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A multiplet analysis of spectra in the presence of broken symmetries

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Abstract. We introduce the notion of a generalised symmetry M of a hamiltonian H . It is a symmetry which has been broken in a very specific manner, involving ladder operators R and R^\dagger . In Theorem 1 these generalised symmetries are characterised in terms of repeated commutators of H with M . Breaking supersymmetry by adding a term linear in the supercharges is discussed as a motivating example.

The complex parameter γ which appears in the definition of a generalised symmetry is necessarily real when the spectrum of M is discrete. Theorem 2 shows that γ must also be real when the spectrum of H is fully discrete and R and R^\dagger are bounded operators.

Any generalised symmetry induces a partitioning of the spectrum of H in what we call M -multiplets. The hydrogen atom in the presence of a symmetry breaking external field is discussed as an example. The notion of stability of eigenvectors of H relative to the generalised symmetry M is discussed. A characterisation of stable eigenvectors is given in Theorem 3.

1. Introduction

In a series of papers [1, 2, 3, 4, 5, 6] the occurrence of supersymmetry in some lattice models has been investigated. From these studies it is clear that supersymmetry is a rather exceptional phenomenon not present in the more common models of solid state physics. On the other hand it was pointed out in [7, 8] that many hamiltonians H satisfy a higher order commutator relation with respect to certain hermitian operators M . In the present paper such an operator M is called a generalised symmetry of the hamiltonian H . This is motivated by showing that when supersymmetry is broken by adding a perturbation term linear in the supercharges then some symmetry operators become a generalised symmetry of the new hamiltonian.

In [7, 8] the ladder operator R and its conjugate R^\dagger are defined as operators satisfying $[R, M] = \gamma R$. The parameter γ is assumed to be real. In the present work complex values of γ are allowed. But one of the results which we show below is that for the study of discrete spectra real-valued γ are of prime importance.

Some of our examples involve unbounded operators. The related problems of the domain of definition of these operators are not discussed. We use the notations

$$[H, M]_n \equiv \underbrace{[\cdots [H, M], \cdots M]}_n. \quad (1)$$

The next Section gives the definition of a generalised symmetry and proves the characterisation theorem. Section 3 motivates our approach from the point of view of

supersymmetry. Section 4 discusses discrete spectra and their decomposition into M -multiplets. The final Section contains a summary of the results.

2. Definition and characterisation

2.1. Definition

We consider triples (H_0, M, R) consisting of a hamiltonian H_0 , a symmetry M , and a ladder operator R . They satisfy

- $H_0 = H_0^\dagger$ and $M = M^\dagger$;
 - $[H_0, M] = 0$;
 - There exists a complex number $\gamma \neq 0$ such that $[R, M] = \gamma R$.
- (2)

Note that if M has a discrete spectrum then γ can be taken real without restriction. Indeed, if ψ is an eigenvector of M with eigenvalue m then either $R\psi = 0$ or $R\psi$ is an eigenvector of M with eigenvalue $m - \gamma$. But the latter is real because M is hermitian. One concludes that either $R = 0$ or γ is real.

An example with complex γ can be constructed in terms of the position and momentum operators \hat{q} and \hat{p} of quantum mechanics. They satisfy the commutation relations $[\hat{q}, \hat{p}] = i\hbar$ where \hbar is Planck's constant. Hence, the choices $M = \hat{p}\hat{q} + \hat{q}\hat{p}$ and $R = \hat{q}$ yield $[R, M] = 2i\hbar R$. Further take $H_0 = M$ to obtain a triple (H_0, M, R) satisfying (2).

Definition 1 *An operator M is a generalised symmetry of the hamiltonian H if the latter can be written as $H = H_0 + R + R^\dagger$, and the triple (H_0, M, R) , satisfies the conditions (2).*

It is quite common to consider as a symmetry of H any operator M commuting with H . These symmetries form a von Neumann algebra which is generated by its hermitian elements. The justification of calling M a symmetry is that when ψ is an eigenvector of H with eigenvalue E then either $M\psi = 0$ or $M\psi$ is again an eigenvector with the same eigenvalue E . The present generalisation allows that the symmetry is broken by adding a symmetry breaking term to the hamiltonian.

Note that the above definition is linear in R . Hence varying the strength of the symmetry breaking term does not change the generalised symmetries of the hamiltonian. Note further that $[R, M] = \gamma R$ implies that $[R^n, M] = n\gamma R^n$. Hence the generalised symmetries of $H_0 + R + R^\dagger$ are also generalised symmetries of $H_0 + \bar{z}R^n + z(R^n)^\dagger$, with complex $z \neq 0$, and $n = 1, 2, \dots$.

A point of discussion is whether one should add in the above definition the requirement that the operators $R^\dagger R$ and RR^\dagger commute with H_0 . This is the case in many examples and has been used in [7] to simplify the analysis of the spectrum of model hamiltonians.

2.2. Commutator relations

Generalised symmetries M of a hamiltonian H can be characterised by relations between higher order commutators of H with M .

Theorem 1 *A hermitian operator M is a generalised symmetry of the hermitian hamiltonian H if and only if one of the following cases occurs.*

(i) *There exist a real number γ_2 with $\gamma_2 \neq 0$ such that*

$$[H, M]_2 = i\gamma_2[H, M]. \quad (3)$$

(ii) *There exists real numbers $\gamma_1 \neq 0, \gamma_2$ for which*

$$[H, M]_3 = 2i\gamma_2[H, M]_2 + (\gamma_1^2 + \gamma_2^2)[H, M]. \quad (4)$$

Note that any genuine symmetry M of H , this is $[H, M] = 0$, satisfies (3) with γ_2 arbitrary and that (3) implies (4) with $\gamma_1 = 0$.

Proof of the theorem

First assume that M is a generalised symmetry of the hamiltonian H .

Let $\gamma = \gamma_1 + i\gamma_2$. If $\gamma_1 = 0$ then one can derive the following equations

$$H = H_0 + R + R^\dagger \quad (5)$$

$$[H, M] = i\gamma_2(R + R^\dagger) \quad (6)$$

$$[H, M]_2 = -\gamma_2^2(R + R^\dagger). \quad (7)$$

They imply (3).

Next assume that $\gamma_1 \neq 0$. A straightforward calculation gives

$$H = H_0 + R + R^\dagger \quad (8)$$

$$[H, M] = \gamma R - \bar{\gamma} R^\dagger \quad (9)$$

$$[H, M]_2 = \gamma^2 R + (\bar{\gamma})^2 R^\dagger \quad (10)$$

$$[H, M]_3 = \gamma^3 R - (\bar{\gamma})^3 R^\dagger. \quad (11)$$

The set of equations (8,9, 10) in the variables H_0 and R has the solution

$$R = \frac{\bar{\gamma}}{2\gamma_1|\gamma|^2} ([H, M]_2 + \bar{\gamma}[H, M]) \quad (12)$$

$$H_0 = \frac{1}{|\gamma|^2} (-[H, M]_2 + 2i\gamma_2[H, M] + |\gamma|^2 H). \quad (13)$$

Taking the commutator of (13) with M yields (4). This ends the first part of the proof.

Next, let us prove the theorem in the other direction. First assume (3). Let

$$H_0 = \frac{i}{\gamma_2}[H, M] + H,$$

$$R = R^\dagger = -\frac{i}{2\gamma_2}[H, M]. \quad (14)$$

They satisfy $H = H_0 + R + R^\dagger$. One verifies that

$$\begin{aligned} [H_0, M] &= \frac{i}{\gamma_2}[H, M]_2 + [H, M] \\ &= 0. \end{aligned} \quad (15)$$

Similarly is

$$\begin{aligned} [R, M] &= -\frac{i}{2\gamma_2}[H, M]_2 \\ &= \frac{1}{2}[H, M] \\ &= i\gamma_2 R. \end{aligned} \quad (16)$$

This proves the first case.

Finally assume that (4) holds. Then operators R and H_0 can be defined by (12) and (13), respectively. One verifies readily that they satisfy $H = H_0 + R + R^\dagger$. Using (4) one obtains

$$[H_0, M] = \frac{1}{|\gamma|^2} (-[H, M]_3 + 2i\gamma_2[H, M]_2 + |\gamma|^2[H, M])$$

$$= 0. \tag{17}$$

Similarly is

$$\begin{aligned} [R, M] &= \frac{\bar{\gamma}}{2\gamma_1|\gamma|^2} ([H, M]_3 + \bar{\gamma}[H, M]_2) \\ &= \gamma R. \end{aligned} \tag{18}$$

This proves the last case. □

From the proof of the Theorem it is also clear that the operators R, R^\dagger and H_0 are uniquely determined by the generalised symmetry M .

A special case of a generalised symmetry occurs when γ is real and different from zero. As noted before, this is the only possibility when the generalised symmetry M has a fully discrete spectrum. Expression (4) becomes

$$[H, M]_3 = \gamma^2[H, M]. \tag{19}$$

This special case has been investigated in [7, 8]. In a slightly different context (19) is called the Dolan-Grady condition [9].

Let us consider again the quantum mechanical example. The hamiltonian of a non-relativistic free particle with mass m reads

$$H = \frac{1}{2m}\hat{p}^2. \tag{20}$$

Let $M = \hat{p}\hat{q} + \hat{q}\hat{p}$ as before. Then one has $[H, M] = -4i\hbar H$. Hence, (3) is satisfied with $\gamma_2 = -4\hbar$. One finds $R = R^\dagger = \frac{1}{2}H$ and $H_0 = 0$. This example shows that the first case of the Theorem contains generalised symmetries which are not genuine symmetries. Note that $M = \hat{p}\hat{q} + \hat{q}\hat{p}$ is not a generalised symmetry of the hamiltonian of the harmonic oscillator,

$$H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega_0^2\hat{q}^2 \tag{21}$$

because $[H, M]_3 = -\gamma^2[H, M]$ holds with $\gamma = 4\hbar$. This relation has the wrong sign to be written in the form (19). This would mean that iM is a generalised symmetry. However, in the present work we require generalised symmetries to be hermitian.

Examples of the second case of the Theorem are discussed in the next Section.

2.3. Existence

An obvious question is whether there exist always generalised symmetries of a given hermitian hamiltonian H , other than the genuine symmetries. The answer is positive since all orthogonal projection operators are generalised symmetries of any hermitian hamiltonian. Indeed, if $M^2 = M$ then one has

$$[H, M]_3 = HM^3 - 3MHM^2 + 3M^2HM - M^3H = [H, M]. \tag{22}$$

Hence, by Theorem 1, M is a generalised symmetry of H . Similarly, if $M^2 = \mathbb{I}$ then

$$[H, M]_3 = HM^3 - 3MHM^2 + 3M^2HM - M^3H = 4[H, M]. \tag{23}$$

Hence orthogonal M are generalised symmetries as well. If one would add to the Definition 1 the requirement that M commutes with $R^\dagger R$ and RR^\dagger then these examples do not work in general. In any case, due to the nonlinear nature of the commutation relations it is unclear what is the structure of the set of generalised symmetries.

3. Motivation

The main motivation for introducing the notion of generalised symmetries is that the operators R and R^\dagger can act as ladder operators — see the next Section. But its origin lies in the study of model hamiltonians and in an effort to generalise the notion of supersymmetry. The idea is that even when a symmetry is broken some traces of this symmetry may remain. The generalised symmetry M of H is a genuine symmetry of the hamiltonian H_0 . The symmetry breaking term $R + R^\dagger$ has special properties which can be used to relate the spectrum of H to that of H_0 .

3.1. Lie superalgebras

The simplest non-trivial Lie superalgebra contains two odd generators Q and Q^\dagger and one even generator $\{Q, Q^\dagger\}$. The latter commutes with all elements of the algebra and is for that reason called supersymmetric. Consider now the hamiltonian

$$H = \{Q, Q^\dagger\} + \bar{z}Q + zQ^\dagger, \quad (24)$$

with complex z . For $z \neq 0$ the supersymmetry is broken. But the hamiltonian still belongs to the Lie superalgebra.

Let us now add to the Lie superalgebra a second even generator, denoted M . The following result is an immediate consequence of the assumption that the new algebra has exactly 4 generators and that we assume a representation as operators in a Hilbert space, with Q^\dagger the adjoint of Q and with $M = M^\dagger$.

Lemma 1 *There exists a complex number γ such that*

$$[Q, M] = \gamma Q. \quad (25)$$

Proof

Because M is even and Q is odd also $[Q, M]$ is odd. Hence it can be written as

$$[Q, M] = \gamma Q + \xi Q^\dagger \quad (26)$$

with γ and ξ complex numbers. Using $Q^2 = 0$, multiplying once from the left and once from the right, one obtains

$$\begin{aligned} -QMQ &= \xi QQ^\dagger \\ QMQ &= \xi Q^\dagger Q. \end{aligned} \quad (27)$$

Adding the two together gives $0 = \xi\{Q, Q^\dagger\}$. But $\{Q, Q^\dagger\}$ is an even generator of the superalgebra. Hence it cannot vanish. Therefore one concludes that $\xi = 0$. □

Now write $H = H_0 + R + R^\dagger$ with $H_0 = \{Q, Q^\dagger\}$ and $R = \bar{z}Q$. The previous Lemma yields $[R, M] = \gamma R$. Further is

$$[H_0, M] = [QQ^\dagger + Q^\dagger Q, M] = 0 \quad (28)$$

Hence, M is a generalised symmetry of the hamiltonian H . Note that M commutes with $Q^\dagger Q$ and QQ^\dagger as well.

In 1981 Witten [10] introduced his non-relativistic model of supersymmetry. The one-particle hamiltonian reads

$$H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dq^2} + \frac{1}{2}mw^2(q) + \frac{1}{2}\hbar\sigma_z \frac{dw}{dq}. \quad (29)$$

It can be written as

$$H_0 = \{Q, Q^\dagger\} \quad (30)$$

with

$$Q = \frac{1}{\sqrt{2m}} (\hat{p} - imw(q)) \sigma_+, \quad (31)$$

where

$$\hat{p} = \frac{\hbar}{i} \frac{d}{dq} \quad \text{and} \quad \sigma_\pm = \frac{1}{2} (\sigma_x \pm i\sigma_y). \quad (32)$$

($\sigma_\alpha, \alpha = x, y, z$ are the Pauli matrices).

The Pauli matrix $M = \sigma_z$ can be added as a fourth generator to the super algebra generated by Q, Q^\dagger, H_0 . It satisfies $[Q, M] = 2Q$. This illustrates the previous Lemma. That M is a generalised symmetry of $H = H_0 + \bar{z}Q + zQ^\dagger$ is a rather trivial result because $M^2 = \mathbb{I}$ — see the remark in subsection 2.3. A short calculation shows that $Q^\dagger Q$ and QQ^\dagger commute with M .

3.2. Lattice models

Let B_j, B_j^\dagger denote the annihilation and creation operators of a fermionic particle at the lattice site j belonging to a finite subset Λ of \mathbb{Z}^n . They satisfy $\{B_j, B_k\} = \delta_{j,k}$ and are the odd generators of a Lie superalgebra with $2n$ odd generators and one even generator, which is the identity operator. The latter is supersymmetric. The physically interesting hamiltonian is

$$H_0 = -\epsilon \sum_{i,j \in \Lambda}^{|i-j|=1} B_i^\dagger B_j. \quad (33)$$

It describes the hopping of fermions between neighbouring places. ϵ is a numerical constant. One has

$$[H_0, B_i] = \epsilon \sum_j^{|i-j|=1} B_j. \quad (34)$$

Hence, H_0 may be added to the super algebra as an even generator. Of further interest is the operator

$$M = \sum_{i \in \Lambda} B_i^\dagger B_i. \quad (35)$$

It counts the number of fermions and it satisfies $[B_j, M] = B_j$ and $[H_0, M] = 0$. Hence, it can also be added as an even generator. Moreover, the triples (H_0, M, B_i) satisfy the conditions (2) with $\gamma = 1$. Hence M is a generalised symmetry of each of the hamiltonians $H_0 + \bar{z}_i B_i + z_i B_i^\dagger$, where z_i is a complex constant. It is also a generalised symmetry of the hamiltonian

$$H = H_0 + \sum_i (\bar{z}_i B_i + z_i B_i^\dagger). \quad (36)$$

However, this hamiltonian is physically not so interesting because fermions are usually conserved in number, while now this symmetry is broken. But by taking $z_i = 0$ except at the borders one can model the effect that fermions leave or enter the system.

3.3. Hard core repulsion

Following [1, 2, 4, 5] the model hamiltonian (33) can be made more interesting by including a nearest neighbour exclusion mechanism. Let

$$Q = \sum_i P_i B_i^\dagger \quad \text{with} \quad P_i = \prod_{j, |i-j|=1} B_j B_j^\dagger. \quad (37)$$

Note that $Q^2 = 0$. A supersymmetric hamiltonian is given by

$$\begin{aligned} H_0 = \{Q, Q^\dagger\} &= \sum_{i,j \in \Lambda} \left(P_i B_i^\dagger B_j P_j + B_j P_j P_i B_i^\dagger \right) \\ &= \sum_{i \in \Lambda} P_i + \sum_{i,j \in \Lambda, |i-j|=1} P_i B_i^\dagger B_j P_j. \end{aligned} \quad (38)$$

The last term describes a hopping of spinless electrons between neighbouring sites under the condition that never two electrons occupy neighbouring positions. The first term counts the numbers of sites which have all neighbouring sites unoccupied. The perturbation term

$$\bar{z}Q + zQ^\dagger = \sum_i P_i (\bar{z}B_i^\dagger + zB_i) \quad (39)$$

is an external field injecting or removing electrons at sites where they do not have neighbours.

The counting operator $M = \sum_i B_i^\dagger B_i$ is a generalised symmetry of this model. Indeed, one has $[Q, M] = -Q$. By adding the source term (39) the conservation of the number of electrons is broken. But it remains a generalised symmetry.

3.4. Jaynes-Cummings model

Another way of making (33) more interesting is by replacing the symmetry-breaking terms $\bar{z}Q + zQ^\dagger$ by an interaction with a bosonic particle. The prototype model is that of Jaynes and Cummings. The hamiltonian is, see for instance [11],

$$H = \frac{1}{2} \hbar \omega_0 \sigma_z + \frac{1}{2} \hbar \omega (c^\dagger c + c c^\dagger) + \hbar \kappa (c^\dagger \sigma_- + c \sigma_+), \quad (40)$$

where c, c^\dagger are the annihilation and creation operators of a harmonic oscillator. They satisfy $[c, c^\dagger] = \mathbb{I}$. The σ_α are the Pauli matrices and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. They satisfy $\{\sigma_+, \sigma_-\} = \mathbb{I}$. The model is said to be at resonance when $\omega = \omega_0$. When at resonance and without interaction term (this is, $\kappa = 0$) the model is supersymmetric with $Q = \sqrt{\hbar \omega} c^\dagger \sigma_-$ (neglecting a shift in the energy scale). The symmetry breaking term $\bar{z}Q + zQ^\dagger$ reproduces the interaction term. Indeed, one obtains

$$H_* = \{Q, Q^\dagger\} + zQ + zQ^\dagger = \hbar \omega \sigma_+ \sigma_- + \hbar \omega c^\dagger c + \hbar \kappa (c^\dagger \sigma_- + c \sigma_+), \quad (41)$$

with $z = \hbar \kappa / \sqrt{\hbar \omega}$. Note that $[Q, \sigma_z] = -Q$ and $[\{Q, Q^\dagger\}, \sigma_z] = 0$. Hence σ_z is a generalised symmetry of H_* . Off resonance the model is not anymore supersymmetric. But even then σ_z remains a generalised symmetry of H . Indeed, one has

$$\begin{aligned} [H, \sigma_z] &= \hbar \kappa (c^\dagger \sigma_- - c \sigma_+) \\ [H, \sigma_z]_2 &= \hbar \kappa (c^\dagger \sigma_- + c \sigma_+) \\ [H, \sigma_z]_3 &= \hbar \kappa (c^\dagger \sigma_- - c \sigma_+) = [H, \sigma_z]. \end{aligned} \quad (42)$$

By Theorem 1 this implies that σ_z is a generalised symmetry of H . One finds $\gamma = 1$ and $R = \hbar \kappa c^\dagger \sigma_-$. Note that $R^\dagger R$ and RR^\dagger both commute with σ_z .

4. The spectrum of H

If $M = M^*$ is a genuine symmetry of the hamiltonian H then the unitary transformations

$$A \rightarrow e^{-i\lambda M} A e^{i\lambda M} \quad (43)$$

leave the hamiltonian invariant. If M is a generalised symmetry then one has

$$e^{-z M} H e^{z M} = H_0 + e^{z\gamma} R + e^{-z\bar{\gamma}} R^\dagger. \quad (44)$$

This similarity transformation does not change the spectrum of the hamiltonian. But it modulates the symmetry breaking term with a complex factor. The eigenvectors transform as $\psi \rightarrow e^{-z M} \psi$. In what follows we explore whether this relation creates a link between eigenvalues of H_0 and of H .

4.1. Eigenvectors

The following result suggests that for the study of discrete spectra of hamiltonians only generalised symmetries with a real γ are of interest. This is the case which has been studied in [7, 8].

Theorem 2 *Let $M = M^\dagger$ be a generalised symmetry of the hamiltonian $H = H^\dagger$, with associated triple (H_0, M, γ) . Assume that H has a fully discrete spectrum and that M is not a symmetry of H . Assume in addition that R is a bounded operator. Then γ is necessarily real.*

In particular, if H has a fully discrete spectrum and R is a bounded operator then case 1 of Theorem 1 only contains genuine symmetries. It is not clear whether the condition that R is a bounded operator can be removed.

Proof of the theorem

Let us assume that $\Im\gamma \neq 0$. Let ψ be a normalised eigenvector of H , with eigenvalue E .

$$\psi_\alpha = e^{i\alpha M} \psi \quad \text{with } \alpha \text{ real.} \quad (45)$$

Then $\|\psi_\alpha\| = 1$. One calculates using (44)

$$\|(H_0 - E)\psi\| = \|(e^{i\alpha\gamma} R + e^{-i\alpha\bar{\gamma}} R^\dagger)\psi_\alpha\| \leq 2e^{-\alpha\Im\gamma} \|R\|. \quad (46)$$

Depending on the sign of $\Im\gamma$ one lets tend α to $+\infty$ or $-\infty$ and one concludes that ψ is an eigenvector of H_0 with eigenvalue E . Because the spectrum of H is fully discrete this implies that H and H_0 coincide and hence that H commutes with M . But by assumption M is not a genuine symmetry of H . Hence, the assumption that $\Im\gamma \neq 0$ is false. One concludes that γ is real. □

4.2. The multiplet structure

When breaking a symmetry of a hamiltonian degenerate energy levels will split up. After breaking the symmetry the corresponding eigenvectors still transform into each other under the symmetry operations of the original hamiltonian. They form a so-called multiplet. This traditional way of defining multiplets of eigenvalues is now adapted. Starting point is the observation that the definition of a multiplet depends on the broken symmetry under consideration.

Definition 2 *Let M be a generalised symmetry of the hamiltonian H . Two eigenvectors ϕ and ψ of H are said to belong to the same M -multiplet if there exists an invertible function f such that $\phi = f(M)\psi$.*

The above definition is an extension of the existing notion of multiplets. In [8] the example of the harmonic oscillator is considered with the shifted harmonic oscillator as the generalised symmetry M . Then all eigenvalues belong to the same M -multiplet. For instance, the first excited state $|1\rangle$ can be written as $|1\rangle = f(M)|0\rangle$, where $|0\rangle$ is the ground state and $f(x)$ is a linear function.

Note that the similarity transformation (44) maps M -multiplets onto themselves.

4.3. The hydrogen atom in an external potential

An obvious example is the hamiltonian H_0 of the hydrogen atom. It is given by

$$H_0 = \frac{1}{2m} \sum_{\alpha} \hat{p}_{\alpha}^2 - \frac{e^2}{\sqrt{\sum_{\alpha} \hat{q}_{\alpha}^2}}, \quad (47)$$

where e, m are constants and $\hat{q}_{\alpha}, \hat{p}_{\alpha}, \alpha = 1, 2, 3$ are the position and momentum operators of the electron. The hamiltonian is invariant under spatial rotations. Therefore the components L_{α} of the angular momentum commute with H_0 .

Now add to this hamiltonian a symmetry breaking potential. The simplest one is a uniform field in the x -direction

$$H = H_0 - 2g\hat{q}_x \quad (48)$$

with real coupling constant g . This hamiltonian can be written in the form $H = H_0 + R + R^{\dagger}$, with $R = -g(\hat{q}_x + i\hat{q}_y)$. From the commutation relations

$$[L_z, \hat{q}_x] = i\hbar\hat{q}_y, \quad [L_z, \hat{q}_y] = -i\hbar\hat{q}_x, \quad [L_z, \hat{q}_z] = 0 \quad (49)$$

follows immediately that $M = L_z$ is a generalized symmetry of H , with $\gamma = -\hbar$.

The method of classifying eigenvalues into multiplets, as worked out in [7], works well if a basis of common eigenvectors of H_0 and M is explicitly known. This is only partially the case for the hamiltonian of the hydrogen atom with $M = L_z$ since the spectrum of these operators is partly discrete, partly continuous. In addition, the perturbed hamiltonian H introduced above does not have any discrete eigenvalue. Therefore it is meaningless to discuss its multiplet structure. Let us therefore look for an alternative symmetry breaking potential.

A basis of eigenvectors of the hydrogen atom can be labeled with quantum numbers n, l, m with $n = 1, 2, \dots, l = 0, 1, \dots, n-1, m = -l, -l+1, \dots, l$. They satisfy

$$\begin{aligned} H_0|n, l, m\rangle &= E_n|n, l, m\rangle, \\ \sum_{\alpha} L_{\alpha}^2|n, l, m\rangle &= \hbar l(l+1)|n, l, m\rangle, \\ L_z|n, l, m\rangle &= \hbar m|n, l, m\rangle, \end{aligned} \quad (50)$$

with E_n proportional to $-1/n^2$. The conventional ladder operators are $L_{\pm} = L_x \pm iL_y$. They satisfy

$$L_{\pm}|n, l, m\rangle \sim |n, l, m \pm 1\rangle. \quad (51)$$

Consider therefore the hamiltonian

$$H = H_0 - 2gL_x = H_0 + R + R^{\dagger} \quad \text{with } R = -gL_-. \quad (52)$$

Then $M = L_z$ is a generalised symmetry of H with $\gamma = -\hbar$. Note that in this case R commutes with H_0 . Hence the Hilbert space can be restricted to the subspace generated by the eigenvectors

of H_0 . The eigenvectors of H form multiplets indexed by the quantum numbers n, l . Within such a multiplet the relevant operators can be represented as square matrices of size $2l + 1$. In this context any diagonal matrix can be written as a function of $M = L_z$. Therefore all wavefunctions of the form

$$\psi = \sum_{m=-l}^l c_m |n, l, m\rangle \quad (53)$$

for which none of the c_m vanish belong to the same M -multiplet. But the multiplet with quantum numbers n, l with $l \geq 1$ always contains at least one eigenvector ψ of H which satisfies $c_{-m} = -c_m$ and hence $c_0 = 0$ — see below. This makes clear that the multiplet with quantum numbers n, l decomposes into more than one M -multiplet.

The requirement that $H\psi = E\psi$ leads to the set of equations

$$Ec_m = E_n c_m + c_{m+1} \langle m | R | m+1 \rangle + c_{m-1} \langle m | R^\dagger | m-1 \rangle. \quad (54)$$

The matrix elements of R and R^\dagger are given by (see for instance Section VI.40 of [12])

$$\langle m+1 | R^\dagger | m \rangle = \langle m | R | m+1 \rangle = -g \frac{\hbar}{\sqrt{2}} \sqrt{(l-m)(l+m+1)} \quad (55)$$

The ansatz that $c_0 = 0$ implies $c_{-1} = -c_1$. It is then easy to see that the equations can be solved iteratively, maintaining $c_{-m} = -c_m$. The top equation

$$Ec_l = E_n c_l + c_{l-1} \langle l | R^\dagger | l-1 \rangle \quad (56)$$

then fixes the eigenvalue E . If $l = 1$ this forces $E = E_n$ and $(R + R^\dagger)\psi = 0$. The $l = 1$ triplets therefore decompose into an M -singlet and an M -doublet.

The $l = 2$ multiplets can be shown to decompose into two M -doublets and one M -singlet with eigenvector of the form $(x, 0, z, 0, x)^T$. In general, the decomposition into M -multiplets is governed by the occurrence of vanishing coefficients c_m .

4.4. Stable eigenvectors

The following definition is taken from [8].

Definition 3 *The eigenvector ψ of the hamiltonian H is stable relative to the generalised symmetry M if the vectors $\psi, R\psi, R^\dagger\psi$ are linearly dependent.*

Eigenvectors ψ for which $R\psi = 0$ or $R^\dagger\psi = 0$ are clearly stable. The $l = 1$ M -multiplets in the example of the hydrogen atom all contain a pair of stable eigenvectors — the proof follows later on. This is not by accident, as shows the next theorem, which improves on a result from [8].

Theorem 3 *Let be given an eigenvector ψ of a hamiltonian H . Assume that M is a generalised symmetry of H , with real γ . Then ψ is stable relative to M if and only if one of the following possibilities holds.*

- 1) *There exist numbers x and y , not both zero, such that $xR\psi + yR^\dagger\psi = 0$.*
- 2) *ψ is an eigenvector of R .*
- 3) *ψ is an eigenvector of R^\dagger .*
- 4) *ψ is an eigenvector of $R - R^\dagger$.*
- 5) *There exists a complex z such that $\chi = e^{-zM}\psi$ is again an eigenvector of H with eigenvalue different from that of ψ .*

Proof of the theorem

Assume first that ψ is stable relative to M . Then there exist x, y, u not all zero such that

$$xR\psi + yR^\dagger\psi = u\psi. \quad (57)$$

The choice $u = 0$ corresponds with the first possibility. Let us therefore assume that $u \neq 0$. Then we can assume that $u = 1$ without restriction. The possibilities $x = 0$ or $y = 0$ then correspond with cases (2) or (3). Remains the situation when $xy \neq 0$. The case that $x + y = 0$ leads to possibility (4) of the Theorem. Assuming $x + y \neq 0$ leads to the possibility (5). Indeed, there exists z satisfying $e^{z\gamma} = -y/x \neq 1$ such that

$$x(e^{z\gamma} - 1) = y(e^{-z\gamma} - 1). \quad (58)$$

This leads to, assuming $H\psi = E'\psi$,

$$\begin{aligned} H\chi &= He^{-zM}\psi \\ &= (H_0 + R + R^\dagger)e^{-zM}\psi \\ &= e^{-zM}(E' + (e^{-z\gamma} - 1)R + (e^{z\gamma} - 1)R^\dagger)\psi \\ &= e^{-zM}\left(E' + \frac{1}{x}(e^{-z\gamma} - 1)[xR + yR^\dagger]\right)\psi \\ &= e^{-zM}\left(E' + \frac{1}{x}(e^{-z\gamma} - 1)\right)\psi \\ &= E''\chi \quad \text{with } E'' = E' + \epsilon \\ &\quad \text{and } \epsilon = \frac{1}{x}(e^{-z\gamma} - 1) = \frac{1}{y}(e^{z\gamma} - 1). \end{aligned} \quad (59)$$

Note that $\epsilon \neq 0$ because of $e^{z\gamma} \neq 1$. This covers case (5) and ends the proof in one direction.

The other direction of the proof is straightforward except in the last case. Assume that both ψ and $\chi = e^{-zM}\psi$ are eigenvectors of H with corresponding eigenvalues E', E'' . Then one has

$$\begin{aligned} E'\psi &= H\psi \\ &= (H_0 + R + R^\dagger)e^{zM}\chi \\ &= e^{zM}(H_0 + e^{z\gamma}R + e^{-z\gamma}R^\dagger)\chi \\ &= e^{zM}(H + (e^{z\gamma} - 1)R + (e^{-z\gamma} - 1)R^\dagger)\chi \\ &= e^{zM}(E'' + (e^{z\gamma} - 1)R + (e^{-z\gamma} - 1)R^\dagger)\chi \\ &= [E'' + (1 - e^{-z\gamma})R + (1 - e^{z\gamma})R^\dagger]\psi. \end{aligned} \quad (60)$$

Because $E' \neq E''$ this shows that $\psi, R\psi, R^\dagger\psi$ are linearly dependent. Hence ψ is stable relative to M . □

The situation of a ladder operator R annihilating the eigenstate ψ is a subcase of case (1) of the theorem ($x = 1$ and $y = 0$). It is well-known because the vacuum vector of the harmonic oscillator is annihilated by the standard boson annihilation operator b . But in [8] the shifted harmonic oscillator is shown to be a generalised symmetry of the harmonic oscillator. Then the operator R is a linear combination of b and the constant operator. The ground state is then an eigenvector of R . Thus this is an example of case (2) of the theorem.

A special subcase of case (1) of the theorem is when $(R + R^\dagger)\psi = 0$. Then one has $H\psi = H_0\psi = E\psi$. Hence, these ψ are simultaneous eigenvectors of H and H_0 . Examples of such stable eigenvectors are the $l = 0$ singlets of the hydrogen atom in an external field, discussed in Subsection 4.3, because they are annihilated by both R and R^\dagger .

In case (4) is $R\psi = R^\dagger\psi$. From (12) then follows that $[H, M]\psi = 0$ (note that γ is taken to be real). Using $H\psi = E\psi$ then follows that $M\psi$ is again an eigenvector of H also with eigenvalue E . Note that $M\psi$ cannot vanish. This is a rather pathological situation which we did not encounter yet in any example.

The last case of the theorem is the generic case. It predicts the existence of pairs of stable eigenvectors belonging to the same M -multiplet. Examples have been discussed in [8]. Such a pair of stable eigenvectors is also found in the $l = 1$ multiplets of the hydrogen atom in an external field (see Subsection 4.3).

5. Summary

We study a class of symmetries $M = M^\dagger$ which are broken by adding to the hamiltonian H_0 a perturbation of the form $R + R^\dagger$, where R is a ladder operator satisfying $[R, M] = \gamma R$ with complex γ . Because the symmetry is only broken in this special way we call M a *generalised symmetry* of $H = H_0 + R + R^\dagger$.

These generalised symmetries are characterised by a commutator relation — see Theorem 1. While a genuine symmetry M commutes with the hamiltonian H the generalised symmetry satisfies relation (4) which involves also the second and third commutators of H with M . In the case that γ is real this commutator relation has been studied in [7, 8]. In a slightly different context it has been called the Dolan-Grady relation [9].

Generalised symmetries, in the weak form of Definition 1, do always exist and can be helpful in studying the discrete spectrum of quantum hamiltonians. Several examples have been mentioned throughout the paper. Many of these examples involve the case of γ being real, which was already studied in [7, 8]. This not by accident. If the generalised symmetry M has a discrete spectrum then γ must be real. In addition, Theorem 2 shows that hamiltonians with a fully discrete spectrum and satisfying some technical condition, only have generalised symmetries with real γ . As a consequence the present generalisation from real to complex γ does not immediately lead to new applications.

We introduce a formal definition of M -multiplets based on the work of [7, 8]. We show for the example of the hydrogen atom how this definition relates with the usual notion of multiplets of degenerate eigenvectors.

We finally discussed the notion of eigenvectors stable relative to a generalised symmetry. This notion was introduced in [8]. Such stable eigenvectors, if known, can help to find new eigenvectors in a way reminiscent of the Darboux method to find new solutions of nonlinear differential equations out of known solutions.

The theory of generalised symmetries as presented here is probably not yet in its final form. It has many applications one of which is a partition of energy spectra into multiplets in a way depending on the choice of a generalised symmetry. From these applications one can learn how to extend our knowledge about generalised symmetries.

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