proj* C is equivalent to a certain category which is completely defined in terms of C-gr. In [1], Artin observed that this last category also makes sense for a noncommutative positively graded algebra R and used this observation to define *Proj*.

We want to study this *Proj* by developing a kind of scheme theory similar to the commutative theory. It is obvious that this theory will be possible only if the algebra R considered contains "enough" Ore-sets in a sense to be made precise. We call such an algebra schematic.

Once there are enough Ore-sets in the algebra, one thinks one is able to prove Serre's theorem by mimicking the commutative case, i.e. over the intersection of two open sets S and T, one would like to put the localisation at $S \vee T$, the Ore-set generated by S and T. This fails, mainly because the composition of two

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Ore-localisations is not necessarily a localisation. We circumvent this problem by considering more open sets than the ones induced from Ore-sets. This is not a topology since the intersection of two open sets is no longer equal to the intersection of the same open sets in reverse order. However, in Section 3 we prove that our model, called the noncommutative site, satisfies generalisations of the axioms of the categorical definition of a topology, the so-called Grothendieck topology.

The main part of this paper is Section 4: we define sheaves on the noncommutative site and show that there is a functor from R-gr to the category of sheaves. A second price we have to pay for this generalised topology is that not all sections of the structure sheaf, i.e. the sheaf induced by the R-module R, are rings. However, this anomaly does not prevent us to establish our main goal: an equivalence between *Proj* R and the category of coherent sheaves, just as in the commutative case. We can also give a criterion to decide whether there exists an R-module whose localisations at Ore-sets T_i are isomorphic to given $Q_{T_i}(R)$-modules.

This paper will have a sequel [6] in which we provide some general theorems in order to prove that a given algebra is schematic. This allows us to exhibit a lot of interesting schematic algebras.

2. Preliminaries

We will need a minimum of torsion theory throughout this paper, so we provide a short introduction which also settles notation. For more details, we refer to [5].

Let $\mathscr{L}$ be a set of left ideals of an arbitrary ring R. We call $\mathscr{L}$ *a filter* if it satisfies the following three axioms:

T_1 If $I \in \mathscr{L}$ and $I \subseteq J$ then $J \in \mathscr{L}$.
T_2 If $I, J \in \mathscr{L}$ then $I \cap J \in \mathscr{L}$.
T_3 If $I \in \mathscr{L}$ and $a \in R$ then $(I:a) \overset{\text{def}}{=} \{x \in R \mid xa \in I\} \in \mathscr{L}$

The functor $\kappa: R \to R$ defined by $\kappa(M) = \{m \in M \mid \exists I \in \mathscr{L}: Im = 0\}$ is then a left exact preradical, i.e. a left exact subfunctor of the identity functor on R-mod. Modules M with $\kappa(M) = M$ are called *torsion modules* while a module M with $\kappa(M) = 0$ is *torsion free*. The filter $\mathscr{L}$ is *idempotent* when it satisfies:

T_4 If $I \lhd_l R$ and $\exists J \in \mathscr{L}$ such that $\forall a \in J$, $(I:a) \in \mathscr{L}$ then $I \in \mathscr{L}$.

This implies that $\mathscr{L}$ is closed under products and that κ is a radical, meaning that $\kappa(M/\kappa(M)) = 0$ for all R-modules M. In this case, one defines the *module of quotients* of M with respect to κ as follows:

$$Q_\kappa(M) = \varinjlim_{I \in \mathscr{L}} \operatorname{Hom}_R(I, M/\kappa(M)),$$

$Q_\kappa(M)$ turns out to be a module over the ring $Q_\kappa(R)$. The following idempotent filters play an important role in the sequel:

1. If R is a positively graded Noetherian ring and R_+ denotes the two-sided ideal $\bigoplus_{n>0} R_n$ then $\mathscr{L}(\kappa_+) = \{I \lhd_l R \mid \exists n \in \mathbb{N}\colon (R_+)^n \subseteq I\}$ is an idempotent filter. We denote the corresponding left exact radical by κ_+.

2. Let S be a left Ore-set in an arbitrary ring R, then $\mathscr{L}(S) = \{I \lhd_l R \mid I \cap S \neq \emptyset\}$ is an idempotent filter. If κ_S denotes the corresponding radical and $Q_S(M)$ the module of quotients of M then it is easy to see that $Q_S(M)$ is isomorphic to $S^{-1}M$, the usual Ore-localisation of M at S.

3. The noncommutative site

We assume the reader is acquainted with the theory of schemes as it can be found in [2]. Let $C = k \oplus C_1 \oplus \cdots$ be a positively graded *commutative* Noetherian ring generated in degree one. Put $Y = \mathit{Proj}\, C$ and $Y(f) = \{\mathfrak{p} \in Y \mid f \notin \mathfrak{p}\}$, the Zariski open set corresponding to a homogeneous element $f \in C$. Then there is a (finite) subset $\{f_i \mid f_i \in C_1\}$ such that $Y = \bigcup_i Y(f_i)$, in other words: for every choice of $d_i \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ with $(C_+)^n \subseteq \sum_i C f_i^{d_i}$. It follows that for any finitely generated graded C-module M

$$\Gamma_*(M) \stackrel{\text{def}}{=} \bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \widetilde{M(n)}) = Q_{\kappa_+}(M) \stackrel{\text{def}}{=} \varinjlim_p \mathit{Hom}(C_+^p, M/\kappa_+(M))$$
$$= \varprojlim_i Q_{f_i}(M),$$

where $\widetilde{M(n)}$ denotes the sheaf of modules associated to the shifted module $M(n)$ and $Q_{f_i}(M)$ is the localization of M at $\{1, f_i, f_i^2, \ldots\}$. Similarly,

$$Q_f(M) = \varprojlim_i Q_{ff_i}(M),$$

where the inverse systems are defined as follows: $g \leq h \Leftrightarrow Y(g) \subseteq Y(h)$. This result is the key to prove Serre's theorem which says that the category of coherent $\mathcal{O}_Y$-modules is equivalent with a certain quotientcategory.

Now look at a *noncommutative* positively graded Noetherian k-algebra $R = k \oplus R_1 \oplus \cdots$, such that $R = k[R_1]$. In [1], Artin observed that the quotientcategory mentioned in Serre's theorem also makes sense for the noncommutative algebra R and called it $\mathit{Proj}\, R$. Let us explain this definition in some detail. Since the filter $\mathscr{L}(\kappa_+)$ is idempotent, we can form the quotientcategory (R, κ_+)-gr, i.e. the full subcategory of $Q_{\kappa_+}(R)$-gr consisting of modules of the form $Q_{\kappa_+}(M)$ for some graded R-module M. Call a graded R-module M κ_+-closed whenever the canonical maps $M \to \mathrm{Hom}_R(R_+^n, M)$ are isomorphisms for all $n \in \mathbb{N}$. It is then well-known that (R, κ_+)-gr is equivalent to the full subcategory of R-gr consisting of the κ_+-closed modules. There are now three equivalent ways to define $\mathit{Proj}\, R$:

1. The Noetherian objects in (R, κ_+)-gr, i.e. those objects of (R, κ_+)-gr which satisfy the ascending chain condition on subobjects.

2. The full subcategory of $Q_{\kappa_+}(R)$-gr consisting of all modules of the form $Q_{\kappa_+}(M)$ for some finitely generated R-module M.
3. The full subcategory of R-gr consisting of all κ_+-closed modules which are torsion over a finitely generated R-submodule.

Our goal is a description of the objects of *Proj R* by means of objects of usual module categories in the same way as above for commutative algebras, i.e. we want modules to be determined by (Ore-)localisations. It is obvious that for this reason there should be "enough" Ore-sets in R. This prompts the definition of schematic algebras:

Definition 1. We say that R as above is *schematic* if there is a finite set I of homogeneous left Ore-sets of R such that for every $S \in I$, $S \cap R_+ \neq \emptyset$ and such that one of the following equivalent properties is satisfied:

1. for each $(x_S)_{S \in I} \in \prod_{S \in I} S, \exists m \in \mathbb{N}$ such that $(R_+)^m \subseteq \sum_{S \in I} R x_S$,
2. $\bigcap_{S \in I} \mathscr{L}(S) = \mathscr{L}(\kappa_+)$,
3. $\bigcap_{S \in I} \kappa_S(M) = \kappa_+(M)\ \forall M \in R\text{-mod}$,
4. $\bigwedge_{S \in I} \kappa_S = \kappa_+$ where $\bigwedge$ denotes the infimum of torsion theories.

In the sequel [6] to this paper, we will prove that several fancy algebras are schematic, a.o. Rees rings of enveloping algebras of Lie algebras and three-dimensional Sklyanin algebras.

Assume from now on that R is *schematic*. The straightforward generalisation of the commutative scheme does not work: the canonical map

$$Q_{\kappa_S \wedge \kappa_T}(M) \to \varprojlim \begin{pmatrix} Q_S(M) & & \\ & \searrow & \\ & & Q_{S \vee T}(M) \\ & \nearrow & \\ Q_T(M) & & \end{pmatrix}$$

is an isomorphism for all $M \in R$-gr if and only if S and T are compatible, i.e. the functors Q_S and Q_T commute. As the Ore-sets of a schematic algebra are rarely two-by-two compatible, it is clear that we will have to change the inverse system, but we want it to originate from (a kind of) Grothendieck topology. Classically, a Grothendieck topology consists of a category $\mathscr{C}$ such that for each object U in $\mathscr{C}$, there is a set $Cov(U)$ consisting of subsets of morphisms with common target U, satisfying three axioms:

G_1 $\{U \to U\} \in Cov(U)$,
G_2 $\{U_i \to U \mid i \in I\} \in Cov(U)$ and $\forall i \in I$: $\{U_{ij} \to U_i \mid j \in I_i\}$ $\in Cov(U_i)$ then $\{U_{ij} \to U_i \to U \mid i \in I, j \in I_i\} \in Cov(U)$,
G_3 if $\{U_i \to U \mid i \in I\} \in Cov(U)$ and $U' \to U$ then the pull-back $U' \times_U U_i$ exists and $\{U' \times_U U_i \to U' \mid i \in I\} \in Cov(U')$.

Every topology is a Grothendieck topology by taking as objects the open sets. The morphisms are just the inclusions and a set of morphisms $\{U_i \to U\}$ is in $Cov(U)$ if and only if $\bigcup_{i \in I} U_i = U$. For more details on Grothendieck topologies, we refer to [3].

We will now define the Grothendieck topology which induces the right inverse system. Let $\mathcal{O}$ be the set $\{S$ homogeneous left Ore-set of $R \mid 1 \in S, 0 \notin S, S \cap R_+ \neq \emptyset\}$. Denote the free monoid on $\mathcal{O}$ by $\mathcal{W}$. If $W = S_1 \dots S_n \in \mathcal{W}$ then by $w \in W$ we mean an element of R of the form $w = s_1 \dots s_n$ with $s_i \in S_i$. We define a category $\underline{\mathcal{W}}$ as follows: the objects are the elements of $\mathcal{W}$ and for two words $W = S_1 \dots S_n$ and $W' = T_1 \dots T_m$ in $\mathcal{W}$ we put

$$\operatorname{Hom}(W', W) = \begin{cases} \{W' \to W\} & \text{if } \exists \alpha\colon \{1, \dots, n\} \to \{1, \dots, m\} \\ & \text{such that } i < j \Rightarrow \alpha(i) < \alpha(j) \text{ and } S_i = T_{\alpha(i)}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Although there may be many possible increasing injections from the letters of W to the letters of W' we only consider one morphism between W' and W in our category, i.e. $\operatorname{Hom}(W', W)$ is always empty or a singleton.

Because we have to deal with arbitrary compositions of Ore-localisations, we introduce the following *important notation*: if $W = S_1 \dots S_n \in \mathcal{W}$, then we denote

$$Q_W(M) = (Q_{S_n} \circ \cdots \circ Q_{S_1})(M) = Q_{S_n}(R) \otimes_R \cdots \otimes_R Q_{S_1}(R) \otimes_R M$$

each $Q_{S_i}(R)$ being the localisation of R at the Ore-set S_i.

We stress that all modules in this paper will be (graded) *left* R-modules and consequently, all localisations will be on the left.

We also associate a set of left ideals to W, namely $\mathcal{L}(W) = \{I \lhd_l R \mid \exists w \in W$ such that $w \in I\}$. $\mathcal{L}(W)$ is a filter in the sense of the preliminaries. The verification of the three axioms is easy (an induction on the second Ore-condition) but it is useful for the sequel to restate T_2 and T_3: T_2 means that for two elements of the same word, say $w, w' \in W$ there is a common multiple in W, i.e. $\exists w'' \in W$ such that $w'' = aw$ and $w'' = bw'$ for some $a, b \in R$. T_3 tells us that $\mathcal{L}(W)$ satisfies a kind of generalised second Ore-condition: if $w \in W, a \in R$, then $\exists w' \in W, b \in R$ with $w'a = bw$. The corresponding left exact preradical κ_W maps an R-module M to its submodule $\kappa_W(M) = \{x \in M \mid \exists w \in W$ such that $wm = 0\} = \operatorname{Ker}(M \to Q_W(M))$ and induces a left exact preradical on R-gr. In general, $\mathcal{L}(W)$ is not an idempotent filter, e.g. $\mathcal{L}(ST)$ is idempotent if and only if S and T are compatible. Note that, if $W' \to W$ in $\underline{\mathcal{W}}$ then $\mathcal{L}(W) \subseteq \mathcal{L}(W')$ and for any $V \in \mathcal{W}$, $W'V \to WV$ and $VW' \to VW$ are in $\underline{\mathcal{W}}$.

By a *global cover*, we understand a finite subset $\{W_i \mid i \in I\}$ of $\mathcal{W}$ such that $\bigcap_{i \in I} \mathcal{L}(W_i) = \mathcal{L}(\kappa_+)$. The existence of at least one global cover is guaranteed by the schematic condition. Now for $W \in \mathcal{W}$ we define $Cov(W)$ to be the set of all sets of $\underline{\mathcal{W}}$-morphisms of the form $\{W_i W \to W \mid i \in I\}$ where $\{W_i \mid i \in I\}$ is a global cover. Topologically, this means that every cover of an open set is induced by a global cover, but note that this is also the case in the commutative theory. Since it is obvious that the pull-back of two morphisms in $\underline{\mathcal{W}}$ may not exist, we will have to modify the axiom G_3.

Proposition 1. *The category* $\underline{\mathscr{W}}$ *together with the sets* $Cov(W)$ *for* $W \in \underline{\mathscr{W}}$ *satisfies* G_1, G_2 *and*

G_3' *if* $\{U_iW \to W \mid i \in I\} \in Cov(W)$, $W' \to W \in \underline{\mathscr{W}}$, *say* $W = S_1S_2 \dots S_n$, $W' = VS_1V_1S_2V_2 \dots S_nV_n$ *and if we define* $U_iW \times_W W' = U_iW' = U_iVS_1V_1S_2V_2 \dots S_nV_n$ *then* $\{U_iW \times_W W' \to W' \mid i \in I\} \in Cov(W')$.

Therefore, we call $\underline{\mathscr{W}}$ a *noncommutative Grothendieck topology*. The proof of this proposition follows immediately from the next lemma:

Lemma 1. *If* $\{W_i \mid i \in I\}$ *is a global cover, i.e.* $\bigcap_{i \in I} \mathscr{L}(W_i) = \mathscr{L}(\kappa_+)$, *then for all* $V \in \mathscr{W}$: $\bigcap_{i \in I} \mathscr{L}(W_iV) = \mathscr{L}(V)$.

Proof. One inclusion is obvious, so take $L \in \cap \mathscr{L}(W_iV)$, say $w_iv_i \in L$ $\forall i \in I$ with $v_i \in V$, $w_i \in W_i$ $\forall i \in I$. Since I is finite and because of property T_2 for $\mathscr{L}(V)$, we can find a common multiple of the v_i in V, say $v = a_iv_i \in V$. Using property T_3 for $\mathscr{L}(W_i)$ yields that $\exists w_i' \in W_i$ such that $w_i'a_i \in Rw_i$. Because $\{W_i \mid i \in I\}$ is a global cover, we can find a natural number n with $(R_+)^n \subseteq \sum_i Rw_i'$. Consequently, $(R_+)^n v \subseteq L$. If T denotes the first letter of V then there is a $t \in T \cap (R_+)^n$. Now $tv \in L$ and $tv \in V$, yielding that $L \in \mathscr{L}(V)$. □

The category $\underline{\mathscr{W}}$ with this noncommutative Grothendieck topology defined by the $Cov(W)$, $W \in \mathscr{W}$ is called the *noncommutative site* and we denote this again by $\underline{\mathscr{W}}$.

4. Sheaves on the noncommutative site

In this section, sheaves on the category $\underline{\mathscr{W}}$ are defined. We will show that each R-module induces a sheaf in a natural way. Conversely, to each sheaf we will associate its global sections. The notion of a coherent sheaf naturally extends to this noncommutative setting and the main theorem states that the category of coherent sheaves on $\underline{\mathscr{W}}$ is equivalent to the quotient category *Proj R* just as in the commutative theory. An important corollary is that a module over a schematic algebra is still determined by its localisations at the Ore-sets of a global cover. The hypothesis on R are the same as in Section 3.

Definition 2. A *presheaf* $\mathscr{F}$ on $\underline{\mathscr{W}}$ is a contravariant functor from $\underline{\mathscr{W}}$ to the category R-gr such that for all $W \in \mathscr{W}$ the sections $\mathscr{F}(W)$ of $\mathscr{F}$ on W is a graded $Q_S(R)$-module where S denotes the last letter of W.

If R is a commutative algebra, then this just means that $\mathscr{F}$ is a presheaf of graded $\mathscr{O}_X$-modules, $\mathscr{O}_X$ being the graded structure sheaf on $X = Proj\,R$. Of course one could take parts of degree zero in order to establish complete similarity with the (ungraded) $\mathcal{O}_X$-modules one usually works with in the commutative theory. Note that if $W = 1$ we require that the global sections $\mathscr{F}(1)$, which we denote by $\Gamma_*(\mathscr{F})$, is a $Q_{\kappa_+}(R)$-module. We will denote the restriction map $\mathscr{F}(V \to W)$ by $\rho_V^W : \mathscr{F}(W) \to \mathscr{F}(V)$. If $W = 1$, then we will simplify this notation further to $\rho_V : \Gamma_*(\mathscr{F}) \stackrel{\text{def}}{=} \mathscr{F}(1) \to \mathscr{F}(V)$. Recall that a finite subset $\{W_i \mid i \in I\}$ of $\mathscr{W}$ is a global cover if $\bigcap_i \mathscr{L}(W_i) = \mathscr{L}(\kappa_+)$.

Definition 3. A presheaf $\mathscr{F}$ on $\underline{\mathscr{W}}$ is a *sheaf* if and only if it satisfies the following two properties:

1. Separatedness: $\forall W \in \mathscr{W}$, $\forall$ global covers $\{W_i \mid i \in I\}$: $m \in \mathscr{F}(W)$ such that $\forall i \in I$: $\rho_{W_iW}^W(m) = 0$ in $\mathscr{F}(W_iW) \Rightarrow m = 0$.
2. Gluing: $\forall W \in \mathscr{W}$, $\forall$ global covers $\{W_i \mid i \in I\}$: given $(m_i) \in \prod_i \mathscr{F}(W_iW)$ such that:

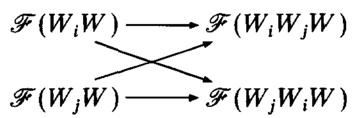
$$\rho_{W_iW_jW}^{W_iW}(m_i) = \rho_{W_iW_jW}^{W_jW}(m_j) \quad \forall (i,j) \in I \times I$$

then there exists an element m in $\mathscr{F}(W)$ such that

$$\rho_{W_iW}^{W}(m) = m_i \quad \forall i \in I.$$

Of course the element m whose existence is guaranteed in 2. is unique by the separatedness condition.

If we let a presheaf $\mathscr{F}$ act on the full subcategory of $\mathscr{W}$ consisting of all W_iW and W_iW_jW for a fixed word W and a fixed global cover $\{W_i \mid i \in I\}$, we get an inverse system represented by the picture

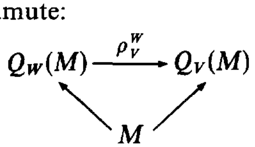


We denote the inverse limit of this system by

$$\varprojlim_{i,j} \mathscr{F}(W_iW).$$

This should not cause any confusion since an element of the inverse limit is already determined by its components in $\mathscr{F}(W_iW)$. We have that $\mathscr{F}$ is a sheaf if and only if, whenever $\{W_i \mid i \in I\}$ is a global cover, then the inverse limit of the above inverse system is isomorphic with $\mathscr{F}(W)$ for every $W \in \underline{\mathscr{W}}$.

Just as in the commutative case, we want a graded module M to determine a sheaf $\tilde{M}$. It is obvious that we have to define $\tilde{M}$ in the following way: $\tilde{M}(1) = Q_{\kappa_+}(M)$, $\tilde{M}(W) = Q_W(M) \forall W \in \mathscr{W} \backslash 1$ and if $V \to W$ then the restriction morphism $\rho_V^W : Q_W(M) \to Q_V(M)$ is the obvious morphism which makes the following diagram commute:



Unlike the commutative case, the sections of the sheaf associated to ${}_RR$ are not necessarily rings, e.g. $Q_TQ_S(R)$ is a ring $\Leftrightarrow$ $Q_TQ_S(R) \cong Q_{S\vee T}(R) \Leftrightarrow S$ and T are compatible. However. $\forall T \in \mathcal{O}$ we have that the sections $Q_T(R)$ are strongly graded rings since R is supposed to be generated in degree 1. In particular, $Q_T(R)$-gr is equivalent to $(Q_T(R))_0$-mod.

Notation. If $W = S_1 \dots S_n$ and $w \in W$, say $w = s_1 \dots s_n$, then the element $1/s_n \otimes \cdots \otimes 1/s_1 \otimes m$ of $Q_W(M)$ will be denoted by m/w. In particular, $m/1$ stands for $1 \otimes m$ in $Q_S(M)$, for $1 \otimes 1 \otimes m$ in $Q_T(Q_S(M))$ etc.; which element is meant depends on the module it belongs to, and that should always be clear from the context.

We need some lemmas in order to prove that $\tilde{M}$ is a sheaf in the above sense.

Lemma 2. *Given $m/w \in Q_W(R)$ and $a \in R$, then $\exists w' \in W$, $b \in R$ such that $w'a = bw$ and $a(m/w) = (bm)/w' \in Q_W(R)$.*

Proof. Suppose $W = S_1 \dots S_n$ and $w = s_1 \dots s_n$ $(s_i \in S_i)$. We put $a_n = a$ and we define inductively the a_i $(i = n, \dots, 0)$ by using repeatedly the second Ore-condition as follows: $s'_i a_i = a_{i-1} s_i$, $s'_i \in S_i$. Then $a(m/w) = a(1/s_n) \otimes \cdots \otimes 1/s_1 \otimes m = 1/s'_n \otimes a_{n-1}$ $(1/s_{n-1}) \otimes \cdots \otimes 1/s_1 \otimes m = \cdots = 1/s'_n \otimes \cdots \otimes 1/s'_n \otimes a_0 m$. It is then easy to see that $b = a_0$ and $w' = s'_1 \dots s'_n$ satisfy both statements above. □

Proposition 2. *The presheaf $\tilde{M}$ is separated.*

Proof. Fix a global cover $\{W_i \mid i \in I\}$. The case $W = 1$ being trivial, it suffices to show $\rho^W_{W_iW}(m) = 0 \Rightarrow m = 0$ for words $W \in \mathcal{W} \setminus \{1\}$. We arrange this by induction on the length of W:

(a) If W is a letter S and we have an element $m/s \in Q_S(M)$ such that $1/s \otimes 1 \otimes m = 0$ in $Q_S(Q_{W_i}(M))$ $\forall i \in I$ then $\exists s' \in S$ such that $s'm \in \bigcap_i \mathrm{Ker}(M \to Q_{W_i}(M))$ but the latter module is $\kappa_+(M)$ because R is schematic. This yields in particular that $s'm$ is S-torsion, so m is S-torsion and consequently $m/s = 0$ in $Q_S(M)$.

(b) Assume that $\rho^V_{W_iV}(m) = 0 \Rightarrow m = 0$ for all words V of length smaller than or equal to n and let W be a word of length $n + 1$, say $W = S_1 \dots S_n S$. Let $1/s \otimes 1/s_n \otimes \cdots \otimes 1/s_1 \otimes m$ be an element of $Q_W(M)$ such that $1/s \otimes 1/s_n \otimes \cdots \otimes 1/s_1 \otimes 1 \otimes m = 0$ in $Q_W(Q_{W_i}(M))$ for all $i \in I$ then there exists a s' in S such that $s'\ 1/s_n \otimes \cdots \otimes 1/s_1 \otimes 1 \otimes m = 0$ in $Q_W(Q_{W_i}(M))$ for all $i \in I$. The previous lemma yields $s'_1 \dots s'_n \in V = S_1 \dots S_n$ and $a \in R$ such that

$$0 = s'(1/s_n) \otimes \cdots \otimes 1/s_1 \otimes 1 \otimes m = 1/s'_n \otimes \cdots \otimes 1/s'_1 \otimes 1 \otimes am$$

in $Q_V(Q_{W_i}(M))$. By induction, we find that $1/s'_n \otimes \cdots 1/s'_1 \otimes am = 0$ in $Q_V(M)$, but this element equals $s'(1/s_n) \otimes \cdots \otimes 1/s_1 \otimes m$. This implies that $1/s \otimes 1/s_n \otimes \cdots \otimes 1/s_1 \otimes m = 0$ in $Q_S(Q_V(M)) = Q_W(M)$. □

Lemma 3. *If $m/w = 1 \otimes \cdots \otimes 1 \otimes n$ in $Q_W(M)$ for some $n \in M$ then there exists $\tilde{w} \in W$ and an $x \in R$ such that $\tilde{w} = xw$ and $\tilde{w}n = xm$.*

Proof. The proof proceeds by induction on the length n of the word $w = s_1 \ldots s_n$, the case $n = 1$ being well-known. Suppose the statement is true for words of length smaller than n. Write $w' = s_1 \ldots s_{n-1}$ then $1/s_n \otimes m/w' = 1 \otimes n/1$. This is an equality in the Ore-localisation of $Q_{W'}(M)$, W' being $S_1 \ldots S_{n-1}$, at the Ore-set S_n, so $\exists a, b \in R$ such that $a(m/w') = bn/1 \in Q_{W'}(M)$, $as_n = b \in S_n$. Using the previous lemma yields $\exists w'' \in W'$, $c \in R$ with $w''a = cw'$ and $bn/1 = a(m/w') = (cm)/w''$. Induction yields that there exists $w''' \in W'$ and $d \in R$ with $w''' = dw''$ and $w'''bn = dcm$. Putting $\tilde{w} = w'''b$ and $x = dc$ finishes the proof. $\square$

Proposition 3. *The separated presheaf $\tilde{M}$ is a sheaf, i.e. it satisfies the gluing condition.*

Proof. Fix a global cover $\{W_i \mid i \in I\}$. It is easy to see that it suffices to prove that the canonical map from $\tilde{M}(W) = Q_W(M)$ to the inverse limit $\Gamma_W(\tilde{M})$ of the inverse system

$$\begin{array}{ccc} Q_W(Q_{W_i}(M)) & \longrightarrow & Q_W(Q_{W_j}(Q_{W_i}(M))) \\ & \times & \\ Q_W(Q_{W_j}(M)) & \longrightarrow & Q_W(Q_{W_i}(Q_{W_j}(M))) \end{array}$$

is an isomorphism. In the particular case that $W = 1$, we will denote the inverse limit by $\Gamma_*(\tilde{M})$ rather than by $\Gamma_1(\tilde{M})$. Because a finite inverse limit and an exact functor commute, we find that $\Gamma_W(\tilde{M}) \cong Q_W(\Gamma_*(\tilde{M}))$, so we are done if we can prove that $\Gamma_*(\tilde{M}) \cong Q_{\kappa_+}(M)$. There is a unique R-linear map $\varphi : M \to \Gamma_*(\tilde{M})$ by the universal property of the inverse limit. It is easily seen that the kernel of this map is exactly $\kappa_+(M)$ due to the fact that R is schematic. If we are able to show that $\operatorname{Coker}\varphi$ is κ_+-torsion, then it will follow from general torsion theory that $Q_{\kappa_+}(M) \cong \Gamma_*(\tilde{M})$.

Consider an element $\xi = (m_i/w_i)_i$ of $\Gamma_*(\tilde{M})$, i.e. $w_i \in W_i$ and $1/w_i \otimes 1 \otimes m_i = 1 \otimes 1/w_j \otimes m_j$ $\forall i,j$. We claim that there exists a natural number n such that $\forall j$: $(R_+)^n\, m_j/w_j$ is contained in the image of the canonical map $M \to Q_{W_j}(M)$. Fix j, then the previous lemma entails for each i the existence of $w_i' = a_i w_i \in W_i$ such that $w_i'(m_j/w_j) = a_i m_i/1 \in Q_{W_j}(M)$. By definition of a covering, we can find a natural number n such that $(R_+)^n\, m_j/w_j$ is contained in the image of $M \to Q_{W_j}(M)$. This n depends on j but since we assume the $(W_i)_i$ to be a finite cover, we can take n big enough to work for all j.

Given a in $(R_+)^n$, then $a\xi = (n_i/1)_i$ for some n_i in M such that $1 \otimes 1 \otimes n_i = 1 \otimes 1 \otimes n_j$ in $Q_{W_i}(Q_{W_j}(M))$ for all i, j. Fix i, then for all j there exists $w_j \in W_i$ such that $w_j(n_i/1) = w_j(n_j/1)$ in $Q_{W_j}(M)$ by the previous lemma. Since we are working with a finite cover, we can find $w \in W_i$ such that $w(n_i/1) = w(n_j/1)$ for all j, i.e. $wa\xi = \varphi(wn_i)$. This being true for all i yields that there exists a natural number $n(a)$ (depending on a) such that $(R_+)^{n(a)} a\xi \subseteq \varphi(M)$. Finally, the ideal $(R_+)^n$ is finitely generated, say by $a_1, \ldots, a_r$. Put $m = \max\{n(a_k) \mid k = 1, \ldots, r\}$ then $(R_+)^{m+n}\xi$ is contained in the image of $\varphi: M \to \Gamma_*(\tilde{M})$. $\square$

Definition 4. An *affine* cover is a finite subset $\{T_i \mid i \in I\}$ of $\mathcal{O}$ such that $\bigcap_{i \in I} \mathcal{L}(T_i) = \mathcal{L}(\kappa_+)$.

Definition 5. A sheaf $\mathcal{F}$ is *quasi-coherent* if and only if there exists an affine cover $\{T_i \mid i \in I\}$ and for each i in I there exists a graded $Q_{T_i}(R)$-module M_i such that for all morphisms $V \to W$ in $\underline{\mathcal{W}}$ we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(T_iW) & \xrightarrow{\rho^{T_iW}_{T_iV}} & \mathcal{F}(T_iV) \\ \downarrow \iota & & \downarrow \iota \\ Q_W(M_i) & \longrightarrow & Q_V(M_i) \end{array}$$

in which the vertical maps are isomorphisms in $R - gr$. $\mathcal{F}$ is called *coherent* if moreover all M_i are *finitely generated* $Q_{T_i}(R)$-modules.

Example 1. The sheaf $\tilde{M}$ is quasi-coherent (resp. coherent) for each graded (resp. finitely generated graded) R-module M.

In fact, we will prove that every quasi-coherent sheaf has to be of the form $\tilde{M}$ for some graded R-module M.

Remark 1. One may ask why we only require isomorphisms between sections on words beginning with a letter of the affine cover. It seems reasonable to ask for isomorphisms $Q_W(M_i) \cong \mathcal{F}(W)$ on all words which admit a morphism to some T_i, i.e. for all words which contain some letter of the affine cover. The reason is that one can then not expect that each finitely generated module induces a coherent sheaf: in fact we can prove that if each $\tilde{M}$ were coherent in this way then all T_i and T_j have to be compatible.

Theorem 1. *If $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{W}$ and $\Gamma_*(\mathcal{F})$ denotes its global sections $\mathcal{F}(1)$ then $\mathcal{F}$ is isomorphic to $\widetilde{\Gamma_*(\mathcal{F})}$, the sheaf associated to $\Gamma_*(\mathcal{F})$.*

Proof. We have to prove that $\mathcal{F}(W) \cong Q_W(\Gamma_*(\mathcal{F}))$ for all W in $\mathcal{W}$. Let $\{T_i \mid i \in I\}$ be an affine cover as in the definition of a quasi-coherent sheaf and fix a word W in $\underline{\mathcal{W}}$. Now

$$Q_W(\Gamma_*(\mathcal{F})) \cong \varprojlim_i Q_W(\mathcal{F}(T_i)) \quad \text{(finite inverse limit)},$$

$$\mathcal{F}(W) \cong \varprojlim_i \mathcal{F}(T_iW) \quad (\mathcal{F} \text{ is a sheaf}).$$

From the coherence of $\mathscr{F}$, we know that we have isomorphisms $\psi_i: Q_W(\mathscr{F}(T_i)) \to \mathscr{F}(T_iW)$ and $\psi_{ij}: Q_W(\mathscr{F}(T_iT_j)) \to \mathscr{F}(T_iT_jW)$. We will need the explicit form of these canonical maps, that is why we introduce the next convenient notation. If W is $S_1 \dots S_n$ then we define W_i to be $S_1 \dots S_i$. In particular, $W_n = W$ and $W_1 = S_1$. Then

$$\psi_i\left(\frac{1}{s_n}\otimes\cdots\otimes\frac{1}{s_1}\otimes m\right) = s_n^{-1}\rho^{T_iW_{n-1}}_{T_iW_n}(s_{n-1}^{-1}\rho^{T_iW_{n-2}}_{T_iW_{n-1}}(\dots s_2^{-1}\rho^{T_iW_1}_{T_iW_2}(s_1^{-1}\rho^{T_i}_{T_iW_1}(m))\dots))$$

and

$$\begin{aligned}&\psi_{ij}\left(\frac{1}{s_n}\otimes\cdots\otimes\frac{1}{s_1}\otimes m\right)\\ &= s_n^{-1}\rho^{T_iT_jW_{n-1}}_{T_iT_jW_n}(s_{n-1}^{-1}\rho^{T_iT_jW_{n-2}}_{T_iT_jW_{n-1}}(\dots s_2^{-1}\rho^{T_iT_jW_1}_{T_iT_jW_2}(s_1^{-1}\rho^{T_iT_j}_{T_iT_jW_1}(m))\dots)).\end{aligned}$$

We want to show that these isomorphisms induce an isomorphism between the inverse limits of the inverse systems $\mathscr{F}(T_iW)$ and $Q_W(\mathscr{F}(T_i))$. Thus we must check that the following diagram commutes:

$$\begin{array}{ccc} Q_W(\mathscr{F}(T_i)) & \xrightarrow{Q_W(\rho^{T_i}_{T_iT_j})} & Q_W(\mathscr{F}(T_iT_j)) \\ \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_{ij}} \\ \mathscr{F}(T_iW) & \xrightarrow{\rho^{T_iW}_{T_iT_jW}} & \mathscr{F}(T_iT_jW) \end{array}$$

Take $\xi = 1/s_n \otimes \cdots \otimes 1/s_n \otimes m$ in $Q_W(\mathscr{F}(T_i))$. On one hand,

$$\begin{aligned}&\psi_{ij}(Q_W(\rho^{T_i}_{T_iT_j})(\xi))\\ &= \psi_{ij}\left(\frac{1}{s_n}\otimes\cdots\otimes\frac{1}{s_1}\rho^{T_i}_{T_iT_j}(m)\right)\\ &= s_n^{-1}\rho^{T_iT_jW_{n-1}}_{T_iT_jW_n}(s_{n-1}^{-1}\rho^{T_iT_jW_{n-2}}_{T_iT_jW_{n-1}}(\dots s_2^{-1}\rho^{T_iT_jW_1}_{T_iT_jW_2}(s_1^{-1}\rho^{T_iT_j}_{T_iT_jW_1}(\rho^{T_i}_{T_iT_j}(m)))\dots)).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\rho^{T_iW}_{T_iT_jW}(\psi_i(\xi))\\ &= \rho^{T_iW}_{T_iT_jW}(s_n^{-1}\rho^{T_iW_{n-1}}_{T_iW_n}(s_{n-1}^{-1}\rho^{T_iW_{n-2}}_{T_iW_{n-1}}(\dots s_2^{-1}\rho^{T_iW_1}_{T_iW_2}(s_1^{-1}\rho^{T_i}_{T_iW_1}(m))\dots))).\end{aligned}$$

Since both $\mathscr{F}(T_iW)$ and $\mathscr{F}(T_iT_jW)$ are $Q_{S_n}(R)$-modules, we find that $\rho^{T_iW}_{T_iT_jW}$ is $Q_{S_n}(R)$-linear. Using this and the fact that

$$\rho^{T_iW}_{T_iT_jW}\circ\rho^{T_iW_{n-1}}_{T_iW_n} = \rho^{T_iT_jW_{n-1}}_{T_iT_jW_n}\circ\rho^{T_iW_{n-1}}_{T_iT_jW_{n-1}},$$

we find that the above element is equal to:

$$s_n^{-1}\rho^{T_iT_jW_{n-1}}_{T_iT_jW_n}(\rho^{T_iW_{n-1}}_{T_iT_jW_{n-1}}(s_{n-1}^{-1}\rho^{T_iW_{n-2}}_{T_iW_{n-1}}(\dots s_2^{-1}\rho^{T_iW_1}_{T_iW_2}(s_1^{-1}\rho^{T_i}_{T_iW_1}(m))\dots))).$$

Using this argument repeatedly, we end with:

$$s_n^{-1}\rho_{T_iT_jW_n}^{T_iT_jW_{n-1}}(s_{n-1}^{-1}\rho_{T_iT_jW_{n-1}}^{T_iT_jW_{n-2}}(\ \dots\ s_2^{-1}\rho_{T_iT_jW_2}^{T_iT_jW_1}(s_1^{-1}\rho_{T_iT_jW_1}^{T_iW_1}(\rho_{T_iW_1}^{T_i}(m)))\ \dots\ ))$$

and of course this is the same as the result of the first calculation. Strictly speaking, we should also prove that the following diagram is commutative:

$$\begin{array}{ccc} Q_W(\mathscr{F}(T_i)) & \xrightarrow{Q_W(\rho_{T_jT_i}^{T_i})} & Q_W(\mathscr{F}(T_jT_i)) \\ \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_{ji}} \\ \mathscr{F}(T_iW) & \xrightarrow{\rho_{T_jT_iW}^{T_iW}} & \mathscr{F}(T_jT_iW) \end{array}$$

but we omit its proof since it is completely similar. Thanks to these commutative diagrams, we conclude that the two inverse limits

$$\varprojlim_{i,j} Q_W(\mathscr{F}(T_i)) \quad \text{and} \quad \varprojlim_{i,j} \mathscr{F}(T_iW)$$

are isomorphic. Consequently, we get an isomorphism $Q_W(\Gamma_*(\mathscr{F})) \cong \mathscr{F}(W)$. We claim that it maps an element $\xi = 1/s_n \otimes \cdots \otimes 1/s_1 \otimes x \in Q_W(\Gamma_*(\mathscr{F}))$ to $s_n^{-1}\rho_{W_n}^{W_{n-1}}(s_{n-1}^{-1}\rho_{W_{n-1}}^{W_{n-2}}(\ \dots\ s_2^{-1}\rho_{W_2}^{W_1}(s_1^{-1}\rho_{W_1}(x))\ \dots\ ))$. It is easy to see that ξ is mapped to the element

$$(s_n^{-1}\rho_{T_iW_n}^{T_iW_{n-1}}(s_{n-1}^{-1}\rho_{T_iW_{n-1}}^{T_iW_{n-2}}(\ \dots\ s_2^{-1}\rho_{T_iW_2}^{T_iW_1}(s_1^{-1}\rho_{T_iW_1}(x))\ \dots\ )))_i \in \varprojlim_i \mathscr{F}(T_iW).$$

Then it suffices to prove that

$$\begin{aligned} &\rho_{T_iW}^{W}(s_n^{-1}\rho_{W_n}^{W_{n-1}}(s_{n-1}^{-1}\rho_{W_{n-1}}^{W_{n-2}}(\ \dots\ s_2^{-1}\rho_{W_2}^{W_1}(s_1^{-1}\rho_{W_1}(x))\ \dots\ ))) \\ &\quad = s_n^{-1}\rho_{T_iW_n}^{T_iW_{n-1}}(s_{n-1}^{-1}\rho_{T_iW_{n-1}}^{T_iW_{n-2}}(\ \dots\ s_2^{-1}\rho_{T_iW_2}^{T_iW_1}(s_1^{-1}\rho_{T_iW_1}(x))\ \dots\ )) \end{aligned}$$

for all i in I. This is done using essentially the same argument as before, i.e. a repeated combination of the $Q_{S_k}(R)$-linearity of the restriction morphism between two words ending on S_k and the fact that $\rho_{TSW}^{SW} \circ \rho_{SW}^{W} = \rho_{TSW}^{TW} \circ \rho_{TW}^{W}$. There remains one thing to be proved, namely that these maps patch together to a sheaf isomorphism, that is we have to prove that for each morphism $V \to W$ in $\underline{\mathscr{W}}$ the following diagram commutes:

$$\begin{array}{ccc} Q_W(\Gamma_*(\mathscr{F})) & \longrightarrow & Q_V(\Gamma_*(F)) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ \mathscr{F}(W) & \xrightarrow{\rho_V^W} & \mathscr{F}(V) \end{array}$$

We omit the proof since this is again done in the same way, up to a minor modification taking into account the imbedding of W into V. □

Now we are ready for:

Theorem 2. *The category of quasi-coherent sheaves is equivalent to* (R, κ_+)-gr.

Proof. As in the commutative case, the functors establishing the equivalence will be $\tilde{\ }$ and Γ_*. We still have to prove that the global sections functor takes its values in (R, κ_+)-gr, i.e. that $\Gamma_*(\mathscr{F})$, the global sections of a quasi-coherent sheaf $\mathscr{F}$, is a κ_+-closed R-module. Choose a natural number n and consider the canonical map $\varphi : \Gamma_*(\mathscr{F}) \to \mathrm{Hom}_R((R_+)^n, \Gamma_*(\mathscr{F}))$. φ is injective because $\mathrm{Ker}\,\varphi \subseteq \kappa_+(\Gamma_*(\mathscr{F})) = \bigcap_{i \in I} \kappa_{T_i}(\Gamma_*(\mathscr{F})) = 0$, the latter equality holds because $\mathscr{F}$ is separated. To prove the surjectivity of φ, we take an element f in $\mathrm{Hom}_R((R_+)^n, \Gamma_*(\mathscr{F}))$. Composing this map with the restriction morphisms yields elements $\rho_{T_i} \circ f$ in $\mathrm{Hom}_R((R_+)^n, \mathscr{F}(T_i)) \cong \mathscr{F}(T_i)$. Thus for each $i \in I$, there is a $x_i \in \mathscr{F}(T_i)$ such that $\rho_{T_i}(f(a)) = a x_i$ for all $a \in (R_+)^n$. The calculation $a \rho^{T_i}_{T_i T_j}(x_i) = \rho^{T_i}_{T_i T_j}(\rho_{T_i}(f(a))) = \rho^{T_j}_{T_i T_j}(\rho_{T_j}(f(a))) = a \rho^{T_j}_{T_i T_j}(x_j)$ for all $a \in (R_+)^n$ yields that $\rho^{T_i}_{T_i T_j}(x_i) = \rho^{T_j}_{T_i T_j}(x_j)$. The gluing condition on $\mathscr{F}$ implies that $\exists x \in \Gamma_*(\mathscr{F})$ such that $\rho_{T_i}(x) = x_i$ and of course this x is mapped to f by φ. We conclude that $\Gamma_*(\mathscr{F})$ is a κ_+-closed module. At this moment, we know that there is a functor from the category of quasi-coherent sheaves to (R, κ_+)-gr and one in the reverse direction. The fact that these functors are equivalences follows readily from Proposition 3 and Theorem 1. □

Theorem 3. *The category of coherent sheaves is equivalent to Proj R.*

Proof. The coherence of $\mathscr{F}$ yields the existence of a finite number of global sections $x_1, \ldots, x_t \in \Gamma_*(\mathscr{F})$ such that $\forall i \in I$: $\rho_{T_i}(x_1), \ldots, \rho_{T_i}(x_t)$ generate $\mathscr{F}(T_i)$ as a $Q_{T_i}(R)$-module. It is now easy to show that $\Gamma_*(\mathscr{F})$ is torsion over the R-submodule generated by $x_1, \ldots, x_t$. □

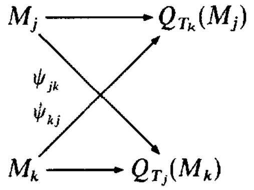
In practice, one is interested in the following question: given some modules over some localisations of the ring R, do they come from an R-module? The next theorem provides a criterion in case R is schematic:

Theorem 4. *Let* $\{T_i \mid i \in I\}$ *be an affine cover and suppose we are given a graded* $Q_{T_i}(R)$*-module* M_i *for each i and homomorphisms* $\psi_{ij} : M_i \to Q_{T_i}(M_j)$ *in* R-gr *such that for each triple* (i, j, k) *the next diagram commutes*:

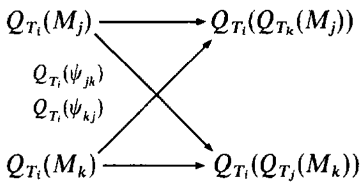
$$\begin{array}{ccc} M_i & \xrightarrow{\psi_{ij}} & Q_{T_i}(M_j) \\ {\scriptstyle \psi_{ik}}\downarrow & & \downarrow{\scriptstyle Q_{T_i}(\psi_{jk})} \\ Q_{T_i}(M_k) & \longrightarrow & Q_{T_i}(Q_{T_j}(M_k)) \end{array}$$

then there exists a graded R-module M such that for each i in I we have that $Q_{T_i}(M) \cong M_i$.

Proof. Put M equal to the inverse limit of the inverse system



Since this is a finite inverse limit, we get that $Q_{T_i}(M)$ is isomorphic with the inverse limit of



and we use this module, denoted by N, to construct an isomorphism to M_i. We have a well-defined map $M_i \to N$ because of the gluing condition above and we have a map ψ from $Q_{T_i}(M)$ to M_i defined by $\psi((m_j)_j/s) = s^{-1}m_i$ where $m_j \in M_j$ such that $\psi_{jk}(m_j) = m_k/1 \in Q_{T_j}(M_k)\ \forall j,k \in I$ and $s \in T_i$. It is now an easy exercise to prove that these two maps are each others inverse. □

The gluing condition above looks the same as in the commutative case, but then it implies that $Q_{T_j}(M_i) \cong Q_{T_i}(M_j)$ as $Q_{T_i \vee T_j}(R)$-modules.

One of the reasons why we did not want to assume the existence of compatible covers is that this property does not lift from the associated graded of a filtered ring to its Rees-ring. At the cost of introducing completions (the so-called micro-localisations), this property does lift and consequently, for a wide variety of algebras is the usual (commutative) Grothendieck-topology sufficient again.

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