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Bornologies and metrically generated theories

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ABSTRACT

Bornologies axiomatize an abstract notion of bounded sets and are introduced as collections of subsets satisfying a number of consistency properties. Bornological spaces form a topological construct, the morphisms of which are those functions which preserve bounded sets. A typical example is a bornology generated by a metric, i.e. the collection of all bounded sets for that metric. In a recent paper [E. Colebunders, R. Lowen, Metrically generated theories, Proc. Amer. Math. Soc. 133 (2005) 1547–1556] the authors noted that many examples are known of natural functors describing the transition from categories of metric spaces to the "metrizable" objects in some given topological construct such that, in some natural way, the metrizable objects generate the whole construct. These constructs can be axiomatically described and are called metrically generated. The construct of bornological spaces is not metrically generated, but an important large subconstruct is. We also encounter other important examples of metrically generated constructs, the constructs of Lipschitz spaces, of uniform spaces and of completely regular spaces. In this paper, the unified setting of metrically generated theories is used to study the functorial relationship between these constructs and the one of bornological spaces.

1. Introduction

Bornologies are meant to axiomatize an abstract notion of bounded sets and are hence introduced as collections of subsets satisfying a number of consistency properties. The theory of bornological spaces plays an important role in functional analysis. A systematic study of this theory was carried out by H. Hogbe-Nlend in [10], including a survey on applications in topological vector spaces, in distribution theory and in differential calculus for non normed spaces. Other applications appear in the work of L. Waelbroeck [13] who used bornologies in the study of topological algebras as well. Bornologies also play a fundamental role in the work of Frölicher and Kriegl on Linear Spaces and Differentiation theory [6]. Bornological spaces form a topological construct, Bor, the morphisms of which are those functions which preserve bounded sets. A typical example is a bornology generated by a pseudometric, i.e. the collection of all bounded sets for that pseudometric. Another standard example is a bornology generated by a uniform space, as it can for instance be found in Bourbaki [2]. As is well known the transition from pseudometric spaces endowed with uniformly continuous maps to the construct Bor of bornological spaces and boundedness-preserving maps is not functorial. However when pseudometric spaces are considered

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with contractions (also called non-expansive maps) between them, then the transition to Bor becomes functorial. Another well-known fact is that the transition from the construct of uniform spaces and uniformly continuous maps to Bor is functorial too.

In this paper we study bornologies for metrically generated theories. Pseudometric spaces with non-expansive maps, uniform spaces with uniformly continuous maps, completely regular topological spaces with continuous maps will be some of the many concrete examples to which the theory can be applied. We will also encounter the so-called Lipschitz structures [7,5] and we show that they also fit into our framework. Metrically generated theories will in fact present a unifying setting for our study of the functorial relation with bornologies. In all cases mentioned the associated bornology will be described as the class of all sets bounded for all pseudometrics in the "gauge" of the structure.

In a previous paper [4] the authors investigated metrically generated theories in a general setting of base categories consisting of generalized metric spaces that allowed application to many examples besides the ones mentioned above. In this paper we will explain the ideas behind metrically generated theories in the classical setting of pseudometric spaces and we will treat several new and old examples. In the main Theorem 2.2, for the specific construct $\mathcal{P}$ of pseudometric spaces, necessary and sufficient conditions on a functor $K : \mathcal{P} \to \mathcal{X}$ will be given in order to characterize when $\mathcal{X}$ can be isomorphically described as a full concretely coreflective subconstruct of the model category $\mathsf{M}^{\mathcal{P}}$ with objects sets structured by collections of pseudometrics and with morphisms functions $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ satisfying the property that $d' \circ f \times f \in \mathcal{D}$ for any $d' \in \mathcal{D}'$. Topological constructs $\mathcal{X}$ for which there exists a functor $K : \mathcal{P} \to \mathcal{X}$ satisfying these necessary and sufficient conditions will be called $\mathcal{P}$-metrically generated or simply metrically generated. Well known examples are easily captured. Creg, the construct of completely regular spaces and continuous maps and Unif, the construct of uniform spaces and uniformly continuous maps are metrically generated, but also the construct Lip [7,5] of Lipschitz spaces is isomorphically described in this setting. Note that in all of these cases, as in all metrically generated constructs, the morphisms are determined in the same way.

After having developed the general setting, in the second part we will focus on bornological spaces. We will show that despite the fact that every pseudometric space (X, d) gives rise to a natural associated bornology $\mathcal{B}_d$, namely the collection of all d-bounded subsets, the functor $B : \mathcal{P} \to \mathsf{Bor} : (X, d) \mapsto (X, \mathcal{B}_d)$, does not satisfy the necessary and sufficient conditions mentioned above. However if we restrict ourselves to the subconstruct of bornological spaces generated by some l^∞-structure as introduced in [6], then we obtain another example of a metrically generated construct. For more examples see [11].

The third and final part of the paper will benefit from having developed the common superconstruct $\mathsf{M}^{\mathcal{P}}$ in which all our metrically generated categories under consideration have isomorphic counterparts. For the natural functor from $\mathsf{M}^{\mathcal{P}}$ to Bor^∞ we study the restriction to Lip, to Unif and to Creg, respectively. Thus we obtain some unified descriptions for the well-known transitions to bornological spaces as for instance described in [10].

2. Metrically generated theories

We refer to [1] for categorical concepts and terminology but recall the few items we frequently require. A *construct* is a category with a forgetful functor to Set. It is called *topological* if arbitrary initial and final structures exist uniquely and certain smallness conditions are fulfilled [1]. If $\mathcal{A}$ is a construct and X a set, then $\mathcal{A}(X)$ stands for the fiber of $\mathcal{A}$-structures on X. If A and B are objects on the same underlying set X then we write $B \leqslant A$ if $\mathrm{id}_X : A \to B$ is a morphism. For a topological construct $\mathcal{A}$ and an object X, $\mathcal{A}(X)$ is a complete lattice with top element the discrete structure and bottom element the indiscrete structure. Sometimes we will use the notation $\underline{X}$ to denote an object in $\mathcal{A}$ with underlying set X.

Given pseudometric spaces (X, d) and (X', d') a map $f : (X, d) \to (X', d')$ is called a *contraction* if $d' \circ f \times f \leqslant d$. $\mathcal{P}$ stands for the construct of pseudometric spaces and contractions, where, in this paper, a pseudometric means the classical concept and thus in particular is finite valued.

We refer the reader to [4] for the basic results on metrically generated theories. In this paper we only recall those definitions and results which we require. We will also recall some main proofs, restricted to the setting of this paper. Met stands for the topological construct with objects all "generalized metric spaces" in the sense that the only condition on the "metric" is that it must be zero on the diagonal, and with morphisms all contractions as defined above. It is clear that $\mathcal{P}$ is a base category in the sense of [4], meaning that it is closed for initial morphisms and contains all Met-indiscrete spaces. Given this base category $\mathcal{P}$, a topological construct $\mathcal{X}$ is called *$\mathcal{P}$-metrically generated* if there exists a concrete functor $K : \mathcal{P} \to \mathcal{X}$ satisfying the following two properties:

(I) K preserves initial morphisms, i.e. for any $f : X \to X'$,

$$f : K(X, d \circ f \times f) \to K(X', d)$$

is initial in $\mathcal{X}$.

(D) $K(\mathcal{P})$ is initially dense in $\mathcal{X}$, i.e. any object of $\mathcal{X}$ is initial in $\mathcal{X}$ for some source having codomains in $K(\mathcal{P})$.

We will then also say that $\mathcal{X}$ is *$\mathcal{P}$-metrically generated by K*. Most often we will refrain from explicitly referring to $\mathcal{P}$, e.g. from now on metrically generated construct will always mean $\mathcal{P}$-metrically generated construct.

Metrically generated constructs can be represented by a model category. In order to explain this we need some preliminary concepts and results.

A $\mathcal{P}$-downset on X is a downset $\mathcal{S} \subset [0,\infty[^{X\times X}$ such that for any $e \in \mathcal{S}$ there exists $d \in \mathcal{P}(X) \cap \mathcal{S}$ with $e \leqslant d$. For any $\mathcal{Q} \subset [0,\infty[^{X\times X}$, we write $\mathcal{Q}\!\downarrow := \{e \in [0,\infty[^{X\times X} \mid \exists d \in \mathcal{Q}\colon e \leqslant d\}$. If $\mathcal{Q} \subset \mathcal{P}(X)$ and $\mathcal{Q}\!\downarrow = \mathcal{S}$ we say that $\mathcal{Q}$ is a *basis* for $\mathcal{S}$.

$\mathsf{M}^{\mathcal{P}}$ stands for the construct with objects, pairs $(X,\mathcal{D})$ where X is a set and $\mathcal{D}$ is a $\mathcal{P}$-downset. $\mathcal{D}$ is called a *$\mathcal{P}$-meter (on X)* and $(X,\mathcal{D})$ a *$\mathcal{P}$-metered space*. If $(X,\mathcal{D})$ and $(X',\mathcal{D}')$ are $\mathcal{P}$-metered spaces then a function $f : X \to X'$ is called a *contraction* if

$$\forall d' \in \mathcal{D}'\colon\ d' \circ f \times f \in \mathcal{D}.$$

The fact that the objects are expressed in terms of meters rather than with collections consisting only of $\mathcal{P}$-metrics has the advantage that the results obtained in this paper also fit in the more general setting of [4], which makes transition to other theories possible.

It is easily verified that $\mathsf{M}^{\mathcal{P}}$ is a topological construct. Given a structured source $(f_j : X \to (X_j,\mathcal{D}_j))_{j\in J}$, the initial structure on X is the meter

$$\{d \circ f_j \times f_j \mid j \in J,\ d \in \mathcal{D}_j\}\!\downarrow.$$

Analogously, given a structured sink $(f_j : (X_j,\mathcal{D}_j) \to X)_{j\in J}$, the final structure on X is the meter

$$\bigl\{d \in \mathcal{P}(X) \bigm| \forall j \in J\colon\ d \circ f_j \times f_j \in \mathcal{D}_j\bigr\}\!\downarrow.$$

In order to easily deal with concretely coreflective subconstructs of $\mathsf{M}^{\mathcal{P}}$ we require the following concept.

We call ξ an *expander on* $\mathsf{M}^{\mathcal{P}}$ if for any X and any meter $\mathcal{D} \in \mathsf{M}^{\mathcal{P}}(X)$, ξ provides us with a meter $\xi(\mathcal{D}) \in \mathsf{M}^{\mathcal{P}}(X)$ in such a way that the following properties are fulfilled:

(E1) $\mathcal{D} \subset \xi(\mathcal{D})$,
(E2) $\mathcal{D} \subset \mathcal{N} \Rightarrow \xi(\mathcal{D}) \subset \xi(\mathcal{N})$,
(E3) $\xi(\xi(\mathcal{D})) = \xi(\mathcal{D})$,
(E4) if $f : Y \to X$ and $\mathcal{D} \in \mathsf{M}^{\mathcal{P}}(X)$ then $\xi(\mathcal{D}) \circ f \times f \subset \xi(\mathcal{D} \circ f \times f\!\downarrow)$.

Here, for any meter $\mathcal{D}$, we have put $\mathcal{D} \circ f \times f := \{d \circ f \times f \mid d \in \mathcal{D}\}$. Given an expander ξ on $\mathsf{M}^{\mathcal{P}}$, we define $\mathsf{M}^{\mathcal{P}}_{\xi}$ as the full subconstruct of $\mathsf{M}^{\mathcal{P}}$ with objects those metered spaces $(X,\mathcal{D})$ for which $\xi(\mathcal{D}) = \mathcal{D}$. For the proof of the next result we refer to [4].

Proposition 2.1. *For any expander ξ on $\mathsf{M}^{\mathcal{P}}$, $\mathsf{M}^{\mathcal{P}}_{\xi}$ is a concretely coreflective subconstruct of $\mathsf{M}^{\mathcal{P}}$ and conversely any concretely coreflective subconstruct of $\mathsf{M}^{\mathcal{P}}$ is so obtained.*

Given $K : \mathcal{P} \to \mathcal{X}$ satisfying (I) and (D), we introduce two concrete functors which will play a crucial role in the sequel.

For any object $(X,\mathcal{D})$ in $\mathsf{M}^{\mathcal{P}}$, with basis $\mathcal{Q}$ for $\mathcal{D}$, we put

$$F_K(X,\mathcal{D}) = \sup_{q\in\mathcal{Q}} K(X,q).$$

Note that this unambiguously defines $F_K : \mathsf{M}^{\mathcal{P}} \to \mathcal{X}$.

For any object $\underline{X}$ in $\mathcal{X}$ we put

$$G_K(\underline{X}) = (X, \mathcal{Q}\!\downarrow)$$

where $\mathcal{Q} = \{q \in \mathcal{P}(X) \mid K(X,q) \leqslant \underline{X}\}$. Note that $\mathcal{Q} \neq \emptyset$. This defines $G_K : \mathcal{X} \to \mathsf{M}^{\mathcal{P}}$. From initiality properties it follows that F_K is a functor and from (I) it follows that G_K is a functor. To avoid confusion, suprema in a topological construct $\mathcal{X}$ will, from now on, sometimes be denoted by $\sup^{\mathcal{X}}$. We now formulate the main theorem and add a sketch of the proof. More details can be found in [4].

Theorem 2.2. *A topological construct is $\mathcal{P}$-metrically generated if and only if it is concretely isomorphic to $\mathsf{M}^{\mathcal{P}}_{\xi}$ for some expander ξ on $\mathsf{M}^{\mathcal{P}}$.*

Proof. Let $K : \mathcal{P} \to \mathcal{X}$ be a functor satisfying (I) and (D). By definition we have that $F_K \circ G_K(\underline{X}) \leqslant \underline{X}$ for any $\mathcal{X}$-object $\underline{X}$. To prove the other inequality, by (D), let $\mathcal{Q}$ be a set of metrics such that $\underline{X} = \sup_{q\in\mathcal{Q}} K(X,q)$. If $G_K(\underline{X}) = (X,\mathcal{D}\!\downarrow)$ with $\mathcal{D}$ as in the definition of G_K then $\mathcal{Q} \subset \mathcal{D}$ and thus $\underline{X} \leqslant \sup_{q\in\mathcal{D}} K(X,q) = F_K(G_K(\underline{X}))$. This proves that $F_K \circ G_K = 1$. That $G_K \circ F_K \geqslant 1$ is proved in the same way.

From the adjointness just established we now have a coreflector

$$\mathsf{M}^{\mathcal{P}} \to \mathsf{M}^{\mathcal{P}}_{\xi}$$

which can be expressed in terms of the following expander. For any X and any meter $\mathcal{D}$ on X let ξ be defined by

$$\big(X, \xi(\mathcal{D})\big) = G_K\big(F_K(X, \mathcal{D})\big).$$

Clearly, the restriction of F_K to $\mathsf{M}^{\mathcal{P}}_{\xi}$ and G_K are mutually inverse isomorphisms between $\mathsf{M}^{\mathcal{P}}_{\xi}$ and $\mathcal{X}$.

Finally, for any expander ξ on $\mathsf{M}^{\mathcal{P}}$, $\mathsf{M}^{\mathcal{P}}_{\xi}$ is metrically generated. Define

$$K_\xi : \mathcal{P} \to \mathsf{M}^{\mathcal{P}}_{\xi} : (X, d) \mapsto \big(X, \xi(d{\downarrow})\big).$$

If $f : (X, \mathcal{D}) \to (X', \mathcal{D}')$ is a morphism in $\mathsf{M}^{\mathcal{P}}_{\xi}$ and $(X', \xi(d{\downarrow})) \leqslant (X', \mathcal{D}')$, then we have that $d \circ f \times f \in \mathcal{D}$ and $\xi((d \circ f \times f){\downarrow}) \subset \xi(\mathcal{D}) = \mathcal{D}$. This proves that (I) is fulfilled. If $(X, \mathcal{D})$ is an object in $\mathsf{M}^{\mathcal{P}}_{\xi}$ with basis $\mathcal{Q}$ of $\mathcal{D}$ then we have that

$$(X, \mathcal{D}) \leqslant \sup^{\mathsf{M}^{\mathcal{P}}}_{q \in \mathcal{Q}} \big(X, \xi(q{\downarrow})\big) \leqslant \sup^{\mathsf{M}^{\mathcal{P}}_{\xi}}_{q \in \mathcal{Q}} \big(X, \xi(q{\downarrow})\big) \leqslant (X, \mathcal{D}),$$

where the second inequality follows from the fact that $\mathsf{M}^{\mathcal{P}}_{\xi}$ is concretely coreflective in $\mathsf{M}^{\mathcal{P}}$ as stated in 2.1. □

3. Some players

In this section we will describe metrically generated constructs which will play a role in the sequel of the present paper. The case of bornological spaces is more intricate and is treated in the next section. Let $(X, \mathcal{D})$ be any metered space.

3.1. Uniform spaces

There is a functor

$$U : \mathcal{P} \to \mathsf{Unif}$$

which with every pseudometric space (X, d) associates the uniform space $(X, \mathcal{U}_d)$ in the usual way. The expander ξ_U given by this functor is determined by:

$e \in \xi_U(\mathcal{M})$ if and only if there exists a pseudometric d such that $e \leqslant d$ and

$$\forall \varepsilon > 0,\ \exists d_1, \ldots, d_n \in \mathcal{M},\ \exists \delta > 0\colon\ \Big\{\sup_{i=1}^{n} d_i < \delta\Big\} \subset \{d < \varepsilon\}.$$

The isomorphic copy of the category of uniform spaces in $\mathsf{M}^{\mathcal{P}}$ is $\mathsf{M}^{\mathcal{P}}_{\xi_U}$ and the associated functors F_U and G_U are given by

$$F_U : \mathsf{M}^{\mathcal{P}} \to \mathsf{Unif} : (X, \mathcal{D}) \mapsto (X, \mathcal{U}_{\mathcal{D}})$$

where $\mathcal{U}_{\mathcal{D}}$ stands for the uniformity generated by the subbase of entourages $\{\{d < \epsilon\} \mid d \in \mathcal{D},\ \epsilon > 0\}$ and

$$G_U : \mathsf{Unif} \to \mathsf{M}^{\mathcal{P}} : (X, \mathcal{U}) \mapsto (X, \mathcal{D}_{\mathcal{U}})$$

where $\mathcal{D}_{\mathcal{U}} := \{d \mid d$ uniformly continuous for $\mathcal{U}\}$. Metered spaces $(X, \mathcal{D})$ for which $\xi_U(\mathcal{D}) = \mathcal{D}$ represent exactly the uniform spaces, as they are described in [8].

3.2. Completely regular spaces

There is also a functor

$$T : \mathcal{P} \to \mathsf{Creg}$$

which with every pseudometric space (X, d) associates the completely regular space $(X, \mathcal{T}_d)$ in the usual way. The expander ξ_T given by this functor is determined by:

$e \in \xi_T(\mathcal{M})$ if and only if there exists a pseudometric d such that $e \leqslant d$ and

$$\forall x \in X,\ \forall \varepsilon > 0,\ \exists d_1, \ldots, d_n \in \mathcal{M},\ \exists \delta > 0\colon\ \Big\{\sup_{i=1}^{n} d_i(x, \cdot) < \delta\Big\} \subset \big\{d(x, \cdot) < \varepsilon\big\}.$$

The isomorphic copy of the category of completely regular spaces in $\mathsf{M}^{\mathcal{P}}$ is $\mathsf{M}^{\mathcal{P}}_{\xi_T}$ and the associated functors F_T and G_T are given by

$$F_T : \mathsf{M}^{\mathcal{P}} \to \mathsf{Creg} : (X, \mathcal{D}) \mapsto (X, \mathcal{T}_{\mathcal{D}})$$

where $\mathcal{T}_{\mathcal{D}}$ is generated by the neighborhoods $\{d(x,\cdot) < \epsilon\}$, $d \in \mathcal{D}$, $\epsilon > 0$ and $x \in X$ and

$$G_T : \mathsf{Creg} \to \mathsf{M}^{\mathcal{P}} : (X, \mathcal{T}) \mapsto (X, \mathcal{D}_{\mathcal{T}})$$

where $\mathcal{D}_{\mathcal{T}} := \{d \mid \forall x \in X,\ d(x,\cdot) \text{ continuous for } \mathcal{T}\}$.

Metered spaces $(X, \mathcal{D})$ for which $\xi_T(\mathcal{D}) = \mathcal{D}$ represent exactly the completely regular topological spaces.

3.3. Lipschitz spaces

The category of Lipschitz spaces was introduced in [7,5]. We recall that a Lipschitz space is a set X together with a collection $\mathcal{L}$ of pseudometrics satisfying the following conditions:

(1) $d_1, d_2 \in \mathcal{L} \Rightarrow d_1 + d_2 \in \mathcal{L}$;
(2) $d \in \mathcal{L}$ and $e \leqslant d \Rightarrow e \in \mathcal{L}$;
(3) $d \wedge 1 \in \mathcal{L} \Rightarrow d \in \mathcal{L}$.

Given two Lipschitz spaces $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ a function $f : X \to X'$ is called a *Lipschitz function* if for each $d' \in \mathcal{L}'$ there exists $d \in \mathcal{L}$, $\delta > 0$ and $K > 0$ such that $d' \circ f \times f \leqslant Kd$ on $\{d < \delta\}$. Lipschitz spaces and functions form a topological category, which, although not in those terms, was actually proved in [7]. There is a functor

$$L : \mathcal{P} \to \mathsf{Lip}$$

which with any pseudometric space (X, d) associates the Lipschitz space $(X, \mathcal{L}_d)$ where

$$e \in \mathcal{L}_d \quad \Leftrightarrow \quad \exists d \in \mathcal{D},\ \delta > 0 \quad \text{and} \quad K > 0\text{: } \{d < \delta\} \subset \{e \leqslant Kd\}.$$

This functor satisfies the conditions (I) and (D) required to turn Lip into a metrically generated construct. It is an easy exercise, which we leave to the reader, to prove that in this setting of metered spaces the conditions (1)–(3) above can be rephrased in terms of an expander ξ_L which is determined by:

$e \in \xi_L(\mathcal{D})$ if and only if there exists a pseudometric d such that $e \leqslant d$ and

$$\exists d_1, \ldots, d_n \in \mathcal{D},\ \delta > 0,\ K > 0\text{: } \left\{\sum_{i=1}^{n} d_i < \delta\right\} \subset \left\{d \leqslant K \sum_{i=1}^{n} d_i\right\}.$$

Further, being a Lipschitz function precisely means being a contraction in this setting. The isomorphic copy of the category of Lipschitz spaces in $\mathsf{M}^{\mathcal{P}}$ is $\mathsf{M}^{\mathcal{P}}_{\xi_L}$ and the associated functors F_L and G_L are given by

$$F_L : \mathsf{M}^{\mathcal{P}} \to \mathsf{Lip} : (X, \mathcal{D}) \mapsto (X, \mathcal{L}_{\mathcal{D}})$$

where $\mathcal{L}_{\mathcal{D}} := \{d \in \xi_L(\mathcal{D}) \mid d \text{ pseudometric}\}$ and

$$G_T : \mathsf{Lip} \to \mathsf{M}^{\mathcal{P}} : (X, \mathcal{L}) \mapsto (X, \mathcal{D}_{\mathcal{L}})$$

where $\mathcal{D}_{\mathcal{L}} := \{e \mid \exists d \in \mathcal{L},\ e \leqslant d\}$.

Metered spaces $(X, \mathcal{D})$ for which $\xi_L(\mathcal{D}) = \mathcal{D}$ represent exactly Lipschitz spaces.

Given two expanders ξ and ξ' on $\mathsf{M}^{\mathcal{P}}$, it is easily seen that $\xi \leqslant \xi'$ if and only if $\mathsf{M}^{\mathcal{P}}_{\xi'}$ embeds into $\mathsf{M}^{\mathcal{P}}_{\xi}$ via the identity functor, in which case the embedding is coreflective. We will then refer to this as the *natural embedding*.

Proposition 3.1. Creg *is a coreflective subconstruct of* Unif *and* Unif *is a coreflective subconstruct of* Lip *by the natural embeddings.*

Proof. Let $\mathcal{D}$ be a meter. Suppose that $e \in \xi_L(\mathcal{D})$ and $\epsilon > 0$, and take $d_1, \ldots, d_n \in \mathcal{D}$, $\delta > 0$ and $K > 0$ as assured by the formula of ξ_L. Put $d := \sup_i d_i$ and $\delta' := \min\{\frac{\delta}{n}, \frac{\epsilon}{nK}\}$. Then it follows from $d(x, y) < \delta'$ that

$$e(x, y) \leqslant K \sum_{i=1}^{n} d_i(x, y) \leqslant Kn\delta' \leqslant \epsilon$$

and thus $e \in \xi_U(\mathcal{D})$, proving that $\xi_L \leqslant \xi_U$.

Now suppose that $e \in \xi_U(\mathcal{D})$, $x \in X$ and $\epsilon > 0$ and let $d_1, \ldots, d_n \in \mathcal{D}$, $\delta > 0$ be as assured by the formula for ξ_U. Then it follows at once that

$$\left\{\sup_{i=1}^{n} d_i(x,\cdot) < \delta\right\} = \left\{\sup_{i=1}^{n} d_i < \delta\right\}(x) \subset \{d < \varepsilon\}(x) = \left\{d(x,\cdot) < \varepsilon\right\}$$

proving that $e \in \xi_T(\mathcal{D})$ and hence that $\xi_U \leqslant \xi_T$. $\square$

4. Bornological spaces

In this section we investigate in greater detail the example of bornological spaces.

Definition 4.1. A bornological space $(X, \mathcal{B})$ is a set X structured by a collection $\mathcal{B}$ of subsets of X satisfying the following conditions:

(1) if $x \in X$ then $\{x\} \in \mathcal{B}$;
(2) if $B_1 \subset B_2$ and $B_2 \in \mathcal{B}$ then $B_1 \in \mathcal{B}$;
(3) if $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$ then $B_1 \cup B_2 \in \mathcal{B}$.

The elements of $\mathcal{B}$ are called bounded sets. A function $f : (X, \mathcal{B}) \to (Y, \mathcal{B}')$ is said to be a bornological map if it preserves bounded sets.

Bor is the construct of bornological spaces and bornological maps. As is well known, it is a topological construct, see for instance [12], from which we recall the formulation of initial structures.

Let $(f_j : X \to (X_j, \mathcal{B}_j))_{j\in J}$ be a source with all $\mathcal{B}_j's$ being bornologies. Then $B \subset X$ is bounded in the initial bornological structure $\mathcal{B}$ on X if and only if for all $i \in J$ the set $f_i(B)$ is bounded.

Let $(f_j : (X_j, \mathcal{B}_j) \to X)_{j\in J}$ be a sink with all $\mathcal{B}_j's$ being bornologies then $B \subset X$ is bounded in the final bornological structure $\mathcal{B}$ on X if and only if B is a finite union of sets belonging to $\{f_i(A_i) \mid i \in J,\ A_i \in \mathcal{B}_i\} \cup \{\{x\} \mid x \in X\}$.

In spite of the fact that every metric space (X, d) gives rise to a natural associated bornology $\mathcal{B}_d$, namely the collection of all d-bounded subsets, the functor

$$B : \mathcal{P} \to \mathsf{Bor} : (X, d) \mapsto (X, \mathcal{B}_d)$$

does not satisfy the condition (D). But also other possible functors cannot generate Bor.

Proposition 4.2. *The construct* Bor *is not metrically generated.*

Proof. Suppose that $K : \mathcal{P} \to \mathsf{Bor}$ is a functor satisfying (I) and (D), mapping (X, d) to $K(X, d) = (X, \mathcal{K}_d)$. Let X be infinite and let $\mathcal{F}$ be a non-principal ultrafilter on X. Then $\mathcal{B}_\mathcal{F} := \{B \subset X \mid X \setminus B \in \mathcal{F}\}$ is a bornology on X. Actually, barring the indiscrete bornology on X, $\mathcal{B}_\mathcal{F}$ is a minimal element in the lattice of bornologies on X. Indeed, suppose $\mathcal{B}_\mathcal{F} \subset \mathcal{B}$ such that there exists $A \in \mathcal{B} \setminus \mathcal{B}_\mathcal{F}$. Then, necessarily, $X \setminus A \in \mathcal{B}_\mathcal{F}$ and hence $X = A \cup (X \setminus A) \in \mathcal{B}$, i.e. $\mathcal{B}$ is the indiscrete bornology. From (D) it follows that there exists a collection $\mathcal{D}$ of metrics such that $(X, \mathcal{B}_\mathcal{F}) = \sup^{\mathsf{Bor}}_{d\in\mathcal{D}} K(X, d) = (X, \bigcap_{d\in\mathcal{D}} \mathcal{K}_d)$. By minimality of $\mathcal{B}_\mathcal{F}$ there hence exists d such that $\mathcal{B}_\mathcal{F} = \mathcal{K}_d$. The number of bornologies of type $\mathcal{B}_\mathcal{F}$ for $\mathcal{F}$ an ultrafilter on X is $2^{2^{|X|}}$, however the number of metrics on X is dominated by $2^{|X|}$. This is a contradiction. □

In order to develop a differentiation theory for linear spaces, A. Frölicher and A. Kriegl had to restrict to those bornological spaces coming from so-called l^∞-structures [6]. Bornological spaces coming from an l^∞ structure have been characterized by the previous authors in various ways. We use the following characterization as definition and call such bornologies, l^∞-bornologies.

Definition 4.3. A bornology is an l^∞-bornology if every unbounded subset contains a countably infinite subset, the only bounded subsets of which are the finite ones.

We denote by Bor^∞ the full subconstruct of Bor with objects all l^∞-bornological spaces.

Proposition 4.4. Bor^∞ *is a concretely reflective subconstruct of* Bor, *which moreover is stable under the formation of coproducts in* Bor.

Proof. Let $(f_j : (X, \mathcal{B}) \to (X_j, \mathcal{B}_j))_{j\in J}$ be an initial source in Bor with all $\mathcal{B}_j's$ being l^∞-bornologies. Let $A \subset X$ be unbounded, then there exists $i \in J$ with $f_i(A)$ unbounded. Thus, there exists a countably infinite subset $D \subset f_i(A)$ the only bounded subsets of which are finite. For every $d \in D$ choose an element in $f_i^{-1}(d) \cap A$ and let E be the set of those points. Then E is a countably infinite subset of A, the only bounded subsets of which are finite. This proves that Bor^∞ is a concretely reflective subconstruct of Bor. Next let $X := \coprod_{j\in J}(X_j, \mathcal{B}_j)$ be a coproduct in Bor with all $\mathcal{B}_j's$ being l^∞-bornologies. Let $A \subset X$ be unbounded. Either $K := \{j \in J \mid (X_j \times \{j\}) \cap A \neq \emptyset\}$ is finite, and then for some $k \in K$, $(X_k \times \{k\}) \cap A$ is unbounded and contains a countably infinite subset the only bounded subsets of which are finite, or K is infinite and then in an infinite countable number of these nonempty intersections we choose a point. The set of these points then again satisfies the required condition. □

Bor^∞ is not stable under the formation of quotients in Bor as the following example shows.

Example 4.5. Consider $\mathbb{N}$ and a fixed ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then it is easily seen that the ultrafilter bornological space $(\mathbb{N}, \mathcal{B}_\mathcal{U})$ of 4.2 is not an l^∞-bornological space.

Now for any $U \in \mathcal{U}$ let $X_U := \mathbb{N} \setminus U$ be equipped with the indiscrete bornology and put

$$X := \coprod_{U \in \mathcal{U}} X_U.$$

Again, it is easily seen that this is an l^∞-bornological space. Now consider the map

$$f : X \to \mathbb{N}$$

where for each $U \in \mathcal{U}$, $f|_{X_U}$ is the canonical injection and let $\mathcal{B}_{\text{fin}}$ stand for the final bornology on $\mathbb{N}$.

(1) $\mathcal{B}_\mathcal{U} \subset \mathcal{B}_{\text{fin}}$: indeed, if $B \in \mathcal{B}_\mathcal{U}$ then $\mathbb{N} \setminus B \in \mathcal{U}$ and hence $X_\mathcal{U} = B$ is bounded in X. Thus $B = f(B) \in \mathcal{B}_{\text{fin}}$.
(2) $\mathcal{B}_{\text{fin}} \neq \mathcal{P}(\mathbb{N})$: we show that $\mathbb{N}$ is not bounded for $\mathcal{B}_{\text{fin}}$. If $\mathbb{N} = f(A)$ then $A \cap X_U \neq \emptyset$ for infinitely many $U \in \mathcal{U}$ since $A \subset \bigcup_{i=1}^n X_{U_i}$ implies $f(A) \subset \bigcup_{i=1}^n f(X_{U_i})$ and hence

$$f(A) \subset \bigcup_{i=1}^n \mathbb{N} \setminus U_i = \mathbb{N} \setminus \bigcap_{i=1}^n U_i \subsetneq \mathbb{N}.$$

(3) The maximality of $\mathcal{B}_\mathcal{U}$ gives that $\mathcal{B}_\mathcal{U} = \mathcal{B}_{\text{fin}}$.

As a consequence, Bor^∞ is not a coreflective subconstruct of Bor. Nevertheless, being a concretely reflective subconstruct of Bor, it is a topological construct in its own right.

Proposition 4.6. *Metric bornologies are l^∞-bornologies.*

Proof. Let (X, d) be a metric space and suppose that D is unbounded. Take $x_0 \in D$ arbitrary. Suppose that points $x_0, \ldots, x_n$ have been found with the property that for all $k < l$, $d(x_l, x_k) \geqslant l - k$. Then take $x_{n+1} \in D \setminus \bigcup_{m=0}^n B(x_m, n+1-m)$. Put $C := \{x_n \mid n \in \mathbb{N}\}$. In this set, for all $k < l$, $d(x_l, x_k) \geqslant l - k$. Then C is unbounded, and hence also infinitely countable, by construction. Moreover it is clear that its only bounded subsets are the finite ones. Hence the condition to be an l^∞-bornology is fulfilled. $\square$

The following result can be deduced indirectly from [6], but below we include a direct proof. We use the notation diam_d to denote the diameter with respect to d.

Theorem 4.7. *The construct* Bor^∞ *is metrically generated.*

Proof. Consider the following concrete functor:

$$B : \mathcal{P} \to \mathsf{Bor}^\infty : (X, d) \mapsto (X, \mathcal{B}_d),$$

where $\mathcal{B}_d$ stands for the bornology generated by d.

Clearly, for $f : (X, d \circ f \times f) \to (X', d)$ and $A \subset X$, $\text{diam}_d f(A) = \text{diam}_{d \circ f \times f} A$, which implies that B preserves initial morphisms. Next we prove that $B(\mathcal{P})$ is initially dense in Bor^∞.

Consider, for $(X, \mathcal{B})$ an l^∞-bornological space, the full source

$$f : \big((X, \mathcal{B}) \to (Y, \mathcal{B}_d)\big)_{(Y,d) \in \mathcal{P}, f \text{ bornological}}.$$

Suppose $B \subset X$ is unbounded, choose an infinite countable subset $D \subset B$ the only bounded subsets of which are the finite ones. Put $D := \{b_n \mid n \geqslant 1\}$ with all $b_m \neq b_n$ for $m \neq n$. Define the map

$$f : (X, \mathcal{B}) \to (\mathbb{N}, \mathcal{B}_{d_E})$$

by

$$f(x) := \begin{cases} 0 & \text{if } x \notin D, \\ n & \text{if } x = b_n. \end{cases}$$

Clearly f is bornological and $f(B)$ is unbounded. This proves that the source is initial. $\square$

As a corollary of our Theorem 2.2 we can now formulate an isomorphic description of the construct Bor^∞ by means of a suitable expander ξ_B on $\mathsf{M}^\mathcal{P}$. The explicit form of the expander is described as follows.

Proposition 4.8. Bor^∞ *is isomorphic to* $\mathsf{M}^{\mathcal{P}}_{\xi_B}$ *where the expander* ξ_B *is determined by*

$$e \in \xi_B(\mathcal{D}) \quad \textit{iff} \quad \forall B\colon \operatorname{diam}_e(B) < \infty \quad \textit{whenever}\ \forall d \in \mathcal{D},\ \operatorname{diam}_d(B) < \infty.$$

Note that the expander ξ_B which we encounter here is quite different from the expanders ξ_T, ξ_U and ξ_L we used in order to describe isomorphic copies of the constructs Creg, Unif or Lip. For instance, unlike the expanders ξ_T, ξ_U and ξ_L the expander ξ_B does not satisfy the condition that for the zero metric 0 on X, $\xi(\{0\}) = \{0\}$. A condition that was introduced in [3] in order to develop a separation theory for metrically generated theories. Every bounded metric on X belongs to $\xi_B(\{0\})$. In fact, applying to Bor the usual categorical definition for an object X to be separated, in the sense that all morphisms from the two point indiscrete space to X are constant implies that the only separated bornological spaces are the singleton spaces.

5. Functors to the construct of bornological spaces

As before consider the following concrete functor:

$$B : \mathcal{P} \to \mathsf{Bor}^\infty : (X, d) \mapsto (X, \mathcal{B}_d),$$

where $\mathcal{B}_d$ stands for the bornology generated by d and with the notations of 5.7 consider the functor

$$F_B : \mathsf{M}^{\mathcal{P}} \to \mathsf{Bor}^\infty.$$

Explicitly, for a metered space $(X, \mathcal{D})$ the associated bornology is defined by $F_B(X, \mathcal{D}) = (X, \mathcal{B}_{\mathcal{D}})$ with

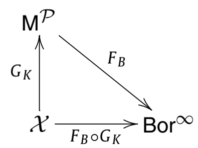
$$\mathcal{B}_{\mathcal{D}} = \bigl\{A \bigm| \operatorname{diam}_d(A) < \infty,\ \forall d \in \mathcal{D}\bigr\}.$$

Proposition 5.1. *With the real line* $\mathbb{R}$ *endowed with the Euclidean metric* d_E *and for any* $(X, \mathcal{D})$ *in* $\mathsf{M}^{\mathcal{P}}$ *we have*: $A \in \mathcal{B}_{\mathcal{D}}$ *if and only if* $f(A)$ *is* d_E*-bounded whenever* $f : (X, \mathcal{D}) \to (\mathbb{R}, \{d_E\}\!\downarrow)$ *is a contraction in* $\mathsf{M}^{\mathcal{P}}$.

Proof. One implication follows immediately from the fact that F_B is a functor. For the other implication, let d be any metric in $\mathcal{D}$. Fix x_0 in X then clearly $f : (X, \mathcal{D}) \to (\mathbb{R}, \{d_E\}\!\downarrow)$ defined by $f(x) = d(x_0, x)$ is a contraction. From the d_E-boundedness of $f(A)$ we immediately conclude that $\operatorname{diam}_d(A)$ is finite. □

Now let $\mathsf{M}^{\mathcal{P}}_{\xi}$ be an arbitrary coreflective subconstruct of $\mathsf{M}^{\mathcal{P}}$. For a metered space in $\mathsf{M}^{\mathcal{P}}_{\xi}$ we define the associated bornological space by restricting F_B to $\mathsf{M}^{\mathcal{P}}_{\xi}$.

Definition 5.2. For a construct $\mathcal{X}$, metrically generated by $K : \mathcal{P} \to \mathcal{X}$ we define the *natural bornology* associated to any object by first going to its equivalent description in the model category and second applying F_B.



From the universality of the coreflector described by ξ and from 5.1 we immediately get the following:

Proposition 5.3. *With the real line* $\mathbb{R}$ *endowed with the Euclidean metric* d_E *and for any* $(X, \mathcal{D})$ *in* $\mathsf{M}^{\mathcal{P}}_{\xi}$ *we have*: $A \in \mathcal{B}_{\mathcal{D}}$ *if and only if* $f(A)$ *is* d_E*-bounded whenever* $f : (X, \mathcal{D}) \to (\mathbb{R}, \xi(\{d_E\}\!\downarrow))$ *is a contraction in* $\mathsf{M}^{\mathcal{P}}_{\xi}$.

Corollary 5.4. *In the case of* Creg, *metrically generated as in Section* 3.2, *the natural bornology associated with a given completely regular space* $(X, \mathcal{T})$ *consists of all subsets of* X *on which all real-valued continuous functions on* $(X, \mathcal{T})$ *are bounded.*

This bornology is one of the examples formulated in [10] and plays an important role in study of $\mathcal{C}(X)$ as a topological vector-space. In general this bornology differs from the collection of all relatively compact subsets of $(X, \mathcal{T})$ as is for instance the case on the ordinal space $W(\omega_1)$ [8].

Corollary 5.5. *In the case of* Unif, *metrically generated as in Section* 3.1, *the natural bornology associated with a given uniform space* $(X, \mathcal{U})$ *consists of all subsets of* X *on which all real-valued uniformly continuous functions on* $(X, \mathcal{U})$ *are bounded.*

In [9] it was shown that this bornology can also be obtained as the collection of subsets A of X satisfying the following condition expressed in terms of entourages (see also Bourbaki [2]).

(B) for every $U \in \mathcal{U}$ there exist a finite set $F \subset X$ and some natural number n such that $A \subset U^n(F)$.

Remark that given an arbitrary metered space $(X, \mathcal{D})$ the natural bornology associated with it does not necessarily coincide with the bornology of the associated uniformity $\mathcal{U}_\mathcal{D}$. For instance the set $\mathbb{R}$ is bounded for the metered space $(\mathbb{R}, \{d_E \wedge 1\}\!\downarrow)$ but it is not bounded for the associated uniformity which is the usual one.

It means that the fact that $\mathrm{diam}_d(A) < \infty$ for every $d \in \mathcal{D}$ is generally not strong enough to imply that A fulfills condition (B) for $\mathcal{U}_\mathcal{D}$.

However, we will now prove that as in the case of Unif also for the larger construct Lip the entourages of the associated uniformity determine the natural bornology.

Theorem 5.6. *Given an arbitrary metered space $(X, \mathcal{D})$ and A a subset of X the following are equivalent*:

(1) $\mathrm{diam}_d(A) < \infty$ *for every* $d \in \xi_L(\mathcal{D})$;
(2) *for every $U \in \mathcal{U}_\mathcal{D}$ there exist a finite set $F \subset X$ and some natural number n such that $A \subset U^n(F)$.*

Proof. The only nontrivial implication is that (1) implies (2). The proof of this implication relies heavily on the construction made in the proof of Theorem 1.12 in [9]. Let $U \in \mathcal{U}_\mathcal{D}$ be a symmetric entourage. Consider an equivalence relation on X by putting $x \sim y$ if there exists a natural number n such that $x \in U^n(y)$, and let $(P_\alpha)_{\alpha \in M}$ be the associated partition of X. By definition of $\mathcal{U}_\mathcal{D}$ there exist pseudometrics $d_1, \ldots, d_n$ in $\mathcal{D}$ and $\epsilon > 0$ such that $\{d_1 \vee \cdots \vee d_n < \epsilon\} \subset U$. It is easily seen that $\xi_L(\mathcal{D})$ is saturated for taking finite sups a multiples of pseudometrics. Hence the pseudometric $\psi := \frac{1}{\epsilon} d_1 \vee \cdots \vee d_n$ belongs to $\xi_L(\mathcal{D})$. Moreover it fulfills $\{\psi < 1\} \subset U$.

By means of ψ another function φ is now defined on $\bigcup_{\alpha \in M}(P_\alpha \times P_\alpha)$. For x and y in P_α all possible U-chains are considered connecting x with y, i.e. finite sequences $x = x_1, x_2, \ldots, x_n, x_{n+1} = y$ such that $(x_i, x_{i+1}) \in U$ for $i = 1, \ldots, n$. Put $\varphi(x, y) = \inf \Sigma_{i=1}^n \psi(x_i, x_{i+1})$ where the infimum is taken over all possible U-chains connecting x and y. Clearly this new function φ coincides with ψ on U.

Let $A \subset X$ and put $M_A = \{\alpha \in M \mid P_\alpha \cap A \neq \emptyset\}$. If M_A is finite put $\mu(\alpha) = 1$ for each $\alpha \in M_A$. If M_A is infinite choose an injective sequence $(\alpha_n)_n$ in M_A and put $\mu(\alpha_n) = n$, $\mu(\alpha) = 1$ for $\alpha \in M_A$ and different from all α_n. Choose $v_\alpha \in P_\alpha$ for each $\alpha \in M$.

The function φ is now extended to ρ on the whole space X. For $x \in P_\alpha$ and $y \in P_\beta$

$$\rho(x, y) = \varphi(x, y) \quad \text{if } \alpha = \beta,$$

$$\rho(x, y) = \varphi(x, v_\alpha) + \varphi(y, v_\beta) + \mu(\alpha) + \mu(\beta) \quad \text{if } \alpha \neq \beta.$$

In [9] it is shown that this function is a pseudometric on X.

Next we prove that ρ belongs to $\xi_L(\mathcal{D})$. In the formula in 3.3 defining $\xi_L(\mathcal{D})$ take $e = d = \rho$ and $d_1 = \psi$, $\delta = 1$ and $K = 1$. Then, for x and y in X, we have

$$\begin{aligned} \psi(x, y) < 1 \quad &\Rightarrow \quad (x, y) \in U \\ &\Rightarrow \quad x \text{ and } y \text{ are in the same } P_\alpha,\ \psi(x, y) = \varphi(x, y) \\ &\Rightarrow \quad \rho(x, y) = \psi(x, y). \end{aligned}$$

Since ρ belongs to $\xi_L(\mathcal{D})$ the assumption in (1) implies that $\mathrm{diam}_\rho(A) < \infty$. The rest of the proof now follows exactly as in [9]. □

Corollary 5.7. *In any metrically generated construct $\mathcal{X} \approx \mathsf{M}_\xi^\mathcal{P}$ which is a subconstruct of* Lip *the natural bornology of an object $(X, \mathcal{D})$ of $\mathcal{X}$ is determined by the associated uniformity $\mathcal{U}_\mathcal{D}$.*

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories, John Wiley & Sons, New York, 1990.
[2] N. Bourbaki, Topologie Générale, chap. 2, Stuctures Uniformes, Hermann, Paris, 1965.
[3] V. Claes, E. Colebunders, A. Gerlo, Epimorphisms and co-wellpoweredness for separated metrically generated theories, Acta Math. Hungar. 114 (2007) 133–152.
[4] E. Colebunders, R. Lowen, Metrically generated theories, Proc. Amer. Math. Soc. 133 (2005) 1547–1556.
[5] R.B. Fraser Jr., Axiom systems for Lipschitz structures, Fund. Math. 62 (1969) 15–25.
[6] A. Frölicher, A. Kriegl, Linear Spaces and Differentiation Theory, Pure Appl. Math., John Wiley & Sons, 1988.
[7] M.A. Geraghty, On Lipschitz structures, J. Math. Anal. Appl. II (1965) 191–198.
[8] L. Gillman, M. Jerison, Rings of Continuous Functions, Springer, 1976.

[9] J. Hejcman, Boundedness in uniform spaces and topological groups, Czechoslovak Math. J. 84 (1959) 544–563.
[10] H. Hogbe-Nlend, Théorie des bornologies et applications, Lecture Notes in Math., vol. 213, Springer, 1971.
[11] R. Lowen, Approach Spaces: The Missing Link in the Topology Uniformity Metric Triad, Oxford Math. Monogr., Oxford University Press, 1997.
[12] G. Preuss, Theory of Topological Structures, Math. Appl., Kluwer Academic Publishers, 1988.
[13] L. Waelbroeck, Topological Vector Spaces and Algebras, Lecture Notes in Math., vol. 230, Springer, 1971.