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The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations

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Abstract

The tanh (or hyperbolic tangent) method is a powerful technique to search for travelling waves coming out from one-dimensional nonlinear wave and evolution equations. In particular, in those problems where dispersive effects, reaction, diffusion and/or convection play an important role. To show the strength of the method, an overview is given to find out which kind of problems are solved with this technique and how in some nontrivial cases this method, adapted to the problem at hand, still can be applied. Single as well as coupled equations, arising from wave phenomena which appear in different scientific domains such as physics, chemical kinetics, geochemistry and mathematical biology, will be treated. Next, attention is focussed towards approximate solutions. As a result, solitary- and shock-wave profiles are derived together with the associated width and velocity. The same method can be easily extended so that difference-differential equations can be similarly solved. Finally, some extensions of the method are discussed.

Keywords: Nonlinear travelling waves

1. General introduction

In many domains of physics (fluid dynamics [24], plasma physics [4], solid-state physics [21], chemistry (chemical kinetics [7]), mathematical biology (population dynamics for instance [19]) nonlinear wave phenomena frequently appear.

Due to the increased interest in those problems, a whole range of analytical solution methods such as Hirota's bilinear technique [9], inverse scattering transform [1], Painlevé analysis [3], etc. were developed. Moreover, numerical simulation with even more powerful computers have revolutionised

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current methods of computation. Those analytical methods, however, are in most cases difficult to handle and require a thorough knowledge of its properties and possibilities before one is able to apply them to the problem one tries to solve.

Our aim is to give an overview of the possibilities this tanh method offers together with some selected examples. It is based on the research we have undertaken during the last decade and the progress made by other researchers in the field. More details of the basics can be found back in [14–16], where numerous other examples are treated. For the sake of completeness, we should mention that this technique is restricted to the search for stationary waves and we thus essentially deal with shock and/or solitary type of solutions. Moreover, we are inherently limited to one dimension (or direction of propagation). Nevertheless, numerous equations have been already solved in this way and, due to some generalisations of the method, this number is still increasing.

This paper is organised as follows. In Section 2, we recall the main features of the tanh method in the cases with and without boundary conditions together with some well-chosen examples. In Section 3, some exceptional cases are treated as well as coupled equations. In Section 4, this technique is applied to get approximate solutions while in Section 6 other generalisation of this technique are highlighted. Finally, Section 6 is devoted to concluding remarks.

2. Outline of the tanh method: exact solutions

The main properties of the method will be explained and then applied to particular and well-chosen examples.

2.1. Theoretical analysis

The nonlinear wave and evolution equations we want to investigate (in principle one dimension) are commonly written as

$$u_t = [u, u_x, u_{xx}, \dots] \quad \text{or} \quad u_{tt} = [u, u_x, u_{xx}, \dots]. \tag{1}$$

We like to know whether travelling waves (or stationary waves) are solutions of (1). The first step is to unite the independent variables x and t into one particular variable through the definition $\xi = c(x - vt)$. Here c (>0) represents the wave number and V the (unknown) velocity of the travelling wave. We recall that the wave number c is inversely proportional to the width of the wave. Depending on the problem under study, this quantity will be determined or will remain a free and arbitrary parameter. Typically for nonlinear waves, both velocity V as well as amplitude will be functions of this parameter c. Accordingly, the quantity $u(x,t)$ is replaced by $U(\xi)$, so that we deal with ODEs rather than with PDEs. In this way, equations like (1) are transformed into

$$-cV\frac{\mathrm{d}U}{\mathrm{d}\xi} = \left[U, c\frac{\mathrm{d}U}{\mathrm{d}\xi}, c^2\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}, \dots\right] \quad \text{or} \quad c^2V^2\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} = \left[U, c\frac{\mathrm{d}U}{\mathrm{d}\xi}, c^2\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}, \dots\right]. \tag{2}$$

Our main goal is to derive exact or at least approximate solutions, if possible, for those ODEs. For this purpose, we introduce a new variable $Y = \tanh\xi$, which is used throughout this paper, into the ODE. This latter then depends solely on Y, because all derivatives $\mathrm{d}/\mathrm{d}\xi$ are now replaced by $(1 - Y^2)\mathrm{d}/\mathrm{d}Y$ in (2). The solution(s) we are looking for will be written as a finite power series in Y, limiting them to solitary- and shock-wave profiles. Depending on the fact whether boundary conditions are involved or not, we distinguish two different cases.

1. *No boundary conditions are imposed.* The following solution of (2) is proposed:

$$F(Y) = \sum_{n=0}^{N} a_n Y^n. \tag{3}$$

To determine N (highest order of Y), the following balancing procedure is used. At least two terms proportional to Y^N must appear after substitution of ansatz (3) into the equation under study. As a result of this analysis, we definitely require $a_{N+1} = 0$ and $a_N \neq 0$ for a particular N. It turns out that $N = 1$ or 2 in most cases. This balance (and thus N) is obtained by comparing the behaviour of Y^N in the highest derivative against its counterpart within the nonlinear term(s). As soon as N is determined in this way, we get after substitution of (3) into (2) (transformed to the Y variable) algebraic equations for a_n $(n=0,1,\ldots,N)$. Depending on the problem under study, the wave number c will remain fixed or undetermined. As already mentioned, the velocity V of the travelling wave is always a function of c. If one is able to find nontrivial values for a_n $(n = 0,1,\ldots,N)$, in terms of known quantities, a solution is ultimately obtained.

2. *Implementation of boundary conditions.* To reduce superfluous calculations, boundary conditions may be implemented a priori. We restrict the analysis to problems with vanishing front or tail. If the solution must vanish for $\xi \to +\infty$ $(Y \to +1)$, series (3) must admit the form

$$F(Y) = (1 - Y)^m \sum_{n=0}^{N-m} a_n Y^n, \quad \text{where } m = 1,\ldots,N. \tag{4}$$

The integer m is not yet determined since we do not know how fast the proposed solution (4) vanishes for $Y \to +1$. Several options need then to be investigated because $m = 1,2,\ldots, \leqslant N$. At the other hand, lesser coefficients a_n $(n = 0,1,\ldots,N - m)$ are involved.

If the solution must vanish for $\xi \to -\infty$ $(Y \to +1)$, we should start with

$$F(Y) = (1 + Y)^m \sum_{n=0}^{N-m} a_n Y^n, \quad \text{where } m = 1,\ldots,N. \tag{5}$$

In both cases a shock wave (or kink) type solution is expected.

If the solution should vanish at both sides, i.e., $\xi \to \pm\infty$ $(Y \to \pm 1)$, one must require

$$F(Y) = (1 - Y)^p (1 + Y)^q \sum_{n=0}^{N-p-q} a_n Y^n, \quad \text{where } p(\neq 0) + q(\neq 0) = 2,\ldots,N. \tag{6}$$

This form is at least proportional to $(1-Y^2)$ so that a solitary-wave profile appears. Depending on the problem one wants to study, one can choose between (3), (4), (5) or (6). The balancing procedure obviously is independent of this choice. There is no guarantee, however, that exact solutions can be obtained for all cases. Remark further that the velocity V can be determined from the asymptotic behaviour of (4) (i.e., for $Y \to +1$), (5) (i.e., for $Y \to -1$) or (6) (see [15]). This feature is very useful when searching for approximate solutions (see Section 3.1).

2.2. Examples

To illustrate this technique and the possibilities it offers, we shall investigate the Korteweg–de Vries (KdV)–Burgers equation, a combination of two famous and fundamental nonlinear wave equations. We thus start with

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - a\,\frac{\partial^2 u}{\partial x^2} + b\,\frac{\partial^3 u}{\partial x^3} = 0. \tag{7}$$

The positive parameters a and b refer, respectively, to a dissipative and dispersive effect. For $a=0$ we get the KdV equation and for $b=0$ the Burgers equation. Following the above proposed procedure we first transform (7) into

$$-cV\,\frac{dU}{d\xi} + U\,\frac{dU}{d\xi} - ac^2\,\frac{d^2U}{d\xi^2} + bc^3\,\frac{d^3U}{d\xi^3} = 0. \tag{8}$$

Next, we introduce $Y = \tanh(\xi)$ so that we arrive at

$$-cV(1-Y^2)\,\frac{dF(Y)}{dY} + cF(Y)(1-Y^2)\,\frac{dF(Y)}{dY} - ac^2(1-Y^2)\,\frac{d}{dY}\left[(1-Y^2)\,\frac{dF(Y)}{dY}\right]$$
$$+\,bc^2(1-Y^2)\,\frac{d}{dY}\left\{(1-Y^2)\,\frac{d}{dY}\left[(1-Y^2)\,\frac{dF(Y)}{dY}\right]\right\} = 0. \tag{9}$$

1. *No boundary conditions*. After substitution of (3) into (9), it turns out that highest power Y^N evolves in (6) as Y^{2N+1} in the second term and as Y^{N+3} in the last term of (9) (note: nth derivative gives order $N+n$). The balancing procedure then leads to the fundamental relation $2N+1=N+3$ or $N=2$. A possible solution can thus be found by using (3) for $N=2$. After substitution into (9) we get the following result, after some algebra (preferably performed with symbolic software):

$$c = \frac{a}{10b} \quad \text{and} \quad F(Y) = a_0 - \frac{6}{25}\,\frac{a^2}{b}\,Y - \frac{3}{25}\,\frac{a^2}{b}\,Y^2$$

with

$$V = a_0 - \frac{3}{25}\,\frac{a^2}{b}, \tag{10}$$

where the arbitrary constant a_0 affects the solution (and therefore its boundary condition) as well as the velocity of the stationary wave.

Note that the wave number $c(=a/10b)$ is now completely defined. This result may be regarded as a precise balance between dispersive and dissipative effects.

For $a=0$ (KdV case) the same analysis can be performed and we ultimately get

$$N = 2 \quad \text{and} \quad F(Y) = a_0 - 12bc^2Y^2 \quad \text{with } V = a_0 - bc^2. \tag{11}$$

The wave number c remains a free and arbitrary parameter, in contrast with foregoing case.

For $b=0$ (Burgers' case) we similarly obtain

$$N = 1 \quad \text{and} \quad F(Y) = a_0 - 2acY \quad \text{with } V = a_0, \tag{12}$$

where c is again not fixed.

2. *With boundary conditions*. We make here the particular choice that $U(\xi)$ (and its derivatives) vanish for $\xi \to +\infty$ ($y \to +1$). Then we are able to integrate (8):

$$-cVU(\xi) + \frac{1}{2}cU(\xi)^2 - ac^2\frac{du}{d\xi} + bc^3\frac{d^2u}{d\xi^2} = 0, \tag{13}$$

a simpler equation than in previous case. After substitution of ansatz (3) into (13) (first transformed to the Y variable), we get after some algebra

$$F(Y) = \frac{3}{25}\frac{a^2}{b}(1-Y)\left(1+\frac{Y}{3}\right) = \frac{3}{25}\frac{a^2}{b}(1-Y^2) + \frac{6}{25}\frac{a^2}{b}(1-Y)$$

with

$$V = \frac{6}{25}\frac{a^2}{b}$$

and

$$c = \frac{a}{10b}. \tag{14}$$

The same result (a combination of a solitary wave and a shock wave) may be derived from (10), imposing the relation $a_0 = \frac{9}{25}(a^2/b)$ so that this solution, as required, vanishes for $\xi \to +\infty$ ($Y \to +1$).

For $a = 0$ (KdV case) we get

$$N = 2 \quad \text{and} \quad F(Y) = 12bc^2(1-Y)(1+Y) = 12bc^2(1-Y^2), \tag{15}$$

the well-known solitary wave in bell-shape form.

For $b = 0$ (Burgers' case) we get almost immediately

$$N = 1 \quad \text{and} \quad F(Y) = 2ac(1-Y) \quad \text{with } V = 2ac. \tag{16}$$

The very same results as (15) and (16) can be obtained by imposing the corresponding boundary condition $F(Y \to \pm 1)$ in (11) and (12), respectively.

3. Exceptional cases

The just-described technique can be generally used and, up to now, many nonlinear equations has been successfully solved in this way. There are, however, some cases for which ansatz (3)–(6) gives no immediate results. It turns out that in many cases a (slight) modification, transformation or generalisation, without changing the basic principles, may eventually lead to solutions. Some of these cases are listed below.

3.1. Matching problem

If a balancing procedure leads to $N = 1$, ansatz (6) cannot be used, since p and q (both different from zero) cannot behave as integers. For such cases, a possible alternative is

$$(1-Y)^{1/2}(1+Y)^{1/2}\sum_{n=0}^{N-1} a_n Y^n = (1-Y^2)^{1/2}\sum_{n=0}^{N-1} a_n Y^n. \tag{17}$$

A typical example in this situation is the modified KdV equation and related equations. The mKdV(+) equation is written as

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} = 0. \tag{18}$$

With ansatz (3) the balancing procedure gives $N = 1$, but no solution of type (3), (4) or (5) is found. Ansatz (6) is prohibited as seen before. Applying (17) gives the well-known solitary-wave solution

$$c\sqrt{6b}(1 - Y^2)^{1/2} \quad \text{with } V = bc^2. \tag{19}$$

Note that, in contrast with (18), the following mKdV(−) equation

$$\frac{\partial u}{\partial t} - u^2 \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} = 0 \tag{20}$$

has the solution

$$c\sqrt{6b}Y \quad \text{with } V = -2bc^2, \tag{21}$$

acquired with the aid of ansatz (3) with $N = 1$. In contrast with the former mKdV(+) equation, (20) has no solution like (19).

3.2. Change of the dependent variable

A transformation of the dependent variable is necessary, before the solution procedure can be applied. For instance, the wave equation describing foam drainage, due to Weaire and Verbist [22], is a typical example. It reads like

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[u^2 - \frac{\sqrt{u}}{2}\frac{\partial u}{\partial x}\right]. \tag{22}$$

To eliminate the square root in the r.h.s. of (22) we obviously define

$$u(x,t) = w^2(x,t). \tag{23}$$

Then a shock-like solution is easily found with ansatz (3) to be

$$u(x,t) = c^2 Y^2 \quad \text{with } V = c^2. \tag{24}$$

Note that the same transformation can also be used for the mKdV(+) equation (18), so that the previous special procedure can be avoided.

3.3. Very particular case

The Gardner equation or combined KdV–mKdV equation poses some difficulty at first sight because two different nonlinear terms appear.

This (rescaled) equation reads

$$\frac{\partial u}{\partial t} - 2Bu\frac{\partial u}{\partial x} - 3Cu^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{25}$$

Taking into account vanishing boundary conditions, we get after transformation to the ξ variable and one integration

$$-cVU - cBU^2 - cCU^3 + c^2\frac{d^2U}{d\xi^2} = 0. \tag{26}$$

Finally, we arrive at

$$-VF(Y) - BF(Y)^2 - CF(Y)^3 + c^2(1-Y^2)\frac{d}{dY}\left[(1-Y^2)\frac{dF(Y)}{dY}\right] = 0. \tag{27}$$

Ansatz (6), with vanishing front and tail, fails to work in this case (as well as the other cases, including (17)), because the balancing procedure gives $N = 1$. At the other hand, it is clear from (27) that a possible solution must be proportional to $(1 - Y^2)$. To keep the balance between terms, an intrinsic property of the tanh technique, we could start the analysis with a solution proportional to $(1 - Y^2)/f(Y)$ where $f(Y)$ must be of order 2 in Y. For more details we refer to [15].

3.4. Coupled equations

This technique obviously can also be used to find stationary solutions of coupled equations. If one for instance deals with two coupled equations and thus two unknowns, two different expansions like (3)–(6) should then be used. A striking example is the following set, arising in geochemistry (see [8]):

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} - \frac{ur}{\varepsilon^2} \quad \text{and} \quad \frac{\partial r}{\partial t} = \frac{ur}{\varepsilon^2}. \tag{28}$$

The quantity $u(x,t)(>0)$ represents a certain substance in water, moving with a given velocity v and reacting with some concentration $r(x,t)(>0)$. A fast reaction is expected, so the nonlinear reaction term is divided by a small number ε.

The following relevant boundary conditions are put forward:

$$u(x,t) \to u_\infty \quad \text{and} \quad r(x,t) \to 0 \quad \text{as } x \to -\infty, \tag{29}$$

$$u(x,t) \to 0 \quad \text{and} \quad r(x,t) \to r_\infty \quad \text{as } x \to +\infty. \tag{30}$$

With the aid of the independent variable $\xi = c(x - Vt)$ and the new variable $Y = \tanh\xi$, together with an ansatz like (3) for both $U(\tanh\xi) = U(Y) = u(x,t)$ and $R(\tanh\xi) = R(Y) = r(x,t)$, we ultimately get

$$U(Y) = 2c\varepsilon^2 V(1-Y) = 2c\varepsilon^2(v - 2cD)(1-Y), \tag{31}$$

$$R(Y) = 2c^2\varepsilon^2 D(1-Y)^2 \tag{32}$$

and

$$V = v - 2cD \quad (v > V). \tag{33}$$

The free parameters V and c are completely determined by the given boundary conditions

$$V = v\frac{u_\infty}{u_\infty + r_\infty}, \qquad c = \frac{1}{2D}\left(v - \frac{u_\infty}{u_\infty + r_\infty}\right). \tag{34}$$

4. The tanh method as perturbation method

In the case that no exact solutions can be derived, one may try to find approximate solutions. It is clear that, in one way or another, a smallness parameter must be available to use this technique as a perturbation method. Different cases can be distinguished.

4.1. Equation with small term(s)

First, if a small term (or terms) is (are) present in the equation under study, one obviously may first investigate the equation without such terms. If in the lowest order the equation can be solved exactly, a perturbation analysis can be made to take into account the effects of the remaining (small) term(s).

A typical example is again given by the KdV–Burgers equation, already solved exactly. Two different cases can be investigated:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - a\,\frac{\partial^2 u}{\partial x^2} = -b\,\frac{\partial^3 u}{\partial x^3} \sim \mathrm{O}(\varepsilon) \quad \text{with } \varepsilon \ll 1, \tag{35}$$

so that $b = \mathrm{O}(\varepsilon)$, or

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - b\,\frac{\partial^3 u}{\partial x^3} = a\,\frac{\partial^2 u}{\partial x^2} \sim \mathrm{O}(\varepsilon) \quad \text{with } \varepsilon \ll 1, \tag{36}$$

so that $a = \mathrm{O}(\varepsilon)$.

Here the conditions to start with a perturbation approach are fulfilled. In the lowest order, both nonlinear wave equations (35) and (36), with zero r.h.s., are exactly solvable. In higher orders, it turns out that one has to solve linear but inhomogeneous PDEs. As shown in [16], the tanh method can be applied straightforwardly to (35). The integration constants, appearing in higher order, poses a problem at first sight. They can be determined with a minimisation procedure. At the other hand, (36) could not be solved in the same manner due to conflicting boundary conditions.

Note that, as argued in [16], the velocity of the stationary wave can be determined a priori using asymptotics. Moreover, in contrast with the exact solution of (35) obtained earlier, the wave number c here remains a free arbitrary parameter!

4.2. Small wave numbers (limit of long wavelengths)

The wave number c can also be used as a smallness parameter (limit of long wavelengths). This fact is then combined with a smallness assumption of the relevant quantities. Typical example, extensively studied, is the set of equations describing ion-acoustic waves [11]. All quantities, including the velocity (otherwise secular terms appear), are developed in a power series in c. We therefore improved considerably previous results (see [17]) and the so-called dressed solitary-waves appeared. Our results were definitely confirmed in [12].

Discrete cases in the long wavelength limit can also be treated. In this case, we study for instance (see later in Section 5.3) nonlinear lattice vibrations. Taking a anharmonic lattice (leading to the famous Fermi–Pasta–Ulam problem [23]) we then deal with

$$m\,\frac{\partial^2 y_n}{\partial t^2} = \kappa(y_{n+1} - 2y_n + y_{n-1})[1 + \alpha(y_{n+1} - y_{n-1})], \tag{37}$$

which gives in the continuous limit (i.e., $y_n \to y(x)$, $y_{n\pm1} \to y(x \pm \Delta x)$ with $\Delta x = h$)

$$m\,\frac{\partial^2 y}{\partial t^2} = \kappa\left[h^2\,\frac{\partial^2 y}{\partial x^2} + \frac{h^4}{12}\,\frac{\partial^4 y}{\partial x^4} + \cdots\right]\left[1 + \alpha\left(2h\,\frac{\partial y}{\partial x} + \frac{h^3}{3}\,\frac{\partial^3 y}{\partial x^3} + \cdots\right)\right] \tag{38}$$

or

$$m\,\frac{\partial^2 y}{\partial t^2} = c_0^2\left[1 + \varepsilon\,\frac{\partial y}{\partial x}\right]\frac{\partial^2 y}{\partial x^2} + \frac{h^2}{12}\,\frac{\partial^4 y}{\partial x^4} + \mathrm{O}(\varepsilon h^3) \quad \text{with } c_0^2 = \frac{\kappa}{m} \text{ and } \varepsilon = 2\alpha h. \tag{39}$$

If we define $\mu = h/24\alpha$, we get in lowest order the so-called Zabusky equation [25]

$$m\,\frac{\partial^2 y}{\partial t^2} = c_0^2\left[1 + \varepsilon\,\frac{\partial y}{\partial x}\right]\frac{\partial^2 y}{\partial x^2} + \mu\varepsilon\,\frac{\partial^4 y}{\partial x^4}, \tag{40}$$

readily solved with the tanh method. Remark that one has to keep both terms in ε, a nonlinear and dispersive one. Hence

$$y(x,t) = \text{constant} + \frac{h^2 c}{\varepsilon}\,Y$$

with

$$V^2 = c_0^2\left(1 + \frac{h^2 c^2}{3}\right). \tag{41}$$

Amazingly, higher-order corrections cannot be calculated because the balancing procedure fails in this case. But the same problem can be solved approximately without the transformation to the continuous limit.

4.3. Part of the problem can be solved exactly

Such rather exceptional case occurs if one deals with coupled equations: one of them can be solved exactly and the other one only approximately at most. Typical example is the following set, arising in the field of deterministic random walk theory [10]:

$$\frac{\partial u}{\partial t} + \gamma\,\frac{\partial w}{\partial x} = u(1-u) \quad (\gamma > 0) \tag{42}$$

and

$$\frac{\partial w}{\partial t} + \gamma\,\frac{\partial u}{\partial x} = -2\mu w - uw \quad (\mu > 0). \tag{43}$$

Transformation to the ξ variable yields

$$-\,cV\,\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi} + c\gamma\,\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi} - U(\xi)(1 - U(\xi)) = 0, \tag{44}$$

$$-\,cV\,\frac{\mathrm{d}W(\xi)}{\mathrm{d}\xi} + c\gamma\,\frac{\mathrm{d}U(\xi)}{\mathrm{d}\xi} - 2\mu V(\xi) + U(\xi)W(\xi) = 0 \tag{45}$$

with the obvious definitions $u(x,t)=U(\xi)$ ($\geqslant 0$) and $w(x,t)=R(\xi)$ ($\geqslant 0$). The relevant boundary conditions are

$$U(\xi)\to 1 \quad \text{and} \quad W(\xi)\to 0 \quad \text{as } \xi\to -\infty, \tag{46}$$

$$U(\xi)\to 0 \quad \text{and} \quad W(\xi)\to 0 \quad \text{as } \xi\to +\infty, \tag{47}$$

so that for $W(\xi)$ a solitary-wave profile and for $U(\xi)$ a shock-wave structure is expected. Suppose that, after transformation to the Y variable, the highest power of Y in U and W is Y^M and Y^N, respectively, we get from (44) (transformed to the Y variable) the condition $M=(N+1)/2$. Different values are possible, but in view of the boundary conditions (46) and (47), we start with $M=2$ and $N=3$. It turns out that these values give the best possible results. General profiles for both unknowns are then derived and substituted in (45). Since no exact profiles arise from the remaining equation a minimisation procedure to get the possible lowest value of the remaining term after all relevant substitutions was applied. It rendered an approximate value for the wave number c. Good agreement with numerical profiles were found (cf. [10]).

4.4. Mixed case

Some years ago, we studied a coupled set arising in population dynamics [13]. A particular reaction–diffusion scheme [18] in chemical kinetics (catalytic reactions) revealed the same set of equations. In dimensionless quantities, this set was written as

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+w-uw, \tag{48}$$

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}+kw-uw \tag{49}$$

with associated boundary conditions ($\xi=c(x-Vt)$)

$$U(\xi)\to \text{constant} \quad \text{and} \quad W(\xi)\to 0 \quad \text{as } \xi\to -\infty, \tag{50}$$

$$U(\xi)\to 0 \quad \text{and} \quad W(\xi)\to 0 \quad \text{as } \xi\to +\infty. \tag{51}$$

Since last equation in this set resembles the Fisher (or) KPP equation,

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}+kw-w^2=\frac{\partial^2 w}{\partial x^2}+w(k-w), \tag{52}$$

which can be solved directly with the tanh method (see [15]), we used an adapted version of the tanh method to get approximate results for (48) and (49). As a result, a shock-like solution for u and a solitary-wave one for w appears.

5. Generalisations

5.1. *Use of other fundamental functions instead of* tanh

To solve mKdV equation and related equations in a more direct manner we could have started with

$$\sum_{n=0}^{N} a_n S^n, \tag{53}$$

where $S = \mathrm{sech}(\xi)$ instead of $\tanh(\xi)$. Such alternative ansatz is introduced in the symbolic software package developed by Hereman and co-authors [2]. In principle, other hyperbolic functions such as coth for instance could be used as well but they give rise to singular (nonphysical) solutions.

Similarly, also Jacobean functions can serve as basic functions in the series development: $S = \mathrm{sn}\,\xi,\ \mathrm{cn}\,\xi$ (cnoïdal waves) or $\mathrm{dn}\,\xi$. This kind of analysis is recently developed and is currently of great interest (see for instance [5,6] and references therein).

5.2. *Extension to more dimensions*

To solve nonlinear wave equations depending on x, y and z coordinates one could extend the tanh method easily with the new coordinate $\eta = \boldsymbol{k} \cdot \boldsymbol{r} - vt = kx + ly + mz - vt$ and then $Y = \tanh \eta$. This feature is not much elaborated since it essentially leads to a trivial extension: similar solutions as in one dimension are then found (see [20] for instance).

5.3. *Solutions of difference-differential PDEs*

This new extension of the tanh method, first introduced by Hereman and coworkers [2], as a new algorithm to find exact solutions of difference-differential equations is very promising. As already pointed out previously, these kind of problems are always solved in the continuous limit (long-wavelength assumption). Moreover, the very same method can be used as a perturbation method. A typical example is the occurrence of nonlinear lattice vibrations in a Toda lattice (so-called exponential lattice [21]). The basic equation reads

$$m \frac{\partial^2 y_n}{\partial t^2} = -V'(y_n - y_{n-1}) + V'(y_{n+1} - y_n)$$

with

$$V(r) = \frac{a}{b} \mathrm{e}^{-br} + ar. \tag{54}$$

After same transformations, this equation can be written as

$$\frac{\partial^2 s_n}{\partial t^2} = \left[a + \frac{\partial r_n}{\partial t} \right] \frac{b}{m} (s_{n+1} + s_{n-1} - s_n). \tag{55}$$

This equation is exactly solvable. The quantity s_n is a function of $T_n = \tanh[c(n - Vt)]$ (n represents an integer). Further, the quantity $s_{n\pm1}$ is a function of

$$T_{n\pm1} = \tanh[c(n \pm 1) - Vt] = \frac{\tanh[c(n - Vt)] \pm \tanh c}{1 \pm \tanh c \tanh[c(n - Vt)]} = \frac{T_n \pm d}{1 \pm dT_n} \quad \text{with } d = \tanh c, \tag{56}$$

based on the shifting property of tanh. Hence, s_n is then represented by ansatz (3) in which Y is replaced by T_n. Similarly, S_{n+1} and S_{n-1} are represented by the same series with T_{n+1} and T_{n-1} as basic functions. The usual balancing procedure leads to $N = 1$, so that we ultimately arrive at

$$S_n = a_0 - \frac{mcV}{b}\,Y$$

with

$$V^2 = \frac{ab}{mc^2}\,\frac{d^2}{(1 - d^2)}. \tag{57}$$

As already mentioned in Section 4.2, equations like

$$m\,\frac{\partial^2 y_n}{\partial t^2} = \kappa(y_{n+1} - 2y_n + y_{n-1})[1 + \alpha(y_{n+1} - y_{n-1})] \tag{58}$$

were studied and approximately solved in the continuous limit since no other solution method was available. It turns out, however, that the above given scheme can also be implemented for such cases. Hence, (58) can then be directly tackled. These results will be published shortly in a forthcoming paper.

6. Conclusion

It is shown with the aid of some typical examples that the tanh technique and its generalisations are a powerful solution method to find analytical expressions for stationary waves. Its strength is its ease of use to find which solitary wave structures and/or shock-wave (kinks) profiles satisfy nonlinear wave and evolution equations. The just-described technique allows to develop an algorithm, exploited by Hereman and co-workers [2] to develop symbolic software packages so that nonlinear PDEs and differential-difference equations can be investigated automatically whether (or not) they possess travelling wave solutions.

Moreover, its use as a perturbation tool certainly broadens its possibilities. Not only in the field of nonlinear PDEs but also in the field of difference-differential equations. Wave profiles obtained in this way may be of some importance for those who will investigate these problems only numerically.

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