Building multivariate Sato models with linear dependence

Reference:
Boen Lynn, Guillaume Florence.- Building multivariate Sato models w ith linear dependence
Quantitative finance - ISSN 1469-7688 - 19:4(2019), UNSP 1850007
Full text (Publisher's DOI): https://doi.org/10.1080/14697688.2018.1523547
To cite this reference: https://hdl.handle.net/10067/1550780151162165141

Building multivariate Sato models with linear dependence

Lynn Boen[*], Florence Guillaume[†‡]

December 19, 2017

Abstract

The increased trading in multi-name financial products has required the development of state-of-the-art multivariate models. These models should be computationally tractable and, at the same time, flexible enough to explain the stylized facts of asset log-returns and of their dependence structure. The popular class of multivariate Lévy models provides a variety of tractable models, but suffers from one major shortcoming: Lévy models can replicate single-name derivative prices for a given time-to-maturity, but not for the whole range of quoted strikes and maturities, especially during periods of market turmoil. Moreover, there is a significant discrepancy between the moment term structure of Lévy models and the one observed in the market. Sato processes on the other hand exhibit a moment term structure that is more in line with empirical evidence and allow for a better replication of single-name option price surfaces. In this paper, we propose a general framework for multivariate models characterized by independent and time-inhomogeneous increments, where the asset log-return processes at unit time are modeled as linear combinations of independent self-decomposable random variables, where at least one self-decomposable random variable is shared by all the assets. As examples, we consider two general subclasses within this new framework, where we assume a normal variance-mean mixture with a one-sided tempered stable mixing density or a difference of one-sided tempered stable laws for the distribution of the risk factors. Particular attention is given to the models' ability to explain the asset dependence structure. A numerical study reveals the advantages of these new types of models.

Keywords: *multivariate asset pricing models, Sato processes, space-scaled self-decomposable laws, calibration*

1 Introduction

The ever-growing demand for basket structured products has stimulated the search for and the development of more realistic multivariate asset pricing models. These multivariate financial models should be at once computationally tractable and flexible enough to replicate the stylized facts of both single-name asset log-returns and their dependence structure, and this whatever the level of market fear. Although many alternatives to the univariate Black-Scholes model have been suggested over the past twenty years to model single-name asset prices, so far only a few extensions

[*]University of Antwerp, Middelheimlaan 1 , B-2020 Antwerpen, Belgium. E-mail: lynn.boen@uantwerpen.be

[†]University of Antwerp, Middelheimlaan 1 , B-2020 Antwerpen, Belgium. E-mail: florence.guillaume@uantwerpen.be

[‡]The authors acknowledge the support of the Research Fund (BOF) of the University of Antwerp (41/FA070300/3/FFB150337).

have been proposed in the multivariate setting, most of them being characterized by Lévy marginal processes. Among these ones, we can distinguish two broad classes of models, based on the way dependence between single-name assets is introduced. In the first category, dependence is achieved by subordinating a multivariate Brownian motion by a linear combination of subordinators, among which at least one is common to all single-name assets. For example, in the multivariate VG model of Madan and Seneta [30], which was later extended by Luciano and Schoutens [27] to incorporate skewness, the multivariate Brownian motion is subordinated by a univariate Gamma distributed time-change. However, due to the uniqueness of the business clock, this multivariate VG model does not allow for independence of the asset log-returns. Hence, Semeraro proposed the αVG model as an extention, where the business clock is modeled as a multivariate subordinator constructed as the weighted sum of two independent Gamma processes with the same shape parameters. Relaxing the constraints on the Gamma subordinators leads to the Generalized αVG model proposed by Guillaume [16], where the marginals are still Lévy but no longer VG distributed. Due to this modification, the volatility of the asset log-returns, as opposed to the volatility under the original αVG model, depends on both the common and the idiosyncratic risk factors, which is more in line with the empirical evidence of the existence of a common as well as an individual business clock (see [19] and [26]). Other examples of multivariate models within this model class can be found in e.g. [28] or [35]. In the second class, dependence is introduced by considering linear combinations of independent univariate Lévy processes, among which at least one is a driving factor of each underlying asset composing the pool. This factor approach was introduced by Vasicek [38] for Brownian motions and was later adopted to the case of Lévy processes, for example in [20] and [31]. A general framework for this class of Lévy models was recently developed by Ballotta and Bonfiglioli [2], where they assume each single-name asset log-return to be driven by a linear combination of one systemic and one idiosyncratic risk factor. Whereas Ballotta and Bonfiglioli considered only one systemic risk factor, Marfé [32] elaborated a multivariate Lévy framework where each asset log-return is driven by a linear combination of one idiosyncratic component and multiple systemic risk factors. The dependence in positive and negative jumps is modeled separately, which allows for a higher flexibility in capturing non-linear dependence structures. Note that in both model classes dependence is introduced by considering marginal asset log-return processes that are driven by at least one common Lévy process, which is required since, unlike Brownian motions which are always active, independent pure jump Lévy processes are active at a disjoint set of times [12].

Although Lévy models, unlike the Black-Scholes model, can accommodate the empirical evidence of leptokurtosis, of semi-heavy tails and of the presence of jumps in asset log-returns, they typically fail to explain option prices across both the strike and the time-to-maturity spectra. Such discrepancy with the market reality typically occurs during periods of financial distress, such as the global financial crisis of 2008, as highlighted by Guillaume (see [15] and [16]). Moreover, Konikov and Madan [21] empirically determined the moment term structures of asset log-returns and observed a significant mismatch with the moment term structures of the Lévy processes. Sato processes on the other hand, which are processes with time-inhomogeneous increments, exhibit moment term structures that are more in line with market observations. Moreover, they are better able at reproducing quoted option prices across both the strike and time-to-maturity dimensions, as illustrated in, for example, [7], [10] and [15].

In this paper, we propose a general framework for linearly dependent multivariate models characterized by independent and time-inhomogeneous increments. The asset log-return processes at unit time are modeled as linear combinations of independent self-decomposable random variables, leading to multivariate models with Sato marginal processes, since self-decomposability is preserved under convolution. Dependence between the asset log-returns is introduced by considering for each asset an idiosyncratic and a systemic risk factor. Both positive and negative correlations can be accommodated by assigning positive or negative weights to the common driving factor. The proposed approach can be seen as an adaptation of the Lévy framework developed in [2], where the time-homogeneous increment property is relaxed. Given the ability of Sato models to explain single-name option prices across both the strike and the time-to-maturity dimensions, this adjustment is expected to improve the goodness of fit of the single-name option price surfaces when compared to multivariate Lévy models. Besides, due to its more flexible correlation structure, the proposed model class will constitute a suitable alternative to the multivariate Sato model built by time-changing a multivariate Brownian motion as proposed in [15]. As examples, we consider normal variance-mean mixtures with a one-sided tempered stable mixing density and differences of one-sided tempered stable laws for the distribution of the risk factors. A numerical study reveals the advantages of this new type of multivariate models.

The paper is organized as follows: Sections 2 and 3 recall the fundamental properties of Lévy and Sato processes, respectively. The general model framework is given in Section 4, where we elaborate some specific multivariate models as well by choosing popular self-decomposable distributions for the risk factors. The calibration procedure is discussed in Section 5, together with the calibration instruments used to calibrate the univariate option surfaces and the dependence structure. Section 6 compares the dependence structure of the multivariate linear VG Lévy and Sato models to the one of the αVG Lévy and Sato models proposed in [15] and [16], whereas the calibration results are discussed in Section 7. Section 8 concludes.

2 Lévy processes

A Lévy process is a stochastic process $X = \{X_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following properties (see [24]):

- The paths of X are $\mathbb{P}$-almost surely càdlàg;
- $X_0 = 0$ $\mathbb{P}$-almost surely;
- For $0 \leq s \leq t$, it holds that $X_t - X_s \stackrel{d}{=} X_{t-s}$;
- For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u \ : \ u < s\}$.

The last two conditions imply that Lévy processes have stationary and independent increments. Lévy processes are the dynamic counterpart of infinitely divisible distributions, i.e. if X is a Lévy process, then X_t has an infinitely divisible distribution $\forall t \geq 0$ and conversely, for any infinitely divisible law $\mathcal{L}$, we can define a Lévy process $X = \{X_t, t \geq 0\}$ such that $X_1 \sim \mathcal{L}$.

Definition 1. *The distribution of a random variable X is infinitely divisible if, for every integer n, the characteristic function $\phi(u) = \mathbb{E}[\exp(\mathrm{i}uX)]$ of X can be written as the nth power of a characteristic function $\phi_n(u)$: $\phi(u) = (\phi_n(u))^n$.*

The Lévy-Khintchine representation characterizes infinitely divisible distributions, and thus Lévy processes, in terms of their characteristic exponent $\Psi_X(u) = \ln(\phi_X(u))$:

Definition 2. *The characteristic exponent of any infinitely divisible distribution can be written in terms of the Lévy-Khintchine representation:*

$$\Psi_X(u) = \mathrm{i}\gamma u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} (\exp(\mathrm{i}ux) - 1 - \mathrm{i}ux\mathbf{1}_{|x|<1})\nu(\mathrm{d}x), \quad u \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$ is called the drift and $\sigma \geq 0$ the diffusion coefficient. ν is a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}} \min(1, |x|^2)\nu(\mathrm{d}x) < \infty$ and is called the Lévy measure.

A Lévy process is uniquely defined by its Lévy triplet (γ, σ, ν). A detailed overview of Lévy processes can be found, for instance, in [11], [33] or [35].

3 Sato processes

Sato processes are closely linked to the class of self-decomposable distributions.

Definition 3. *A random variable X has a self-decomposable law if, for any constant c, $0 < c < 1$, X has the same probability law as the sum of a down-scaled version of itself and an independent random variable X_c:*

$$X \overset{d}{=} cX + X_c. \tag{3.1}$$

Self-decomposable distributions are infinitely divisible distributions with a Lévy-Khintchine representation of the form (see [24]):

$$\Psi_X(u) = \mathrm{i}\gamma u - \frac{\sigma^2}{2}u^2 + \int_{\mathbb{R}} \left(\exp(\mathrm{i}ux) - 1 - \mathrm{i}ux\mathbf{1}_{|x|<1}\right) \frac{h(x)}{|x|} dx, \quad u \in \mathbb{R}, \tag{3.2}$$

where $h(x) \geq 0$ is a function that is decreasing for positive x and increasing for negative x.

A Sato process can be constructed from any self-decomposable distribution: the law of X_t is obtained by space-scaling the law of a self-decomposable random variable X (see [10]):

$$X_t \overset{d}{=} t^\gamma X,$$

where γ is called the self-similar exponent. Sato processes are thus defined as self-similar processes with a self-decomposable law at unit time. It can be shown that a Sato process is additive (for a proof, see [10]). Hence, Sato processes have independent, but time-inhomogeneous increments. One can easily check that, for a Sato process $X = \{X_t, t \geq 0\}$, it holds that $\mathrm{Var}[X_t] = t^{2\gamma}\mathrm{Var}[X]$ and that the skewness and kurtosis are constant over the term. In general, we have:

$$\mu_n[X_t] = \mu_n[X], \quad n > 2,$$

where μ_n is the nth standardized moment. Consequently, Sato processes have moment term structures that are more consistent with the market implied moment term structures than Lévy processes. Indeed, Konikov and Madan [21] empirically showed that the skewness (in absolute value) and the kurtosis of market asset log-returns are constant or rise slightly over the term, while they scale like $1/\sqrt{t}$ and $1/t$, respectively, for Lévy processes.

The following theorem implies that we can build new Sato processes based on any linear combination of independent self-decomposable random variables and will be used to build multivariate linear factor models with marginal Sato processes:

Theorem 1. *Let X and Z be independent self-decomposable random variables with Lévy triplet $(\gamma_X, \sigma_X, \nu_X)$ and $(\gamma_Z, \sigma_Z, \nu_Z)$, respectively, where $\nu_X(\mathrm{d}x) = \frac{h_1(x)}{|x|}\mathrm{d}x$ and $\nu_Z(\mathrm{d}x) = \frac{h_2(x)}{|x|}\mathrm{d}x$. Then $Y = X + aZ$, $a \in \mathbb{R}\setminus\{0\}$ is again a self-decomposable random variable with Lévy triplet $(\gamma_Y, \sigma_Y, \nu_Y)$, where*

- $\gamma_Y = \gamma_X + a\gamma_Z + \int_{\mathbb{R}} \left(x \left(\mathbf{1}_{\{|x|<1\}} - \mathbf{1}_{\{|x|<|a|\}} \right) \right) \frac{h_2(\frac{x}{a})}{|x|} \mathrm{d}x$,
- $\sigma_Y^2 = \sigma_X^2 + a^2 \sigma_Z^2$,
- $\nu_Y(\mathrm{d}x) = \left(\frac{h_1(x)+h_2(\frac{x}{a})}{|x|} \right) \mathrm{d}x$.

Proof. See Appendix A. □

For additional information about Sato processes, we refer the reader to [33].

4 Multivariate models built as linear combinations of Lévy and Sato processes

4.1 General Lévy framework

Ballotta and Bonfiglioli [2] propose a class of multivariate Lévy models with dependent marginals, built as linear combinations of independent Lévy processes. In their approach, dependence is introduced by considering for each asset an idiosyncratic and a systemic risk factor. We shortly revisit the construction and properties of this class of multivariate Lévy models and refer the reader to [2] for further details.

Let $Z = \{Z_t, t \geq 0\}$ and $X^{(j)} = \{X_t^{(j)}, t \geq 0\}$, $j = 1, \ldots, n$ be independent Lévy processes. Then for $a_j \in \mathbb{R} \setminus \{0\}$, the process $Y = \{Y_t, t \geq 0\}$ with

$$\mathbf{Y}_t = (Y_t^{(1)}, \ldots, Y_t^{(n)})' = (X_t^{(1)} + a_1 Z_t\ , \ldots,\ X_t^{(n)} + a_n Z_t)' \tag{4.1}$$

is a multivariate Lévy process with characteristic function

$$\phi_{\mathbf{Y}}(\mathbf{u}; t) = \phi_Z\left(\sum_{j=1}^n a_j u_j; t\right) \prod_{j=1}^n \phi_{X^{(j)}}(u_j; t), \quad \mathbf{u} \in \mathbb{R}^n. \tag{4.2}$$

Since dependence is introduced linearly, the pairwise linear correlation coefficient ρ correctly describes the dependence between components of the multivariate Lévy process $\mathbf{Y}$ at time t and is given by (see [2]):

$$\rho\left(Y_t^{(i)}, Y_t^{(j)}\right) = \frac{a_i a_j \operatorname{Var}[Z_1]}{\sqrt{\operatorname{Var}\left[Y_1^{(i)}\right]}\sqrt{\operatorname{Var}\left[Y_1^{(j)}\right]}}. \tag{4.3}$$

4.2 General Sato framework

The model (4.1) can be adapted to model asset log-returns with time-inhomogeneous increments provided that the marginal distributions of $\mathbf{Y}_1$ are self-decomposable. Theorem 1 implies that if $X_1^{(j)}, j = 1, \ldots, n$ and Z_1 are chosen to be self-decomposable random variables, then $\mathbf{Y}_1$ has self-decomposable marginals as well. In that case, we can construct a multivariate Sato model by assuming that the n-dimensional asset log-return is modeled by

$$\mathbf{Y}_t \stackrel{d}{=} t^{\gamma}\mathbf{Y} = (t^{\gamma_1} Y^{(1)}, \ldots, t^{\gamma_n} Y^{(n)})' = (t^{\gamma_1} X^{(1)} + a_1 t^{\gamma_1} Z, \ldots, t^{\gamma_n} X^{(n)} + a_n t^{\gamma_n} Z)'. \tag{4.4}$$

The characteristic function of $\mathbf{Y}_t$ is then given by

$$\phi_{\mathbf{Y}}(\mathbf{u};t) = \phi_{\mathbf{Y}}(\mathbf{u}t^{\boldsymbol{\gamma}};1) = \phi_Z\left(\sum_{j=1}^{n} a_j u_j t^{\gamma_j};1\right)\prod_{j=1}^{n}\phi_{X^{(j)}}(u_j t^{\gamma_j};1), \quad \mathbf{u}\in\mathbb{R}^n. \tag{4.5}$$

The pairwise linear correlation coefficient is given by (4.3) and correctly describes the dependence between the components of the multivariate Sato model. Besides, the term structure of the standardized co-moments of any multivariate process with Sato marginals, $\mathbf{X} = \{\mathbf{X}_t, t \geq 0\}$ is flat, which follows from combining the fact that $X_t^{(j)} \stackrel{d}{=} t^{\gamma_j} X_1^{(j)}$ and the definition of the (m,n)-th standardized co-moment of two random variables X and Y:

$$\mu_{m,n}(X,Y) = \frac{\mathbb{E}\left[(X-\mathbb{E}[X])^m\,(Y-\mathbb{E}[Y])^n\right]}{\left(\sqrt{\mathrm{Var}[X]}\right)^m\left(\sqrt{\mathrm{Var}[Y]}\right)^n}, \quad m,n\geq 0.$$

This contrasts with the term structure of the standardized co-moments of order $\{(m,n), m+n>2\}$ of multivariate Lévy processes, which are decreasing over the term. In particular, the co-skewness of a Lévy process $\mathbf{X} = \{\mathbf{X}_t, t\geq 0\}$ scales like $1/\sqrt{t}$ and the excess of co-kurtosis like $1/t$. A sketch of the proof is given in Appendix B. Note that, in both general frameworks (i.e. Lévy and Sato), one can impose conditions on the parameters of $\mathbf{X}_1$ and Z_1 to guarantee that $\mathbf{Y}_1$ belongs to the same class of distributions. Such a restriction is sometimes imposed to increase the tractability of the model (see, f.i., [2], [36] or [28]).

We now discuss popular examples of self-decomposable laws, namely one-sided tempered stable distributions and normal variance-mean mixtures with a one-sided tempered stable mixing density. We will use such distributions to build specific multivariate Sato option pricing models in Section 4.4 and 4.5, respectively.

4.3 One-sided tempered stable distributions

This section recalls the basic properties of one-sided tempered stable distributions. For more information on tempered stable distributions, we refer the reader to [23].

A random variable G is said to follow a one-sided tempered stable distribution, denoted $G \sim \mathrm{TS}(C,\alpha,\lambda)$, with parameters $C\in(0,\infty)$, $\lambda\in(0,\infty)$ and $\alpha\in[0,1)$, if it has a characteristic function of the following form:

$$\phi_G(u) = \exp\left(\int_{\mathbb{R}}(\exp(\mathrm{i}ux)-1)\nu(\mathrm{d}x)\right), \quad u\in\mathbb{R}, \tag{4.6}$$

where the Lévy measure ν is given by:

$$\nu(\mathrm{d}x) = \frac{C}{x^{1+\alpha}}e^{-\lambda x}\mathbf{1}_{(0,\infty)}(x)\mathrm{d}x. \tag{4.7}$$

The characteristic function (4.6) can be rewritten for $\alpha\in(0,1)$ as (see [23]):

$$\begin{aligned}\phi_G(u) &= \exp\left(C\Gamma(-\alpha)[(\lambda-\mathrm{i}u)^\alpha-\lambda^\alpha]\right)\\ &= \exp\left(C\Gamma(-\alpha)\lambda^\alpha\left[\left(1-\frac{\mathrm{i}u}{\lambda}\right)^\alpha-1\right]\right), \quad u\in\mathbb{R},\end{aligned}$$

where Γ denotes the Gamma function. The cumulants of a one-sided tempered stable distribution are given by (see [23]):

$$\kappa_n = \Gamma(n-\alpha)\frac{C}{\lambda^{n-\alpha}}, \quad n \in \mathbb{N}.$$

From the Lévy measure (4.7), we have that $\nu(\mathrm{d}x) = \frac{h(x)}{|x|}\mathrm{d}x$ where

$$h(x) = \begin{cases} 0 & x \leq 0, \\ Ce^{-\lambda x}x^{-\alpha} & x > 0. \end{cases} \tag{4.8}$$

It follows that the one-sided tempered stable distribution is self-decomposable. Hence, one can build Sato processes with a one-sided tempered stable distribution at unit time. Further, since one-sided tempered stable distributions are infinitely divisible and concentrated on the positive real line, one can build (zero-drift) subordinators with a $\mathrm{TS}(C, \alpha, \lambda)$ law at unit time.

Definition 4. *If, in the Lévy-Khintchine representation of an infinitely divisible random variable X, we have that $\sigma = 0$, $\nu(-\infty, 0) = 0$, $\int_{|x|<1} |x|\nu(\mathrm{d}x) < \infty$ and*

$$\Psi_X(u) = \mathrm{i}\gamma_0 u + \int_{\mathbb{R}_+} (\exp(\mathrm{i}ux) - 1)\nu(\mathrm{d}x), \quad u \in \mathbb{R},$$

where $\gamma_0 \geq 0$, then $\{X_t, t \geq 0\}$ is a subordinator, i.e. a non-decreasing Lévy process.

Subordinators are not suited to model asset log-returns, since they are non-decreasing. However, subordinators are frequently used to time-change other Lévy processes, such as Brownian motions, where the deterministic calendar time is replaced by some business time, such that the arrival of new information occurs according to a stochastic business clock (see e.g. [1], [30] and [36]). In particular, when the subordinator is associated to a one-sided tempered stable distribution, the subordinated Brownian motion at unit time follows a normal variance-mean mixture with a one-sided tempered stable mixing density, which will be called a normal tempered stable distribution from now on. Alternatively, single-name asset log-returns at unit time can be modeled by the difference of two self-decomposable random variables such that the log-price gains and losses are modeled separately.

4.4 Normal tempered stable distributions

A random variable X follows a normal tempered stable distribution (i.e. a normal variance-mean mixture with a one-sided tempered stable distribution as mixing density) if

$$X = \theta G + \sigma\sqrt{G}B,$$

where $\theta \in \mathbb{R}$, $\sigma > 0$, $G \sim \mathrm{TS}(C, \alpha, \lambda)$ and where $B \sim N(0, 1)$ is independent of G (see [3]). The characteristic function of the random variable X is then given by:

$$\phi_X(u) = e^{\Psi_G(u\theta + \mathrm{i}u^2\sigma^2/2)}, \quad u \in \mathbb{R}, \tag{4.9}$$

where

$$\Psi_G(u) = C\Gamma(-\alpha)\lambda^\alpha \left[\left(1 - \frac{\mathrm{i}u}{\lambda}\right)^\alpha - 1\right], \quad u \in \mathbb{R}. \tag{4.10}$$

Combining (4.9) and (4.10), the characteristic function of X is given by:

$$\phi_X(u) = \exp\left(C\Gamma(-\alpha)\lambda^\alpha\left[\left(1-\frac{\mathrm{i}u\theta}{\lambda}+\frac{u^2\sigma^2}{2\lambda}\right)^\alpha-1\right]\right), \quad u\in\mathbb{R}. \tag{4.11}$$

The Lévy process associated to X is then a normal tempered stable process, i.e. $X=\{X_t, t\geq 0\}$ is constructed as a Brownian motion with drift time-changed by a one-sided tempered stable subordinator $G=\{G_t, t\geq 0\}$ independent of the standard Brownian motion $B=\{B_t, t\geq 0\}$ (see for example [23] for the use of normal tempered stable processes in finance). The Lévy triplet of the normal tempered stable distribution is given by:

Theorem 2. *Let $X=\{X_t, t\geq 0\}$ be a Lévy process on $\mathbb{R}$ with Lévy triplet $(\gamma_X, \sigma_X, \nu_X)$ and $Z=\{Z_t, t\geq 0\}$ be a subordinator with Lévy triplet $(\gamma_Z, 0, \nu_Z)$. Then the Lévy triplet of the subordinated Lévy process $Y=\{Y_t, t\geq 0\}$ arising from them, i.e. $Y_t = X_{Z_t}$, is given by:*

- $\gamma_Y = \gamma_Z\gamma_X + \int_{\mathbb{R}^+}\nu_Z(\mathrm{d}t)\int_{|x|<1} x\mu^t(\mathrm{d}x)$,
- $\sigma_Y = \gamma_Z\sigma_X$,
- $\nu_Y(B) = \gamma_Z\nu_X(B) + \int_{\mathbb{R}^+}\mu^t(B)\nu_Z(\mathrm{d}t), \qquad B\in\mathcal{B}(\mathbb{R}\setminus\{0\})$,

where μ^t is the probability density function of X_t.

Proof. For a proof, we refer the reader to [33]. □

Hence, since the Lévy triplet of a Brownian motion with drift θ is given by $(\theta, \sigma, 0)$ and the Lévy triplet of a one-sided tempered stable distribution by $(\int_{|x|<1} x\nu(\mathrm{d}x), 0, \nu)$, with ν as in (4.7), the Lévy measure $\tilde{\nu}$ of the normal tempered stable distribution becomes:

$$\tilde{\nu}(x) = \frac{C}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}^+}\frac{e^{-\frac{(x-\theta t)^2}{2\sigma^2 t}-\lambda t}}{t^{\alpha+3/2}}\mathrm{d}t, \quad x\in\mathbb{R}\setminus\{0\}. \tag{4.12}$$

Using the following representation of the modified Bessel function of the second kind (see 8.432.6 in [14]):

$$K_p(y) = \frac{1}{2}\left(\frac{y}{2}\right)^p\int_{\mathbb{R}^+} e^{-t-\frac{y^2}{4t}}t^{-p-1}\mathrm{d}t, \quad y>0, p\in\mathbb{R},$$

we can rewrite (4.12) as:

$$\tilde{\nu}(x) = \frac{2C}{\sigma\sqrt{2\pi}}e^{\frac{\theta x}{\sigma^2}}\left(\frac{\sqrt{\theta^2+2\sigma^2\lambda}}{|x|}\right)^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}\left(\frac{|x|\sqrt{\theta^2+2\sigma^2\lambda}}{\sigma^2}\right), \quad x\in\mathbb{R}\setminus\{0\}. \tag{4.13}$$

In what follows, whenever we are working with a normal tempered stable distribution, we impose $\mathbb{E}[G]=1$, leading to $C=\frac{\lambda^{1-\alpha}}{\Gamma(1-\alpha)}$. This restriction makes sure that the subordinator associated to G increases on average as the calendar time t. The characteristic exponent of G_1 then reduces to

$$\Psi_{G_1}(u) = \frac{-\lambda}{\alpha}\left[\left(1-\frac{\mathrm{i}u}{\lambda}\right)^\alpha-1\right], \quad u\in\mathbb{R}, \tag{4.14}$$

and the characteristic function of X_1 to

$$\phi_{X_1}(u) = \exp\left(\frac{-\lambda}{\alpha}\left[\left(1-\frac{\mathrm{i}u\theta}{\lambda}+\frac{u^2\sigma^2}{2\lambda}\right)^\alpha-1\right]\right), \quad u\in\mathbb{R}. \tag{4.15}$$

The parameter $\alpha \in [0, 1)$ is called the *stability parameter* and $k := \frac{1-\alpha}{\lambda}$ the *variance rate*, since

$$\mathrm{Var}[G_t] = \frac{1-\alpha}{\lambda} t = kt.$$

We will denote $X \sim \mathrm{NTS}(\alpha, \sigma, k, \theta)$ if X is a normal tempered stable random variable with a $\mathrm{TS}\left(\frac{\lambda^{1-\alpha}}{\Gamma(1-\alpha)}, \alpha, \lambda\right)$ mixing density.

The $\mathrm{NTS}(\alpha, \sigma, k, \theta)$ law satisfies the following properties:

- Time-scaling property: $X_1 \sim \mathrm{NTS}(\alpha, \sigma, k, \theta) \;\Rightarrow\; X_t \sim \mathrm{NTS}(\alpha, \sqrt{t}\sigma, \frac{k}{t}, t\theta)$,
- Space-scaling property: $X_1 \sim \mathrm{NTS}(\alpha, \sigma, k, \theta) \;\Rightarrow\; cX_1 \sim \mathrm{NTS}(\alpha, c\sigma, k, c\theta)$,

for $t \geq 0$ and $c > 0$.

By the following theorem, which is stated and proved in [34], the normal tempered stable distribution is self-decomposable by construction.

Theorem 3 (Sato). *Let $\{W_t, t \geq 0\}$ be a Brownian motion with drift and let $\{G_t, t \geq 0\}$ be a self-decomposable subordinator. Then the subordinated process $\{X_t, t \geq 0\}$ arising from them is self-decomposable.*

Proof. For the proof we refer the reader to [34]. □

Hence, one can construct a multivariate Sato model of the form (4.4) based on the $\mathrm{NTS}(\alpha, \sigma, k, \theta)$ distribution.

Definition 5. *We define the class of multivariate linear normal tempered stable Sato models by considering $X_1^{(j)}, j = 1, \dots, n$ and Z_1 in (4.4), to be independent normal tempered stable distributions, i.e. $X_1^{(j)} \sim NTS(\alpha, \sigma_j, k_j, \theta_j), j = 1, \dots, n$ and $Z_1 \sim NTS(\alpha, \sigma_Z, k_Z, \theta_Z)$*[1].

The characteristic function (4.5) of $\mathbf{Y}_t$ then becomes:

$$\begin{aligned}\phi_{\mathbf{Y}}(\mathbf{u}; t) = &\exp\left(\frac{(\alpha-1)}{\alpha k_Z}\left[\left(1 - \frac{\mathrm{i}\left(\sum_{j=1}^n a_j u_j t^{\gamma_j}\right)\theta_Z k_Z}{1-\alpha} + \frac{\left(\sum_{j=1}^n a_j u_j t^{\gamma_j}\right)^2 \sigma_Z^2 k_Z}{2(1-\alpha)}\right)^{\alpha} - 1\right]\right) \\ &\times \prod_{j=1}^n \exp\left(\frac{(\alpha-1)}{\alpha k_j}\left[\left(1 - \frac{\mathrm{i} u_j t^{\gamma_j}\theta_j k_j}{1-\alpha} + \frac{u_j^2 t^{2\gamma_j}\sigma_j^2 k_j}{2(1-\alpha)}\right)^{\alpha} - 1\right]\right), \quad \mathbf{u} \in \mathbb{R}^n.\end{aligned}$$

Consequently, the marginal characteristic function of $Y_t^{(j)}, j = 1, \dots, n$ is given by:

$$\begin{aligned}\phi_{Y^{(j)}}(u; t) = &\exp\left(\frac{(\alpha-1)}{\alpha k_Z}\left[\left(1 - \frac{\mathrm{i} a_j u t^{\gamma_j}\theta_Z k_Z}{1-\alpha} + \frac{a_j^2 u^2 t^{2\gamma_j}\sigma_Z^2 k_Z}{2(1-\alpha)}\right)^{\alpha} - 1\right]\right) \\ &\times \exp\left(\frac{(\alpha-1)}{\alpha k_j}\left[\left(1 - \frac{\mathrm{i} u_j t^{\gamma_j}\theta_j k_j}{1-\alpha} + \frac{u_j^2 t^{2\gamma_j}\sigma_j^2 k_j}{2(1-\alpha)}\right)^{\alpha} - 1\right]\right), \quad u \in \mathbb{R}.\end{aligned}$$

[1]Note that we assume the same α for $X_1^{(j)}, j = 1, \dots, n$ and Z_1 in order to have idiosyncratic and systemic risk factors that belong to the same sub-class of normal tempered stable distributions. Note that this restriction can be relaxed to enhance the flexibility of the model, at the cost of a decrease of its parsimony.

Similarly, we can define the class of multivariate linear normal tempered stable Lévy models by considering $X^{(j)} = \{X_t^{(j)}, t \geq 0\}, j = 1, \dots, n$ and $Z = \{Z_t, t \geq 0\}$ in (4.1) to be independent normal tempered stable processes. The characteristic function (4.2) of $\mathbf{Y}_t$ is then given by:

$$\begin{aligned}\phi_{\mathbf{Y}}(\mathbf{u};t) = &\exp\left(\frac{t(\alpha-1)}{\alpha k_Z}\left[\left(1-\frac{\mathrm{i}\left(\sum_{j=1}^n a_j u_j\right)\theta_Z k_Z}{1-\alpha}+\frac{\left(\sum_{j=1}^n a_j u_j\right)^2\sigma_Z^2 k_Z}{2(1-\alpha)}\right)^\alpha - 1\right]\right)\\ &\times\prod_{j=1}^n \exp\left(\frac{t(\alpha-1)}{\alpha k_j}\left[\left(1-\frac{\mathrm{i}u_j\theta_j k_j}{1-\alpha}+\frac{u_j^2\sigma_j^2 k_j}{2(1-\alpha)}\right)^\alpha - 1\right]\right), \quad \mathbf{u}\in\mathbb{R}^n,\end{aligned}$$

with marginal characteristic functions:

$$\begin{aligned}\phi_{Y^{(j)}}(u;t) = &\exp\left(\frac{t(\alpha-1)}{\alpha k_Z}\left[\left(1-\frac{\mathrm{i}a_j u\theta_Z k_Z}{1-\alpha}+\frac{a_j^2 u^2\sigma_Z^2 k_Z}{2(1-\alpha)}\right)^\alpha - 1\right]\right)\\ &\times\exp\left(\frac{t(\alpha-1)}{\alpha k_j}\left[\left(1-\frac{\mathrm{i}u_j\theta_j k_j}{1-\alpha}+\frac{u_j^2\sigma_j^2 k_j}{2(1-\alpha)}\right)^\alpha - 1\right]\right), \quad u\in\mathbb{R}.\end{aligned}$$

In multivariate models, the distribution of the asset log-returns is often restricted to be from the same family of distributions as the one of the underlying risk factors (see, f.i., [2], [28] or [36]). This can be achieved by imposing conditions on the idiosyncratic and the common parameters. In the case of normal tempered stable distributed risk factors, imposing the following conditions (4.16) on the parameters of $X_1^{(j)}, j = 1, \dots, n$ and Z_1 ensures that $Y_1^{(j)}, j = 1, \dots, n$ are normal tempered stable random variables as well, in both the Lévy and the Sato settings, leading to a restricted model:

$$\begin{cases} k_Z a_j \theta_Z = k_j \theta_j, & j = 1, \dots, n,\\ k_Z a_j^2 \sigma_Z^2 = k_j \sigma_j^2, & j = 1, \dots, n.\end{cases} \tag{4.16}$$

With these conditions, $Y_1^{(j)}$ follows a $\mathrm{NTS}(\alpha, \tilde{\sigma}_j, \tilde{k}_j, \tilde{\theta}_j) = \mathrm{NTS}(\alpha, a_j\sigma_Z\sqrt{\frac{k_Z+k_j}{k_j}}, \frac{k_j k_Z}{k_j+k_Z}, a_j\theta_Z\frac{k_j+k_Z}{k_j})$ distribution, where, due to the conditions (4.16), it must hold that

$$\frac{\tilde{\sigma}_j^2}{\tilde{\theta}_j^2\tilde{k}_j} = \frac{\sigma_Z^2}{k_Z\theta_Z} = c, \quad \forall j = 1, \dots, n, \tag{4.17}$$

for some $c > 0$. In what follows, we will scale the parameter σ_Z to 1 without loss of generality. Indeed, due to the space-scaling property of the $\mathrm{NTS}(\alpha, \sigma, k, \theta)$ distribution, multiplying a_j by a constant b and dividing σ_Z and θ_Z by b will not change the distribution of $\mathbf{Y}_t$.

4.4.1 Some specific examples

Different values of the stability parameter α result in different sub-classes of distributions for $X_1^{(j)}, j = 1, \dots, n$ and Z_1 in Definition 5. In particular, the density function of the normal tempered stable distribution is known in explicit form for $\alpha = 0$ and $\alpha = \frac{1}{2}$. With $\alpha = 0$, the $\mathrm{NTS}(\alpha, \sigma, k, \theta)$ distribution reduces to the variance gamma $\mathrm{VG}(\sigma, k, \theta)$ distribution and taking $\alpha = \frac{1}{2}$ leads to the normal inverse Gaussian NIG(σ, k, θ) law (see [11]). The VG(σ, k, θ) distribution has characteristic function

$$\phi_{\mathrm{VG}}(u; \sigma, k, \theta) = \left(1 - \mathrm{i}u\theta k + \frac{u^2\sigma^2 k}{2}\right)^{-1/k}, \quad u \in \mathbb{R}, \tag{4.18}$$

and the characteristic function of the NIG(σ, k, θ) distribution is given by:

$$\phi_{\mathrm{NIG}}(u; \sigma, k, \theta) = \exp\left(\frac{1}{k}\left(1 - \sqrt{1 - 2\mathrm{i}u\theta k + u^2\sigma^2 k}\right)\right), \quad u \in \mathbb{R}. \tag{4.19}$$

The first four moments of the VG and the NIG distributions are given in Table 1.

	VG(σ, k, θ)	NIG(σ, k, θ)
mean	θ	θ
variance	$\sigma^2 + k\theta^2$	$\sigma^2 + k\theta^2$
skewness	$\frac{\theta k\left(3\sigma^2 + 2k\theta^2\right)}{\left(\sigma^2 + k\theta^2\right)^{3/2}}$	$\frac{3k\theta}{\sqrt{\sigma^2 + k\theta^2}}$
kurtosis	$3\left(1 + 2k - \frac{k\sigma^4}{\left(\sigma^2 + k\theta^2\right)^2}\right)$	$3\left(1 + 5k - \frac{4k\sigma^2}{\sigma^2 + k\theta^2}\right)$

Table 1: Moments of the VG and the NIG distribution

.

Taking $X_1^{(j)}$ and Z_1 in (4.4) to be VG distributed, we obtain a multivariate linear VG Sato model. The joint characteristic function of $\mathbf{Y}_t$ is then given by:

$$\begin{aligned}\phi_{\mathbf{Y}}(\mathbf{u}; t) = &\left(1 - \mathrm{i}\left(\sum_{j=1}^{n} a_j u_j t^{\gamma_j}\right)\theta_Z k_Z + \frac{\left(\sum_{j=1}^{n} a_j u_j t^{\gamma_j}\right)^2 \sigma_Z^2 k_Z}{2}\right)^{\frac{-1}{k_Z}} \\ &\times \prod_{j=1}^{n}\left(1 - \mathrm{i}u_j t^{\gamma_j}\theta_j k_j + \frac{u_j^2 t^{2\gamma_j}\sigma_j^2 k_j}{2}\right)^{\frac{-1}{k_j}}, \quad \mathbf{u} \in \mathbb{R}^n.\end{aligned}$$

The corresponding multivariate linear VG Lévy process has characteristic function:

$$\phi_{\mathbf{Y}}(\mathbf{u}; t) = \left(1 - \mathrm{i}\left(\sum_{j=1}^{n} a_j u_j\right)\theta_Z k_Z + \frac{\left(\sum_{j=1}^{n} a_j u_j\right)^2 \sigma_Z^2 k_Z}{2}\right)^{\frac{-t}{k_Z}} \prod_{j=1}^{n}\left(1 - \mathrm{i}u_j\theta_j k_j + \frac{u_j^2\sigma_j^2 k_j}{2}\right)^{\frac{-t}{k_j}}, \quad \mathbf{u} \in \mathbb{R}^n,$$

and the correlation coefficient (4.3) becomes:

$$\rho\left(Y_t^{(i)}, Y_t^{(j)}\right) = \frac{a_i a_j\left(\sigma_Z^2 + k_Z\theta_Z^2\right)}{\sqrt{\sigma_i^2 + k_i\theta_i^2 + a_i^2\left(\sigma_Z^2 + k_Z\theta_Z^2\right)}\sqrt{\sigma_j^2 + k_j\theta_j^2 + a_j^2\left(\sigma_Z^2 + k_Z\theta_Z^2\right)}}.$$

Imposing the conditions (4.16), we obtain a restricted model setting where the marginals $Y_1^{(j)}, j = 1, \ldots, n$ follow a $\mathrm{VG}(a_j\sigma_Z\sqrt{\frac{k_Z+k_j}{k_j}}, \frac{k_jk_Z}{k_j+k_Z}, a_j\theta_Z\frac{k_j+k_Z}{k_j})$ distribution. Making the change of variables:

$$\tilde{\sigma}_j = a_j\sigma_Z\sqrt{\frac{k_Z+k_j}{k_j}}, \quad \tilde{k}_j =, \frac{k_jk_Z}{k_j+k_Z} \quad \text{and} \quad \tilde{\theta}_j = a_j\theta_Z\frac{k_j+k_Z}{k_j}, \tag{4.20}$$

leads to marginal characteristic functions that are independent of the parameters of the common risk factor Z (i.e. σ_Z, k_Z and θ_Z):

$$\phi_{Y^{(j)}}(u;t) = \left(1 - \mathrm{i}u\tilde{\theta}_j\tilde{k}_j + \frac{1}{2}u^2\tilde{\sigma}_j^2\tilde{k}_j\right)^{-t/\tilde{k}_j}, \quad u \in \mathbb{R}, \tag{4.21}$$

and

$$\phi_{Y^{(j)}}(u;t) = \left(1 - \mathrm{i}ut^{\gamma_j}\tilde{\theta}_j\tilde{k}_j + \frac{1}{2}u^2t^{2\gamma_j}\tilde{\sigma}_j^2\tilde{k}_j\right)^{-1/\tilde{k}_j}, \quad u \in \mathbb{R}, \tag{4.22}$$

under the restricted multivariate linear VG Lévy and the restricted multivariate linear VG Sato models, respectively. The correlation coefficient then reduces to

$$\rho\left(Y_t^{(i)}, Y_t^{(j)}\right) = \mathrm{sign}(\tilde{\theta}_i\tilde{\theta}_j)\frac{\sqrt{\tilde{k}_i\tilde{k}_j}}{k_Z} \propto \frac{1}{k_Z}. \tag{4.23}$$

Note that in order for $\tilde{\sigma}_j$ to be positive, it must hold that $a_j > 0$, $\forall j = 1, \ldots, n$ and the dependence is therefore restricted to be positive. Indeed, with $a_j > 0$ the sign of $\tilde{\theta}_i\tilde{\theta}_j$ must be $+1$ due to (4.20). Despite this limitation, the constraints (4.20) allow to perform the decoupling calibration procedure proposed by Leoni and Schoutens [25], where the calibration of the single-name option price surfaces is decoupled from the calibration of the dependence structure. This calibration technique should be used whenever possible, since it leads to a significant decrease in computation time and it reduces the probability to end up in a "bad" local minimum due to the splitting of the parameter space into subspaces. Note that for the present class of models, due to the restriction (4.17), the idiosyncratic parameters have to be calibrated together, which is, besides the positive correlation, the main drawback of this particular model. In Section 4.5.1 we propose an alternative model where one can take full advantage of the benefits of the decoupling calibration procedure.

One can construct multivariate models based on the NIG distribution in a similar manner, where the characteristic function of $\mathbf{Y}_t$ is then obtained by combining (4.19) and (4.2) or (4.5), to end up with a multivariate Lévy or Sato process, respectively.

Instead of modeling the idiosyncratic and systemic risk factors by a normal tempered stable distribution, we can decompose each of them into a positive (favorable) and a negative (unfavorable) risk factor, each being modeled by a one-sided tempered stable random variable, leading to a difference of one-sided tempered stable distributions.

4.5 Difference of one-sided tempered stable distributions

A random variable X is said to follow a difference of one-sided tempered stable distributions if $X = G^{(1)} - G^{(2)}$, where $G^{(1)} \sim \mathrm{TS}(C_1, \alpha, \lambda_1)$ and $G^{(2)} \sim \mathrm{TS}(C_2, \alpha, \lambda_2)$ are independent (see for example

[22] for the use of DTS distributions in finance). This will be denoted by $X \sim \mathrm{DTS}(C_1, C_2, \alpha, \lambda_1, \lambda_2)$. The characteristic function of a DTS random variable X is given by:

$$
\begin{aligned}
\phi_X(u) &= \phi_{G^{(1)}}(u)\phi_{G^{(2)}}(-u) \\
&= \exp\left(C_1\Gamma(-\alpha)\lambda_1^\alpha\left[\left(1-\frac{\mathrm{i}u}{\lambda_1}\right)^\alpha - 1\right]\right)\exp\left(C_2\Gamma(-\alpha)\lambda_2^\alpha\left[\left(1+\frac{\mathrm{i}u}{\lambda_2}\right)^\alpha - 1\right]\right) \qquad (4.24) \\
&= \exp\left(\Gamma(-\alpha)\left(C_1\lambda_1^\alpha\left[\left(1-\frac{\mathrm{i}u}{\lambda_1}\right)^\alpha - 1\right] + C_2\lambda_2^\alpha\left[\left(1+\frac{\mathrm{i}u}{\lambda_2}\right)^\alpha - 1\right]\right)\right), \quad u \in \mathbb{R}. \qquad (4.25)
\end{aligned}
$$

From Theorem 1, it follows that the Lévy triplet of a $\mathrm{DTS}(C_1, C_2, \alpha, \lambda_1, \lambda_2)$ distribution is given by:

$$
\left(\int_0^1 \frac{C_1 e^{-\lambda_1 x} - C_2 e^{-\lambda_2 x}}{x^\alpha}\mathrm{d}x, 0, \frac{C_1 e^{-\lambda_1 x}x^{-\alpha}\mathbf{1}_{(0,\infty)} + C_2 e^{-\lambda_2|x|}|x|^{-\alpha}\mathbf{1}_{(-\infty,0)}}{|x|}\mathrm{d}x\right).
$$

The Lévy process associated to a DTS distribution is called a difference of one-sided tempered stable subordinators or bilateral tempered stable process. The $\mathrm{DTS}(C_1, C_2, \alpha, \lambda_1, \lambda_2)$ distribution satisfies the following scaling properties:

- Time-scaling property: $X_1 \sim \mathrm{DTS}(C_1, C_2, \alpha, \lambda_1, \lambda_2) \;\Rightarrow\; X_t \sim \mathrm{DTS}(tC_1, tC_2, \alpha, \lambda_1, \lambda_2)$,
- Space-scaling property: $X_1 \sim \mathrm{DTS}(C_1, C_2, \alpha, \lambda_1, \lambda_2) \;\Rightarrow\; cX \sim \mathrm{DTS}(\frac{C_1}{c^\alpha}, \frac{C_2}{c^\alpha}, \alpha, c\lambda_1, c\lambda_2)$,

for $t \geq 0$ and $c > 0$.

Theorem 1 implies that the DTS law is self-decomposable. Hence, we can construct a multivariate Sato model of the form (4.4) based on the DTS distribution:

Definition 6. *We define the class of multivariate linear DTS Sato models by considering $X_1^{(j)}$, $j = 1, \ldots, n$ and Z_1 in (4.4) to be independent DTS random variables, i.e. $X_1^{(j)} \sim DTS(C_{1,j}, C_{2,j}, \alpha, \lambda_{1,j}, \lambda_{2,j})$, $j = 1, \ldots, n$ and $Z_1 \sim DTS(C_{1,Z}, C_{2,Z}, \alpha, \lambda_{1,Z}, \lambda_{2,Z})$[2].*

In general, we have:

$$
\begin{cases}
X_1^{(j)} &= G_1^{(1,j)} - G_1^{(2,j)}, \\
Z_1 &= H_1^{(1)} - H_1^{(2)},
\end{cases} \qquad (4.26)
$$

where $G_1^{(1,j)} \sim \mathrm{TS}(C_{1,j}, \alpha, \lambda_{1,j})$, $G_1^{(2,j)} \sim \mathrm{TS}(C_{2,j}, \alpha, \lambda_{2,j})$, for $j = 1, \ldots, n$, $H_1^{(1)} \sim \mathrm{TS}(C_{1,Z}, \alpha, \lambda_{1,Z})$ and $H_1^{(2)} \sim \mathrm{TS}(C_{2,Z}, \alpha, \lambda_{2,Z})$ are mutually independent. The characteristic function of the multivariate Sato process $\mathbf{Y}_t$ is then given by combinating (4.5) and (4.24):

$$
\begin{aligned}
\phi_{\mathbf{Y}}(\mathbf{u}; t) = &\exp\left(C_{1,Z}\Gamma(-\alpha)\lambda_{1,Z}^\alpha\left[\left(1 - \frac{\mathrm{i}\left(\sum_{j=1}^n a_j u_j t^{\gamma_j}\right)}{\lambda_{1,z}}\right)^\alpha - 1\right]\right) \\
&\times \exp\left(C_{2,Z}\Gamma(-\alpha)\lambda_{2,Z}^\alpha\left[\left(1 + \frac{\mathrm{i}\left(\sum_{j=1}^n a_j u_j t^{\gamma_j}\right)}{\lambda_{2,Z}}\right)^\alpha - 1\right]\right) \\
&\times \prod_{j=1}^n \exp\left(\Gamma(-\alpha)\left(C_{1,j}\lambda_{1,j}^\alpha\left[\left(1 - \frac{\mathrm{i}u_j t^{\gamma_j}}{\lambda_{1,j}}\right)^\alpha - 1\right] + C_{2,j}\lambda_{2,j}^\alpha\left[\left(1 + \frac{\mathrm{i}u_j t^{\gamma_j}}{\lambda_{2,j}}\right)^\alpha - 1\right]\right)\right), \quad \mathbf{u} \in \mathbb{R}^n,
\end{aligned} \qquad (4.27)
$$

[2] Note that we assume the same α for $X_1^{(j)}, j = 1, \ldots, n$ and Z_1 in order to have idiosyncratic and systemic risk factors that belong to the same sub-class of difference of one-sided tempered stable distributions. Note that this restriction can be relaxed to enhance the flexibility of the model, at the cost of a decrease of its parsimony.

with marginals

$$
\begin{aligned}
\phi_{Y^{(j)}}(u;t) = &\exp\left(\Gamma(-\alpha)\left(C_{1,Z}\lambda_{1,Z}^{\alpha}\left[\left(1-\frac{\mathrm{i}a_j u t^{\gamma_j}}{\lambda_{1,Z}}\right)^{\alpha}-1\right]+C_{2,Z}\lambda_{2,Z}^{\alpha}\left[\left(1+\frac{\mathrm{i}a_j u t^{\gamma_j}}{\lambda_{2,Z}}\right)^{\alpha}-1\right]\right)\right)\\
&\times\exp\left(\Gamma(-\alpha)\left(C_{1,j}\lambda_{1,j}^{\alpha}\left[\left(1-\frac{\mathrm{i}u t^{\gamma_j}}{\lambda_{1,j}}\right)^{\alpha}-1\right]+C_{2,j}\lambda_{2,j}^{\alpha}\left[\left(1+\frac{\mathrm{i}u t^{\gamma_j}}{\lambda_{2,j}}\right)^{\alpha}-1\right]\right)\right), \quad u\in\mathbb{R}.
\end{aligned}
\tag{4.28}
$$

Analogously, we can build a multivariate Lévy process of the form (4.1) where the risk factors at unit time are DTS distributed. The characteristic function of $\mathbf{Y}_t$ can be found by combining (4.2) and (4.24):

$$
\begin{aligned}
\phi_{\mathbf{Y}}(\mathbf{u};t) = &\exp\left(tC_{1,Z}\Gamma(-\alpha)\lambda_{1,Z}^{\alpha}\left[\left(1-\frac{\mathrm{i}\left(\sum_{j=1}^{n}a_j u_j\right)}{\lambda_{1,Z}}\right)^{\alpha}-1\right]\right)\\
&\times\exp\left(tC_{2,Z}\Gamma(-\alpha)\lambda_{2,Z}^{\alpha}\left[\left(1+\frac{\mathrm{i}\left(\sum_{j=1}^{n}a_j u_j\right)}{\lambda_{2,Z}}\right)^{\alpha}-1\right]\right)\\
&\times\prod_{j=1}^{n}\exp\left(t\Gamma(-\alpha)\left(C_{1,j}\lambda_{1,j}^{\alpha}\left[\left(1-\frac{\mathrm{i}u_j}{\lambda_{1,j}}\right)^{\alpha}-1\right]+C_{2,j}\lambda_{2,j}^{\alpha}\left[\left(1+\frac{\mathrm{i}u_j}{\lambda_{2,j}}\right)^{\alpha}-1\right]\right)\right), \quad \mathbf{u}\in\mathbb{R}^n,
\end{aligned}
\tag{4.29}
$$

with marginals

$$
\begin{aligned}
\phi_{Y^{(j)}}(u;t) = &\exp\left(t\Gamma(-\alpha)\left(C_{1,Z}\lambda_{1,Z}^{\alpha}\left[\left(1-\frac{\mathrm{i}a_j u}{\lambda_{1,Z}}\right)^{\alpha}-1\right]+C_{2,Z}\lambda_{2,Z}^{\alpha}\left[\left(1+\frac{\mathrm{i}a_j u}{\lambda_{2,Z}}\right)^{\alpha}-1\right]\right)\right)\\
&\times\exp\left(t\Gamma(-\alpha)\left(C_{1,j}\lambda_{1,j}^{\alpha}\left[\left(1-\frac{\mathrm{i}u}{\lambda_{1,j}}\right)^{\alpha}-1\right]+C_{2,j}\lambda_{2,j}^{\alpha}\left[\left(1+\frac{\mathrm{i}u}{\lambda_{2,j}}\right)^{\alpha}-1\right]\right)\right), \quad u\in\mathbb{R}.
\end{aligned}
\tag{4.30}
$$

As for the normal tempered stable model, we can ensure that $Y_1^{(j)}$ follows a DTS distribution as well by imposing the following conditions on the parameters of $X_1^{(j)}$ and Z_1:

$$
a_j = \frac{\lambda_{1,Z}}{\lambda_{1,j}} \quad \text{and} \quad a_j = \frac{\lambda_{2,Z}}{\lambda_{2,j}}. \tag{4.31}
$$

We then have that $Y_1^{(j)}$ follows a $\text{DTS}(\tilde{C}_{1,j},\tilde{C}_{2,j},\alpha,\tilde{\lambda}_{1,j},\tilde{\lambda}_{2,j}) = \text{DTS}\left(C_{1,j}+\left(\frac{\lambda_{1,Z}}{\lambda_{1,j}}\right)^{\alpha}C_{1,Z}, C_{2,j}+\left(\frac{\lambda_{2,Z}}{\lambda_{2,j}}\right)^{\alpha}C_{2,Z},\alpha,\lambda_{1,j},\lambda_{2,j}\right)$ distribution, and only positive correlations are possible.

In what follows, we will scale the parameter $\lambda_{1,Z}$ to 1 without loss of generality. Indeed, by the space-scaling property of the DTS distribution, scaling $\lambda_{1,Z}$ to 1 will not change the law of $\mathbf{Y}_t$.

4.5.1 Some specific examples

Popular examples of one-sided tempered stable distributions in financial applications are the inverse Gaussian (IG) ($\alpha = \frac{1}{2}$) and the Gamma ($\alpha = 0$) distributions, leading to a difference of IG and a difference of Gamma distributions, respectively (see also [13] or [22] and [7] for the use of difference of Gamma distributions in single-name and multi-name asset pricing models, respectively). The characteristic function of a Gamma(C,λ) and of an IG(C,λ) distribution is given by:

$$
\phi_{Gamma}(u;C,\lambda) = \left(1-\frac{\mathrm{i}u}{\lambda}\right)^{-C}, \quad u\in\mathbb{R},
$$

and

$$\phi_{IG}(u;C,\lambda) = e^{-2C\sqrt{\pi}(\sqrt{\lambda-\mathrm{i}u}-\sqrt{\lambda})}, \quad u \in \mathbb{R},$$

respectively[3]. Table 2 presents the first four moments of the Gamma and the inverse Gaussian distributions.

	Gamma(C,λ)	IG(C,λ)
mean	$\frac{C}{\lambda}$	$\frac{\sqrt{\pi}C}{\sqrt{\lambda}}$
variance	$\frac{C}{\lambda^2}$	$\frac{\sqrt{\pi}C}{2\lambda^{3/2}}$
skewness	$\frac{2}{\sqrt{C}}$	$\frac{3}{\sqrt{2C}\,(\pi\lambda)^{1/4}}$
kurtosis	$3\left(1+\frac{2}{C}\right)$	$3\left(1+\frac{5}{2C\sqrt{\pi\lambda}}\right)$

Table 2: Moments of the Gamma and the IG distributions.

.

Hence, imposing $\alpha = 0$ in (4.24) and substituting it in (4.5) leads to a linear multivariate difference of Gamma Sato model with the following characteristic function for $\mathbf{Y}_t$:

$$\phi_{\mathbf{Y}}(\mathbf{u},t) = \left(1 - \frac{\mathrm{i}\sum_{j=1}^{n} u_j a_j t^{\gamma_j}}{\lambda_{1,Z}}\right)^{-C_{1,Z}} \left(1 + \frac{\mathrm{i}\sum_{j=1}^{n} u_j a_j t^{\gamma_j}}{\lambda_{2,Z}}\right)^{-C_{2,Z}} \prod_{j=1}^{n}\left(\left(1-\frac{\mathrm{i}u_j t^{\gamma_j}}{\lambda_{1,j}}\right)^{-C_{1,j}}\left(1+\frac{\mathrm{i}u_j t^{\gamma_j}}{\lambda_{2,j}}\right)^{-C_{2,j}}\right), \quad \mathbf{u} \in \mathbb{R}^n.$$

The corresponding Lévy model has a characteristic function of the following form:

$$\phi_{\mathbf{Y}}(\mathbf{u},t) = \left(1 - \frac{\mathrm{i}\sum_{j=1}^{n} u_j a_j}{\lambda_{1,Z}}\right)^{-tC_{1,Z}} \left(1 + \frac{\mathrm{i}\sum_{j=1}^{n} u_j a_j}{\lambda_{2,Z}}\right)^{-tC_{2,Z}} \prod_{j=1}^{n}\left(\left(1-\frac{\mathrm{i}u_j}{\lambda_{1,j}}\right)^{-C_{1,j}}\left(1+\frac{\mathrm{i}u_j}{\lambda_{2,j}}\right)^{-C_{2,j}}\right)^{t}, \quad \mathbf{u} \in \mathbb{R}^n.$$

In this case, the correlation (4.3) is given by:

$$\rho\left(Y_t^{(i)}, Y_t^{(j)}\right) = \frac{a_i a_j \left(\frac{C_{1,Z}}{\lambda_{1,Z}^2} + \frac{C_{2,Z}}{\lambda_{2,Z}^2}\right)}{\sqrt{\frac{C_{1,i}}{\lambda_{1,i}^2} + \frac{C_{2,i}}{\lambda_{2,i}^2} + a_i^2\left(\frac{C_{1,Z}}{\lambda_{1,Z}^2} + \frac{C_{2,Z}}{\lambda_{2,Z}^2}\right)}\sqrt{\frac{C_{1,j}}{\lambda_{1,j}^2} + \frac{C_{2,j}}{\lambda_{2,j}^2} + a_j^2\left(\frac{C_{1,Z}}{\lambda_{1,Z}^2} + \frac{C_{2,Z}}{\lambda_{2,Z}^2}\right)}}.$$

Imposing the conditions (4.31), we obtain a restricted model setting where the marginals follow a difference of Gamma distributions. The charactertistic function of $Y_t^{(j)}, j = 1, \ldots, n$ then reduces to:

$$\phi_{Y^{(j)}}(u,t) = \left(1 - \frac{\mathrm{i}u}{\lambda_{1,j}}\right)^{-t(C_{1,Z}+C_{1,j})} \left(1 + \frac{\mathrm{i}u}{\lambda_{2,j}}\right)^{-t(C_{2,Z}+C_{2,j})}, \quad u \in \mathbb{R}$$

in the Lévy setting and to

$$\phi_{Y^{(j)}}(u,t) = \left(1 - \frac{\mathrm{i}ut^{\gamma_j}}{\lambda_{1,j}}\right)^{-(C_{1,Z}+C_{1,j})} \left(1 + \frac{\mathrm{i}ut^{\gamma_j}}{\lambda_{2,j}}\right)^{-(C_{2,Z}+C_{2,j})}, \quad u \in \mathbb{R}$$

[3] Taking $a = \sqrt{2\pi}C$ and $b = \sqrt{2\lambda}$ leads to Barndorff-Nielsen's [4] parametrization of the IG distribution.

in the Sato setting. The correlation coefficient is then given by:

$$\rho\left(Y_t^{(i)}, Y_t^{(j)}\right) = \frac{C_{1,Z}\lambda_{2,i}\lambda_{2,j} + C_{2,Z}\lambda_{1,i}\lambda_{1,j}}{\sqrt{\left(\left(C_{1,i}+C_{1,Z}\right)\lambda_{2,i}^2 + \left(C_{2,i}+C_{2,Z}\right)\lambda_{1,i}^2\right)\left(\left(C_{1,j}+C_{1,Z}\right)\lambda_{2,j}^2 + \left(C_{2,j}+C_{2,Z}\right)\lambda_{1,j}^2\right)}}.$$

Setting $C_{1,j}^* = C_{1,Z}+C_{1,j}$ and $C_{2,j}^* = C_{2,Z}+C_{2,j}$, $j = 1,\ldots,n$, the marginal characteristic functions become independent of the common parameters such that we can again decouple the calibration of the option price surfaces and the correlation fitting. However, unlike under the restricted linear normal tempered stable model described in Section 4.4, we can now calibrate the idiosyncratic parameters corresponding to the different assets seperately. Indeed, the condition (4.31) is equivalent to $a_j = \frac{1}{\lambda_{1,j}}$ and $\frac{\lambda_{1,j}}{\lambda_{2,j}} = c, \forall j = 1,\ldots,n$, for some $c > 0$, which can be further restricted to $\lambda_{1,j} = \lambda_{2,j}, \forall j = 1,\ldots,n$, allowing to calibrate the single-name option price surfaces separately. The correlation coefficient then reduces to:

$$\rho\left(Y_t^{(i)}, Y_t^{(j)}\right) = \frac{C_{1,Z}+C_{2,Z}}{\sqrt{C_{1,i}^*+C_{2,i}^*}\sqrt{C_{1,j}^*+C_{2,j}^*}}, \tag{4.32}$$

and can only take positive values. For more flexible dependence structures (i.e. that can accommodate both positive and negative correlations), one can consider the general framework (i.e. without imposing (4.31)). However, one can then not resort to the decoupling calibration procedure, which might lead to complex optimization problems due to the high dimensionality of the parameter space.

Remark 4.1. These reduced multivariate difference of Gamma Lévy and Sato models are similar to the Δ-Gamma Lévy and Sato models proposed in [7], but with the extra condition on the difference of Gamma models that

$$\frac{\lambda_{1,j}}{\lambda_{2,j}} = \frac{\lambda_{1,Z}}{\lambda_{2,Z}} = a_j \quad \forall j = 1,\ldots,n.$$

These restriced Δ-Gamma models thus fit in the general framework proposed in this paper as a special case.

Similar expressions can be derived when we assume $X_1^{(j)}, j = 1,\ldots,n$ and Z_1 to follow a difference of IG distributions.

5 Calibration

We calibrate the restricted linear multivariate VG and the restricted linear multivariate difference of Gamma Lévy and Sato models on a total of 68 quoting days ranging between the fourth of January 2007 and the 20th of October 2009, with biweekly quotes (i.e. every two weeks). This period includes different levels of market fear, as indicated by the VIX in Figure 1, and hence allows us to compare the model performance under different market regimes. We consider a basket of three major stocks included in the Dow Jones Industrial Average (DJX), namely Microsoft Corp. (MSFT), General Electric Co. (GE) and Pfizer Inc. (PFE). European option prices are extracted from the quoted American option prices using the iterative Implied Binomial (iIB) tree approach introduced by Tian [37]. To fit the dependence structure, we consider market implied correlations. More precisely, we approximate the correlation between any pair of single-name linear returns by the Average Linear Return Correlation (ALRC) index with a time horizon of one year:

$$\rho^{\mathrm{ALRC}}(t) = \frac{\mathrm{Var}\left(\frac{S_t - S_0}{S_0}\right) - \sum_{j=1}^{n} \left(w_j^*\right)^2 \mathrm{Var}\left(\frac{S_t^{(j)} - S_0^{(j)}}{S_0^{(j)}}\right)}{2\sum_{j<k} w_j^* w_k^* \sqrt{\mathrm{Var}\left(\frac{S_t^{(j)} - S_0^{(j)}}{S_0^{(j)}}\right)}\sqrt{\mathrm{Var}\left(\frac{S_t^{(k)} - S_0^{(k)}}{S_0^{(k)}}\right)}},$$

where $w_j^* = \frac{S_0^{(j)}}{S_0} w_j$ and where the variance of the linear returns is extracted from the single-name

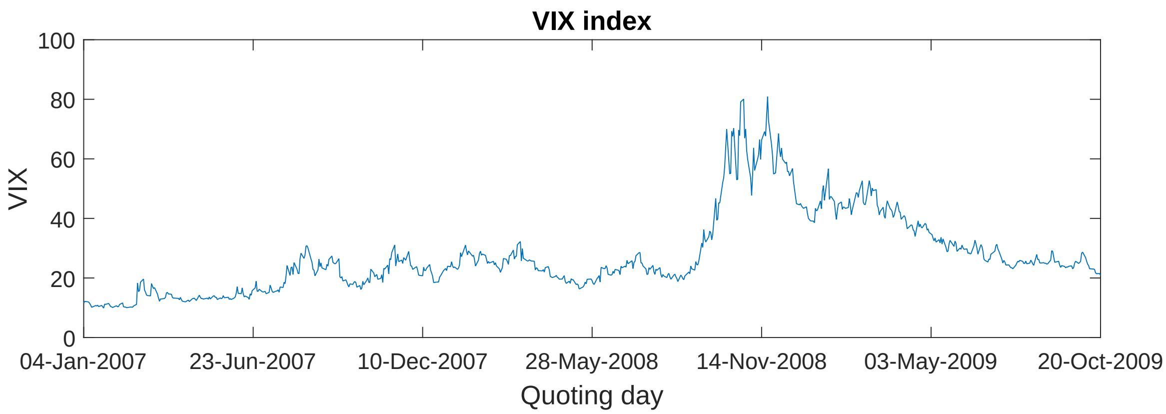


Figure 1: The evolution of the Volatility index (VIX)

option price surfaces and the index option price surface using the option payoff spanning formula of Breeden and Litzenberger [8]. Here, w_j denotes the weight corresponding to the jth stock in the arithmetic market (i.e. $S_0 = \sum_{j=1}^{n} w_j S_0^{(j)}$). For a price-weighted index like the Dow Jones, $w_j = \frac{1}{\delta}$, where δ is the index divisor. The correlation between the asset log-returns can subsequently be approximated by using Taylor series expansions.

Whenever possible, we employ a decoupling calibration procedure. The idiosyncratic parameters are first calibrated on the univariate option price surfaces by minimizing the following objective function:

$$\mathrm{MRMSE} = \sum_{j=1}^{n} \frac{\mathrm{RMSE}^{(j)}}{n} = \sum_{j=1}^{n} \left(\frac{1}{n} \sqrt{\frac{1}{N^{(j)}} \sum_{k=1}^{N^{(j)}} \left(P_k^{(j)} - \hat{P}_k^{(j)} \right)^2} \right), \tag{5.1}$$

where $N^{(j)}$ is the number of quoted option prices for stock j and $P_k^{(j)}$ and $\hat{P}_k^{(j)}$ denote the kth quoted option price and model option price of stock j, respectively. The model option prices are computed using the Carr-Madan formula (see [6]). Recall that we have to impose

$$\frac{\tilde{\sigma}_j^2}{\tilde{k}_j \tilde{\theta}_j^2} = \frac{\sigma_Z^2}{k_Z \theta_Z^2} = c, \quad \forall j = 1, \ldots, n$$

for some $c \in \mathbb{R} \setminus \{0\}$ under the restricted linear normal tempered stable multivariate models and

$$\frac{\lambda_{1,j}}{\lambda_{2,j}} = \frac{\lambda_{1,Z}}{\lambda_{2,Z}} = a_j \quad \forall j = 1, \ldots, n$$

		Lévy setting	Sato setting	Decoupled?
linear NTS	General	$4n+3$	$5n+3$	✗
linear DTS	General	$5n+4$	$6n+4$	✗
linear VG	General	$4n+2$	$5n+2$	✗
	Restricted	$2n+2$	$3n+2$	✓
linear DG	General	$5n+3$	$6n+3$	✗
	Restricted	$3n+2$	$4n+2$	✓

Table 3: Number of model parameters in function of the number of assets n.

under the restricted linear DTS multivariate models. Once the idiosyncratic parameters are calibrated, we calibrate the systemic parameters to match the market implied correlations by minimizing the objective function

$$\mathrm{RMSE}^{\rho} = \sqrt{\frac{1}{\frac{n^2-n}{2}} \sum_{j=1}^{n} \sum_{k>j} \left(\rho_{jk} - \hat{\rho}_{jk}\right)^2}, \tag{5.2}$$

where ρ_{jk} and $\hat{\rho}_{jk}$ denote the market implied and model correlations between the jth and kth log-returns, respectively. Note that the conditions (4.16) and (4.31) imposed on the idiosyncratic parameters to make the decoupled calibration feasible restrict the admissible values of the systemic parameters, and hence the attainable values of the model correlation. However, the decoupling procedure should be used whenever leading to an accurate fit of the dependence structure. Otherwise, one can perform a joint calibration, where all the parameters are calibrated at once by minimizing an objective function of the following form:

$$\mathrm{MRMSEJ} = \sum_{j=1}^{n} \frac{\mathrm{RMSE}^{(j)}}{n} + \alpha^{\rho} \sqrt{\frac{1}{\frac{n^2-n}{2}} \sum_{j=1}^{n} \sum_{k>j} \left(\rho_{jk} - \hat{\rho}_{jk}\right)^2},$$

as proposed by [16][4]. Here, $\alpha^{\rho} \geq 0$ allows the user to specify the relative importance of the correlation matching, where $\alpha^{\rho} = 0$ indicates that correlation fitting is not a desired feature and that the model is calibrated on the univariate option surfaces only, and where we take $\alpha^{\rho} = 1$ throughout this paper. Table 3 summarizes the total number of parameters for each model in terms of the number of assets n, together with an indication of whether the model can be calibrated using the decoupling calibration procedure or not.

6 Comparison with existing multivariate models

Imposing the condition (4.17) on the αVG model of Semeraro [36] and the original αVG Sato model proposed by Guillaume [15] leads to multivariate models that have the same marginal structure as

[4]Note that we omit MRMSE* here to ensure that we attach the same importance to the correlation fitting in the Lévy and the Sato settings.

the restricted multivariate Lévy and Sato linear VG models described in Section 4.4.1, but with a different dependence structure. Under the αVG models, the asset log-returns at unit time are modeled by a normal variance-mean mixture, where the mixing density is a weighted sum of a common and an idiosyncratic Gamma random variable, i.e. $Y_1^{(j)} = \theta_j G_1^{(j)} + \sigma_j \sqrt{G_1^{(j)}} W^{(j)}, j = 1, \ldots, n$, where $G_1^{(j)} = X_1^{(j)} + \alpha_j Z_1$ with $Z_1 \sim \text{Gamma}(c_1, 1)$ and $X_1^{(j)} \sim \text{Gamma}(a_j, 1/\alpha_j)$, $\{\theta_j \in \mathbb{R}, \sigma_j > 0, c_1 > 0, a_j > 0, \alpha_j > 0,\ j = 1, \ldots, n\}$ independent Gamma random variables and where $W^{(j)}, j = 1, \ldots, n$ are independent standard Brownian motions that are independent of $X^{(j)}, j = 1, \ldots, n$ and Z. The joint characteristic function then takes the form (see [36] and [15], respectively):

$$\phi_{\mathbf{Y}}(\mathbf{u}; t) = \left(1 - \mathrm{i} \sum_{j=1}^{n} \alpha_j \left(\theta_j u_j + \mathrm{i} \frac{1}{2} \sigma_j^2 u_j^2\right)\right)^{-c_1 t} \prod_{j=1}^{n} \left(1 - \mathrm{i}\alpha_j \left(\theta_j u_j + \mathrm{i}\frac{1}{2}\sigma_j^2 u_j^2\right)\right)^{-a_j t}, \mathbf{u} \in \mathbb{R}^n$$

and

$$\phi_{\mathbf{Y}}(\mathbf{u}; t) = \left(1 - \mathrm{i} \sum_{j=1}^{n} \alpha_j \left(\theta_j u_j t^{\gamma_j} + \mathrm{i} \frac{1}{2} \sigma_j^2 u_j^2 t^{2\gamma_j}\right)\right)^{-c_1} \prod_{j=1}^{n} \left(1 - \mathrm{i}\alpha_j \left(\theta_j u_j t^{\gamma_j} + \mathrm{i}\frac{1}{2}\sigma_j^2 u_j^2 t^{2\gamma_j}\right)\right)^{-a_j}, \mathbf{u} \in \mathbb{R}^n$$

under the original αVG and αVG Sato models, respectively. We then have that $G_1^{(j)} \sim \text{Gamma}(a_j + c_1, \frac{1}{\alpha_j})$ and the following condition makes sure that, in the Lévy setting, the business time increases on average as the calendar time t:

$$a_j = \frac{1}{\alpha_j} - c_1.$$

The marginal characteristic function then reduces to:

$$\phi_{Y^{(j)}}(u; t) = \left(1 - \mathrm{i}\alpha_j \theta_j u + \frac{1}{2}\alpha_j \sigma_j^2 u^2\right)^{-t/\alpha_j}, u \in \mathbb{R} \tag{6.1}$$

under the Lévy setting and to

$$\phi_{Y^{(j)}}(u; t) = \left(1 - \mathrm{i}\alpha_j \theta_j u t^{\gamma_j} + \frac{1}{2}\alpha_j \sigma_j^2 u^2 t^{2\gamma_j}\right)^{-1/\alpha_j}, u \in \mathbb{R} \tag{6.2}$$

under the Sato setting and the correlation coefficient is given by (see [15]):

$$\rho\left(Y_t^{(i)}, Y_t^{(j)}\right) = \frac{\theta_i \theta_j \alpha_i \alpha_j}{\sqrt{\left(\sigma_i^2 + \theta_i^2 \alpha_i\right)\left(\sigma_j^2 + \theta_j^2 \alpha_j\right)}} c_1. \tag{6.3}$$

Comparing the marginal characteristic functions (6.1) and (6.2) with the characteristic function of the restricted linear VG Lévy (4.21) and Sato (4.22) models, it is clear that we can write the characteristic function of the αVG models in terms of the parametrization used in Section 4.4.1 by taking $\theta_j = \tilde{\theta}_j$, $\sigma_j = \tilde{\sigma}_j$ and $\alpha_j = \tilde{k}_j$. As for the linear VG model, we impose

$$\frac{\tilde{\sigma}_j^2}{\tilde{\theta}_j^2 \tilde{k}_j} = c, \quad \forall j = 1, \ldots, n, \tag{6.4}$$

such that we can compare the dependence structure under the two model settings (i.e. the restricted αVG type models versus the restricted linear VG type models), since the marginals then coincide. The correlation coefficient for the restricted αVG models then reduces to

$$\rho\left(Y_t^{(i)}, Y_t^{(j)}\right) = \text{sign}(\tilde{\theta}_i \tilde{\theta}_j) \frac{\sqrt{\tilde{k}_i \tilde{k}_j} c_1}{c + 1} \propto c_1,$$

where again $\text{sign}(\tilde{\theta}_i\tilde{\theta}_j) = +1$, since we assume the marginals under the restricted αVG models to be the same as under the restricted linear VG models. Note that if we perform a decoupled calibration procedure, the constraint

$$c_1 < \min_{j=1,\dots,n}\left(\frac{1}{\tilde{k}_j}\right) \tag{6.5}$$

must hold under the αVG models to ensure the positivity of the idiosyncratic parameters $a_j = \frac{1}{\tilde{k}_j} - c_1, \forall j = 1,\dots,n$. Besides, the constraint

$$\frac{1}{k_Z} < \min_{j=1,\dots,n}\left(\frac{1}{\tilde{k}_j}\right) \tag{6.6}$$

must hold under the restricted linear VG models to ensure the positivity of the idiosyncratic parameters $k_j = \frac{1}{\tilde{k}_j} - \frac{1}{k_Z}, j = 1,\dots,n$. Hence, the upperbound on the common parameter (i.e. on c_1 and $1/k_Z$) imposed by decoupling the calibration is the same under both the restricted αVG and the restricted linear VG models, allowing for a comparison of the maximal attainable values of the correlation coefficient ($\mu_{1,1}^{\max}$), the coskewnesses ($\mu_{1,2}^{\max}$ and $\mu_{2,1}^{\max}$) and the excesses of co-kurtosis[5] ($C_{1,3}^{\max}$, $C_{2,2}^{\max}$ and $C_{3,1}^{\max}$). In particular, one can prove that $\forall i \neq j$ (see Appendix C for the proof):

$$\frac{\mu_{1,1}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{\mu_{1,1}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} < 1,$$

$$\frac{\mu_{1,2}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{\mu_{1,2}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} < 1 \quad \text{and} \quad \frac{\mu_{2,1}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{\mu_{2,1}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} < 1,$$

and

$$\frac{C_{1,3}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{C_{1,3}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} < 1, \quad \frac{C_{2,2}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{C_{2,2}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} < 1 \text{ and } \frac{C_{3,1}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{C_{3,1}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} < 1,$$

where the superscript αVG or $linVG$ indicates whether the restricted αVG or the restricted linear VG models are considered. These results imply that, for given marginals (assuming that (6.4) holds), the restricted linear VG models have a wider range of maximal attainable values for the correlation, as well as for the co-skewness and the excess of co-kurtosis. Hence, they can capture a broader range of linear and non-linear dependence structures than the restricted αVG models. It follows that the upper bound ((6.5) or (6.6)) on the common parameter is expected to be reached more often under the restricted αVG models than under the restricted linear VG models. Note that a similar comparison of the Δ-Gamma models of [7] and of the αVG models can be found in [17]. Again, provided that the assets are skewed in the same direction, the Δ-Gamma models allow for a more flexible dependence structure than the αVG models. The same conclusions hold for the αNIG and linear NIG models.

[5] The excesses of co-kurtosis can be written in terms of the standardized co-moments $\mu_{m,n}$ as follows:
$C_{1,3}(Y_t^{(j)}, Y_t^{(k)}) = \mu_{1,3}(Y_t^{(j)}, Y_t^{(k)}) - 3\mu_{1,1}(Y_1^{(j)}, Y_1^{(k)})$,
$C_{3,1}(Y_t^{(j)}, Y_t^{(k)}) = \mu_{3,1}(Y_t^{(j)}, Y_t^{(k)}) - 3\mu_{1,1}(Y_1^{(j)}, Y_1^{(k)})$ and
$C_{2,2}(Y_t^{(j)}, Y_t^{(k)}) = \mu_{2,2}(Y_t^{(j)}, Y_t^{(k)}) - 1 - 2(\mu_{1,1}(Y_1^{(j)}, Y_1^{(k)}))^2$.

7 Calibration Results

As numerical study, we first calibrate the multivariate restricted αVG, restricted linear VG, original αVG and the restricted linear DG Lévy and Sato models using the decoupling calibration procedure described in Section 5.

7.1 Calibration of the marginals

The evolution of the MRMSE for the different models under consideration is shown in Figure 2. As expected, the option price surface goodness of fit is the same for both the restricted linear and the restricted αVG models, as the marginal characteristic functions are the same. The original αVG models have a slightly lower MRMSE value, due to the relaxation of the constraint (6.4). The restricted linear DG models have an option price surface goodness of fit that is comparable to the models with VG marginals, although they have one extra degree of freedom per underlying. Moreover, as expected from previous studies (see f.i. [7], [10], [15]), it is clear that the Sato models outperform the Lévy models in terms of marginal goodness of fit, especially during periods of market turmoil. This is further observed on Figure 3, where the option price surface goodness of fit is shown for the GE stock for the 11th of December 2008 under the restricted linear VG and αVG models in the Lévy (left) and the Sato (right) settings.

7.2 Calibration of the dependence structure

Figures 4 and 5 show the evolution of the correlation RMSE (i.e. RMSE$^\rho$) under the Lévy and the Sato settings, respectively, while Figures 6 and 7 display the value of the common parameters and their upper bound. Comparing the restricted linear VG models to the restricted αVG models in terms of correlation fitting, one observes that the restricted linear VG models outperform the restricted αVG models in both the Lévy and the Sato settings whenever the upper bound on c_1 is reached (under the restricted αVG models), which is in line with the theoretical results obtained in Section 6. This is the case for 79.41% (respectively 61.76%) of the quoting days with an average improvement[6] of 49.71% (respectively 45.51%) in the Lévy (respectively Sato) setting. On the other hand, when the upper bound on c_1 is not reached, the correlation goodness of fit under the restricted αVG models is approximately the same as under the restricted linear VG models[7]. However, the values of RMSE$^\rho$ are still considerably large (an average of 16% and 15% of the maximal attainable value of the correlation RMSE[8], under the linear VG Lévy and Sato models). This can be explained by the too low number of common parameters (i.e. one) to match the correlation coefficients between three pairs of assets at the same time.

The condition (6.4) was imposed on the αVG models to allow for a comparison of the dependence structure under both construction methods, i.e. building multivariate models as a linear combination of Lévy and Sato processes versus building multivariate models by time-changing a multivariate Brownian motion. For the sake of complete comparison, we also calibrated the original αVG Lévy and Sato models using the decoupling calibration procedure. We observe that the restricted linear VG models outperform the original αVG models in terms of the correlation goodness of fit on 66.18%, respectively 67.65% of the quoting days with an average relative reduction of 50.83%, respectively 56.87% of RMSE$^\rho$ in the Lévy, respectively Sato, setting. Moreover, note that the

[6] $\text{mean}\big[(\text{RMSE}^\rho_{aVG} - \text{RMSE}^\rho_{linVG})/\text{RMSE}^\rho_{aVG}\big]$

[7] Because $1 + c \approx 1$ in this case, where c is given in (6.4), such that $\mu_{1,1}^{\alpha VG} \approx \mu_{1,1}^{linVG}$.

[8] max(RMSE$^\rho$)=1 since we assume that the market correlation is positive (otherwise the restricted models would not be appropriate).

common parameter reaches its upper bound more often under the original αVG models than under the restricted linear VG models. The analytical and numerical results indicate that building multivariate models as a linear combination of Lévy and Sato processes constitutes a suitable alternative to constructing multivariate models by time-changing a multivariate Brownian motion[9].

Under the restricted linear difference of Gamma Lévy and Sato models, the idiosyncratic parameters corresponding to different assets can be calibrated separately, in contrast to the restricted linear VG models, where all the idiosyncratic parameters had to be calibrated simultaneously because of the restriction (6.4). Due to the decoupling calibration procedure, the constraints:

$$0 < C_{1,Z} < \min_{j=1,\dots,n} (C^*_{1,j}) \quad \text{and} \quad 0 < C_{2,Z} < \min_{j=1,\dots,n} (C^*_{2,j})$$

must hold to ensure the positiveness of the parameters $C_{1,j} = C^*_{1,j} - C_{1,Z}$ and $C_{2,j} = C^*_{2,j} - C_{2,Z}$ for $j = 1, \dots, n$. We observe a significant improvement in the correlation goodness of fit when compared to the models with VG marginals, which might be due to the extra common parameter to fit the correlation structure (see Figures 4 and 5). Table 4 lists the percentage of quoting days for which the restricted linear DG models outperform the different VG-type models in terms of the correlation fitting, together with the average relative gain[10] that is achieved. One observes that the extra flexibility in the restricted linear DG models significantly improves the correlation goodness of fit. However, the values of the correlation RMSE are still rather high (on average 10.69%, respectively 7.94%, of the maximal value of RMSE$^\rho$ under the restricted linear DG Lévy, respectively Sato, model.).

	res linVG	res αVG	αVG	res linVGS	res αVGS	αVGS
% of quoting days	69.12	97.06	85.29	79.41	91.18	89.71
av. gain (%)	57.60	64.43	61.69	64.65	74.28	72.45

Table 4: Percentage of quoting days for which the linear DG Lévy(left)/Sato(right) model outperforms the different VG-type models in terms of the correlation goodness of fit and the average relative gain.

7.3 Investigation of the co-moments

For the original αVG models, the sign of the standardized co-moments is determined by the sign of the idiosyncratic parameters $\tilde{\theta}_j, j = 1, \dots, n$. Under the restricted VG models, only the sign of the co-skewness is affected by the sign of $\tilde{\theta}_j, j = 1, \dots, n$, while the correlation and the excesses of co-kurtosis are positive by construction. Since the idiosyncratic parameters are calibrated first under the decoupling calibration procedure and since the $\tilde{\theta}_j, j = 1, \dots, n$ turn out to be all negative, only positive correlations can be achieved for the original αVG models for the data considered. Furthermore, the sign of the co-skewness then is negative, while the sign of the excess of co-kurtosis is positive for the data considered (see formulas in Appendix C), as is the case for the restricted linear VG models. For the restricted linear DG models, the excess of co-kurtosis is proportional to $(C_{1,Z} + C_{2,Z})$ and hence, the excess of co-kurtosis will always be positive. The co-skewness

[9] A similar comparison can be done for models with NIG marginals. The results are similar and available to the interested reader on demand.

[10] $\text{mean}\big[(\text{RMSE}^\rho_{VG} - \text{RMSE}^\rho_{linDG})/\text{RMSE}^\rho_{VG}\big]$

however can achieve both positive and negative values, since it is proportional to $(C_{1,Z} - C_{2,Z})$. Figures 8, 9 and 10 show the maximal attainable values per pair of assets for the correlation coefficient, the co-skewness $\mu_{1,2}$ and the symmetric excess of co-kurtosis $\mu_{2,2}$, respectively[11]. These maximal attainable values are obtained by taking the common parameter equal to its upper bound $\min_j(1/\tilde{k}_j)$, $j \in \{\text{GE, MSFT, PFE}\}$ in the corresponding co-moment formula. It is clear that the upper bound on the common parameter (i.e. on c_1 or $1/k_Z$) translates into a strict upper bound on the maximal attainable correlation, which is unfavorable. Comparing the maximal attainable dependence values between the restricted linear VG and the restricted αVG models, it is confirmed that the restricted linear VG models have a wider range of attainable values for the correlation, the co-skewness and the excess of co-kurtosis than the restricted αVG models. Moreover, it can be seen that for 80.88% of the quoting days under consideration the restricted linear VG Lévy model has a larger maximal attainable correlation than the original αVG Lévy model for all the asset pairs. Under the Sato setting, this is the case for 75% of the quoting days. Comparing the higher order co-moments (i.e. $\mu_{1,2}$, $\mu_{2,1}$, $\mu_{1,3}$, $\mu_{2,2}$ and $\mu_{3,1}$) individually, it can be seen that the restricted linear VG models outperform the original αVG models on roughly half of the quoting days in our dataset. However, when combining the co-skewnesses and excesses of co-kurtosis, it can be seen that at least one of the co-skewnesses $\mu_{1,2}$ and $\mu_{2,1}$ of the restricted linear VG model has a larger maximal attainable value than under the original αVG model for all the asset pairs on 75% (respectively 70.59%) of the quoting days under the Lévy (respectively Sato) setting. For the excesses of co-kurtosis $\mu_{1,3}, \mu_{2,2}$ and $\mu_{3,1}$, this holds for 76.47% (respectively 72.06%) of the quoting days. When comparing the maximal attainable dependence values between the restricted linear DG and the restricted linear VG models, it can be seen that there is no unambiguous overall winner, although the maximal attainable dependence values of the restricted linear DG models exceed the ones of the restricted linear VG models on slightly more quoting days than vice versa, except for the excesses of co-kurtosis in the Lévy setting. However, when comparing the restricted linear DG models to the original αVG models, it is clear that the restricted linear DG models have larger maximal attainable dependence values on the majority of quoting days that are considered in this paper. Table 5 summarizes the results.

	1. linVG ↔ 2. oaVG		1. linVG ↔ 2. linDG		1. linDG ↔ 2. linVG		1. linDG ↔ 2. oaVG	
	Lévy	Sato	Lévy	Sato	Lévy	Sato	Lévy	Sato
$\mu_{1,1}$	80.88%	75.00%	17.65%	13.24%	35.29%	39.71%	95.59%	86.76%
$\mu_{1,2}$	48.53%	54.41%	16.18%	5.88%	29.41 %	36.76%	82.35%	77.94%
$\mu_{2,1}$	63.24%	60.29%	17.65%	5.88%	30.88%	30.88%	82.35%	79.41%
$\mu_{1,2}$ or $\mu_{2,1}$	75%	70.59%	19.12%	5.88%	35.29%	39.71%	97.06%	85.29%
$\mu_{1,3}$	47.06%	52.94%	32.35%	11.76%	14.71%	29.41%	73.53%	79.41%
$\mu_{2,2}$	47.06%	52.94%	29.41%	14.71%	11.76%	23.53%	80.88%	77.94%
$\mu_{3,1}$	61.76%	55.88%	29.41%	13.24%	8.82%	23.53%	70.59%	75.00%
$\mu_{1,3}$, $\mu_{2,2}$ or $\mu_{3,1}$	76.47%	72.06%	35.29%	19.12%	17.65%	38.24%	89.71%	88.24%

Table 5: Percentage of quoting days for which (one or at least one of) the maximal attainable dependence value(s) is bigger under model 1 than under model 2 for all the asset pairs under consideration. Notation: linVG = restricted linear VG models, linDG = restricted linear DG models, oaVG = original αVG models.

[11]The results for $\mu_{2,1}$, $\mu_{1,3}$ and $\mu_{3,1}$ are similar and available to the interested reader on demand.

Finally, we compare the market implied correlation values to the maximal attainable correlation values, both being displayed in Figure 8. The maximal attainable correlations are higher than the market implied correlations more often under the restricted linear DG models than under the different VG-type models. Whenever the maximal attainable correlation for the VG-type models is higher than the market implied correlation for each pair of assets, the upper bound on the common parameter will not be reached and the bad correlation fit is solely due to the insufficient number of common parameters (i.e. one parameter to fit three correlations). When not all of the asset pairs have a maximal attainable correlation higher than the corresponding market implied correlation, the upper bound on the common parameter might also explain the bad correlation fit. In this case, whenever the upper bound on the common parameter is reached, the bad fit arises from the strict upper bound on the common parameter, while the too low number of common parameters is the main cause when the upper bound is not reached. A natural extension would be to add extra common parameters, and the restricted linear DG models can be considered as such an extension. However, under this model at least one of the common parameters reaches its upper bound for 82.35% of the quoting days in the Lévy setting and 69.12% of the quoting days in the Sato setting, indicating that the strict upper bound is the main cause for the relatively high value of RMSE^{ρ}.

Remark 7.1. As the previous results show, the decoupling calibration procedure might lead to a bad correlation fit due to the upper bound on the common parameters. A joint calibration procedure as described in Section 5 does not suffer from this limitation, since all the parameters are calibrated together on both the option price surfaces and the correlation structure. Applying such a joint calibration might however be computationally challenging, due to the high dimensionality of the parameter space. As an example, we calibrated the restricted linear VG models using the joint calibration procedure on a selection of five quoting days exhibiting different levels of market turmoil, as indicated by the VIX in Table 6. The results of the decoupled calibration are summarized in Table 7 and those of the joint calibration in Table 8. It can be observed that there is a significant improvement in the correlation goodness of fit, while retaining a similar option price surface goodness of fit.

	08/01/2008	20/05/2008	04/06/2008	28/10/2008	11/12/2008
VIX	25.43	17.58	20.80	66.96	55.78

Table 6: Volatility index (VIX).

	08/01/2008		20/05/2008		04/06/2008		28/10/2008		11/12/2008	
	MRMSE	RMSE^{ρ}	MRMSE	RMSE^{ρ}	MRMSE	RMSE^{ρ}	MRMSE	RMSE^{ρ}	MRMSE	RMSE^{ρ}
Res. lin.VG	0.1724	0.1333	0.1286	0.2567	0.1065	0.3007	0.3034	0.1152	0.2998	0.1410
Res. lin.VGS	0.0920	0.1193	0.0813	0.2080	0.0775	0.2492	0.1364	0.1541	0.1071	0.1229

Table 7: Value of the objective functions MRMSE and RMSE^{ρ} for the restricted linear VG Lévy and Sato models under the decoupled calibration.

	08/01/2008		20/05/2008		04/06/2008		28/10/2008		11/12/2008	
	MRMSE	RMSE$^\rho$	MRMSE	RMSE$^\rho$	MRMSE	RMSE$^\rho$	MRMSE	RMSE$^\rho$	MRMSE	RMSE$^\rho$
Res. lin.VG	0.1745	9.8919e-11	0.1403	1.8112e-11	0.1121	2.7512e-9	0.3045	2.8172e-11	0.3040	2.539e-10
Res. lin.VGS	0.1015	7.9121e-11	0.1024	6.6849e-12	0.0997	2.4183e-10	0.1472	1.1531e-11	0.1204	8.3133e-9

Table 8: Value of the objective functions MRMSE and RMSE$^\rho$ for the restricted linear VG Lévy and Sato models using the joint calibration technique.

8 Conclusion

This paper proposed a general framework for multivariate Sato models with a linear dependence structure. The asset log-return processes at unit time are modeled as linear combinations of independent self-decomposable random variables. Dependence is introduced by considering at least one self-decomposable random variable common to all the assets. The proposed framework can be seen as an extension of the Lévy framework developed in [2], where the time-homogeneous property of the increments is relaxed. For the distribution of the risk factors, we considered a normal variance-mean mixture with a one-sided tempered stable mixing density and a difference of one-sided tempered stable distributions. In particular, we elaborated the specific examples of Variance Gamma (VG) and difference of Gamma (DG) distributions as an illustration. In order to overcome the difficulties of a joint calibration procedure, we imposed conditions on the model parameters such that the decoupling calibration procedure proposed in [25] could be employed.

We have proven that the dependence structure under the restricted linear VG models is more flexible than the dependence structure under the restricted αVG models, while having identical marginal characteristic functions and hence an identical option price surface goodness of fit. This, together with the numerical comparison of the proposed linear models to the original αVG models developed in [36] and [15], indicates that building multivariate Sato models with a linear dependence structure is a good alternative to considering multivariate Sato models where the dependence is introduced through subordination, providing further evidence to the conclusion of Guillaume [17]. Indeed, the dependence structure of the restricted linear VG models and the restricted linear DG models turns out to be more flexible than the one of the original αVG models on the majority of the quoting days under consideration, while retaining a comparable option price surface goodness of fit.

Tables and Figures

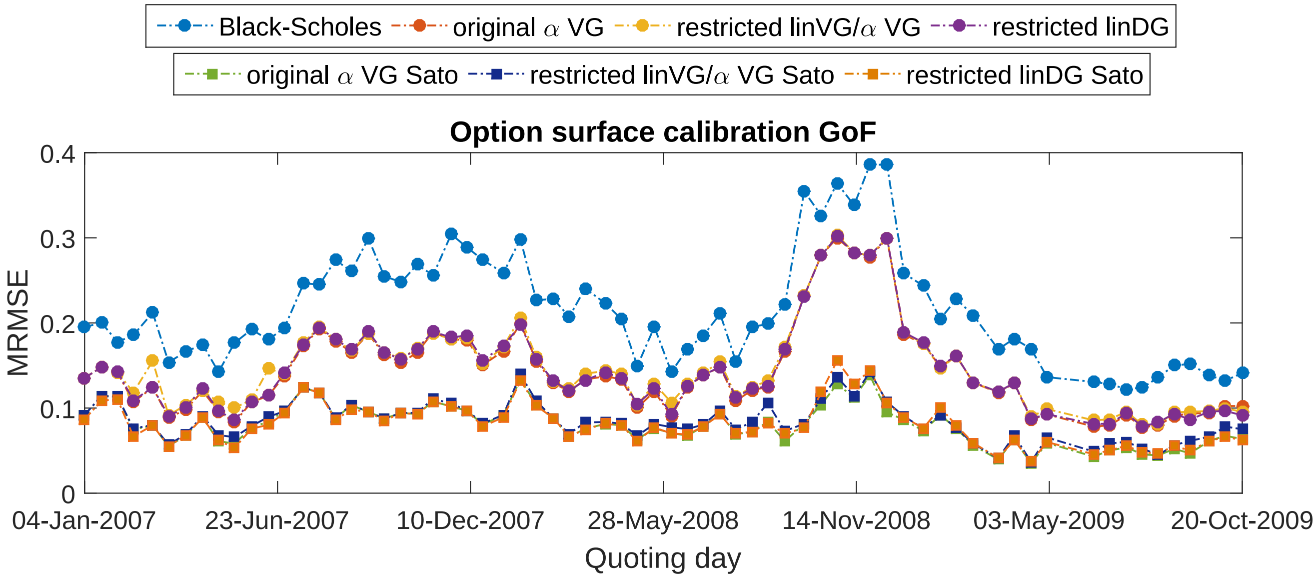
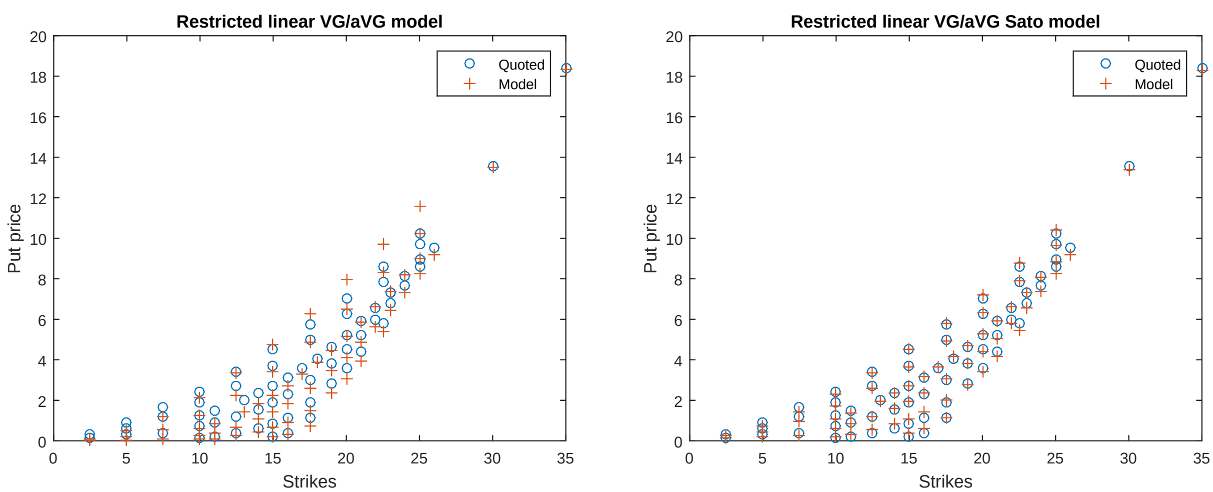


Figure 2: Comparison of the univariate option price surface goodness of fit under the different Lévy and Sato models.



(a)

(b)

Figure 3: Option price surface goodness of fit for the GE stock on 11/12/2008, under the restricted linear VG and restricted αVG models in the Lévy (a) and the Sato (b) settings.

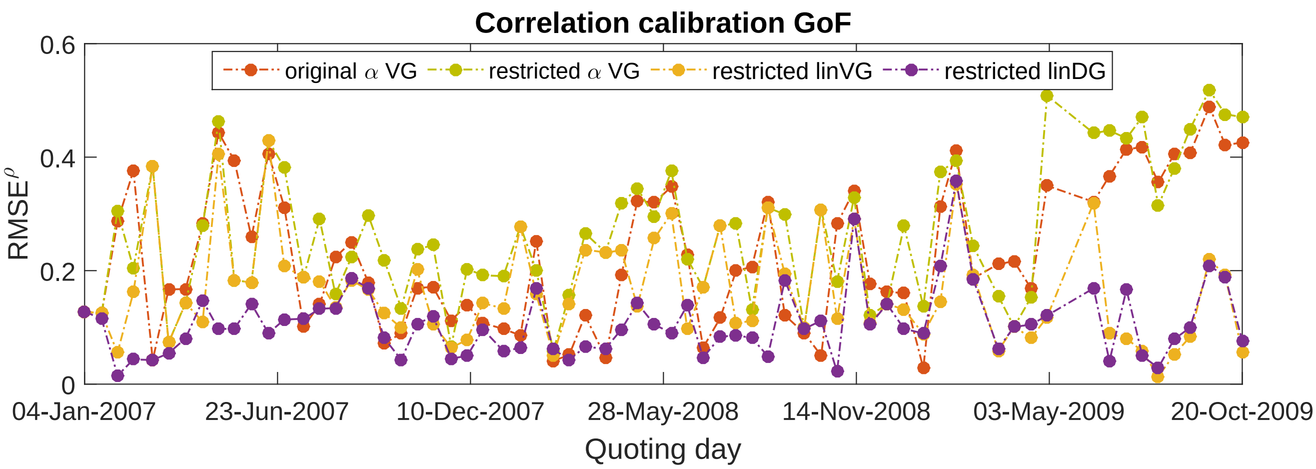


Figure 4: Comparison of the correlation goodness of fit under the Lévy models.

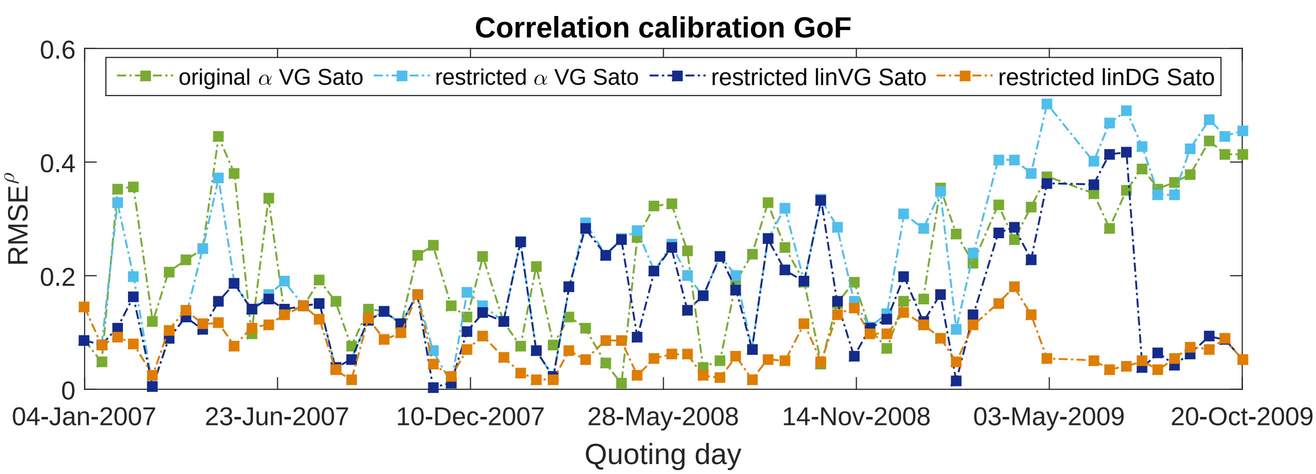


Figure 5: Comparison of the correlation goodness of fit under the Sato models.

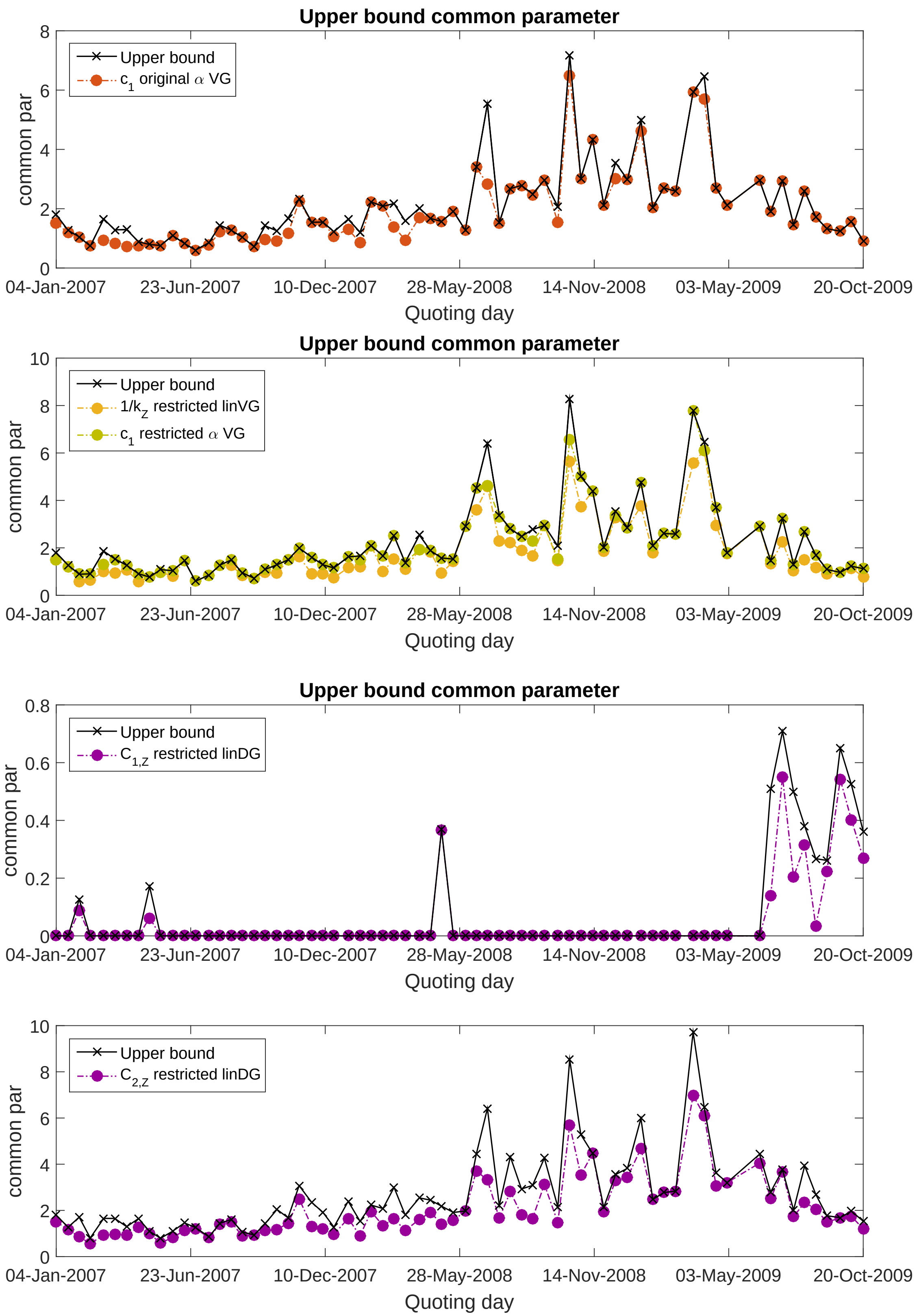


Figure 6: Common parameters under the different Lévy models and their upper bound.

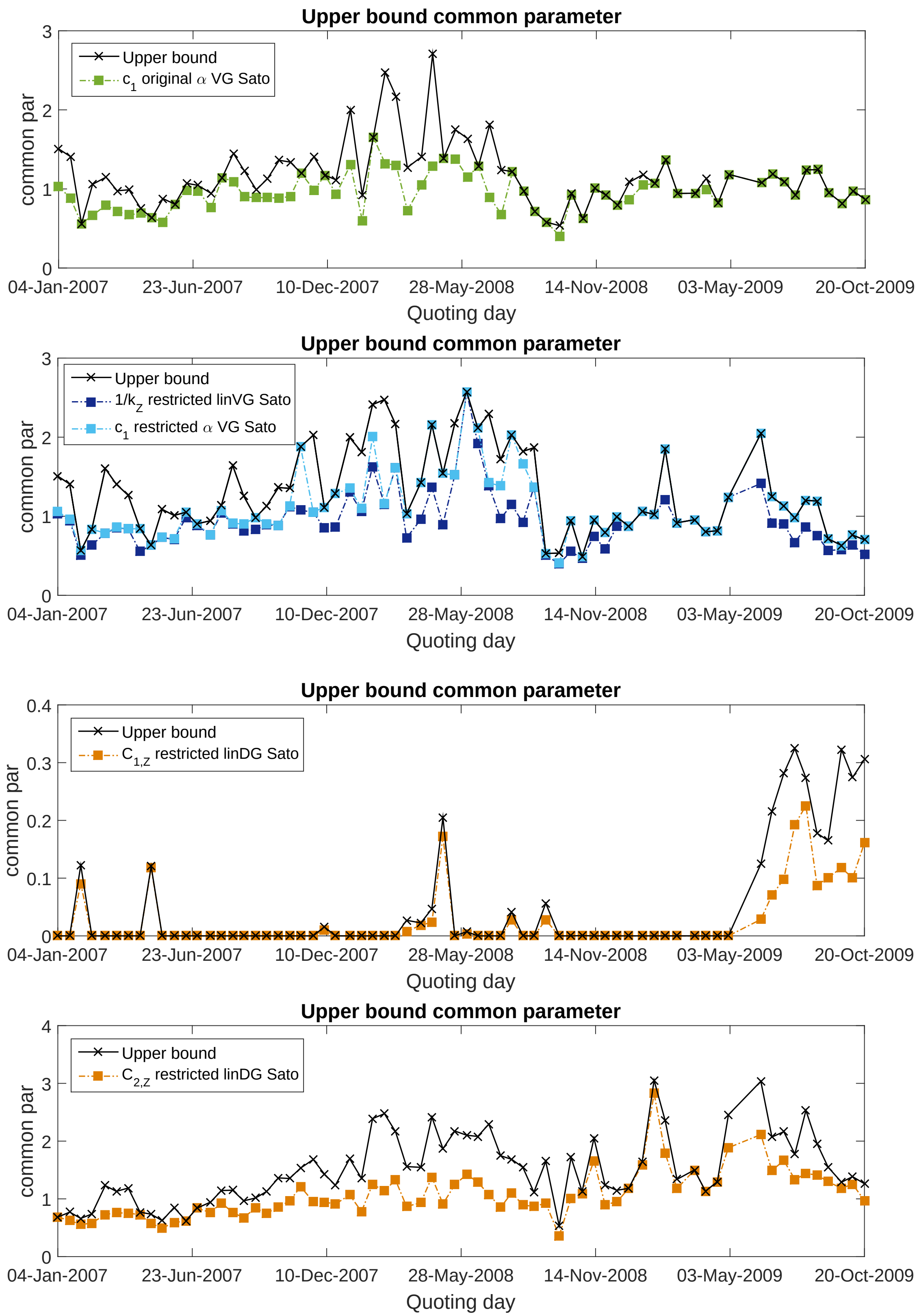


Figure 7: Common parameters under the different Sato models and their upper bound.

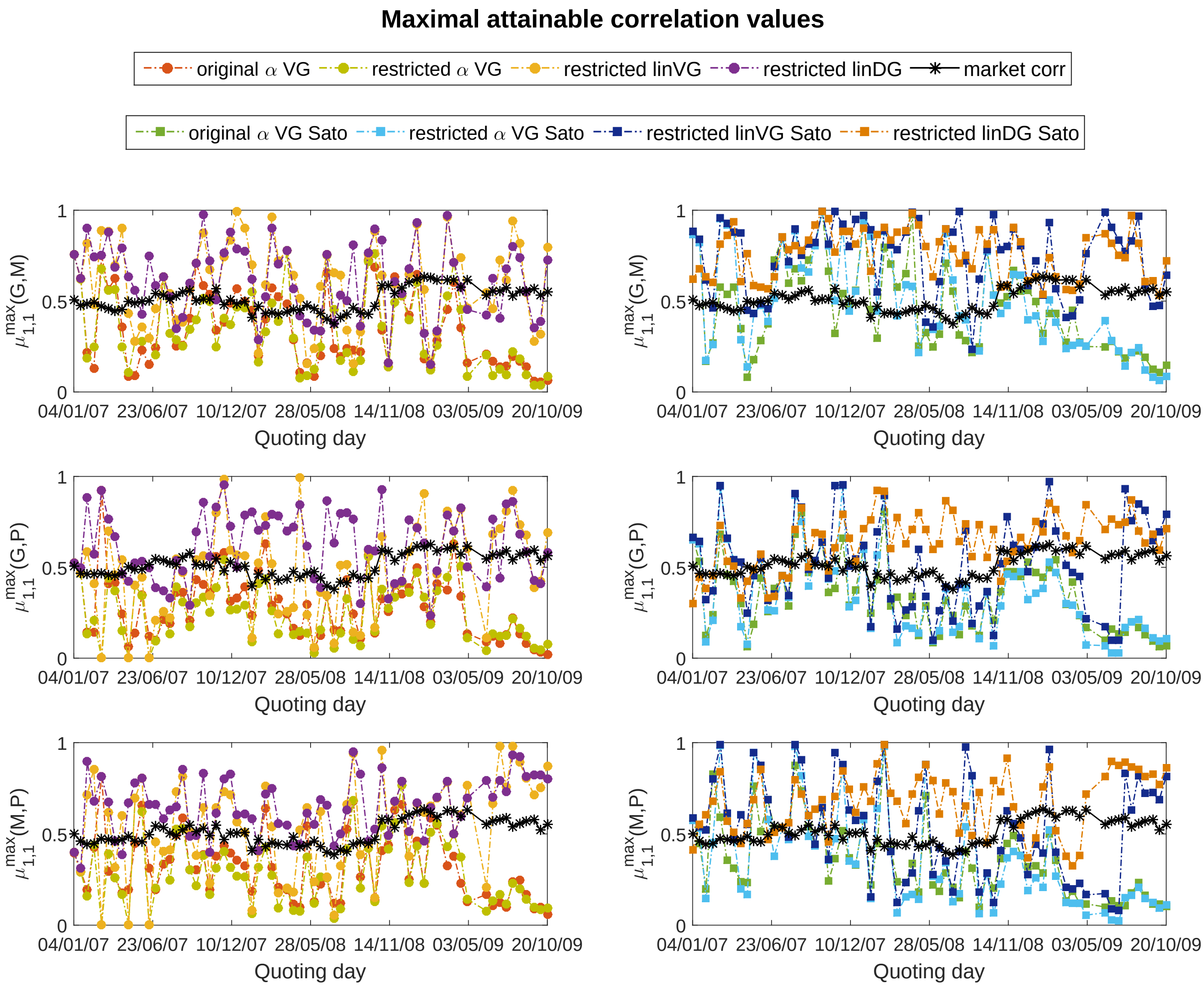


Figure 8: Comparison of the maximal attainable correlation $\mu_{1,1}^{\max}$ values under the Lévy (left) and Sato (right) models.

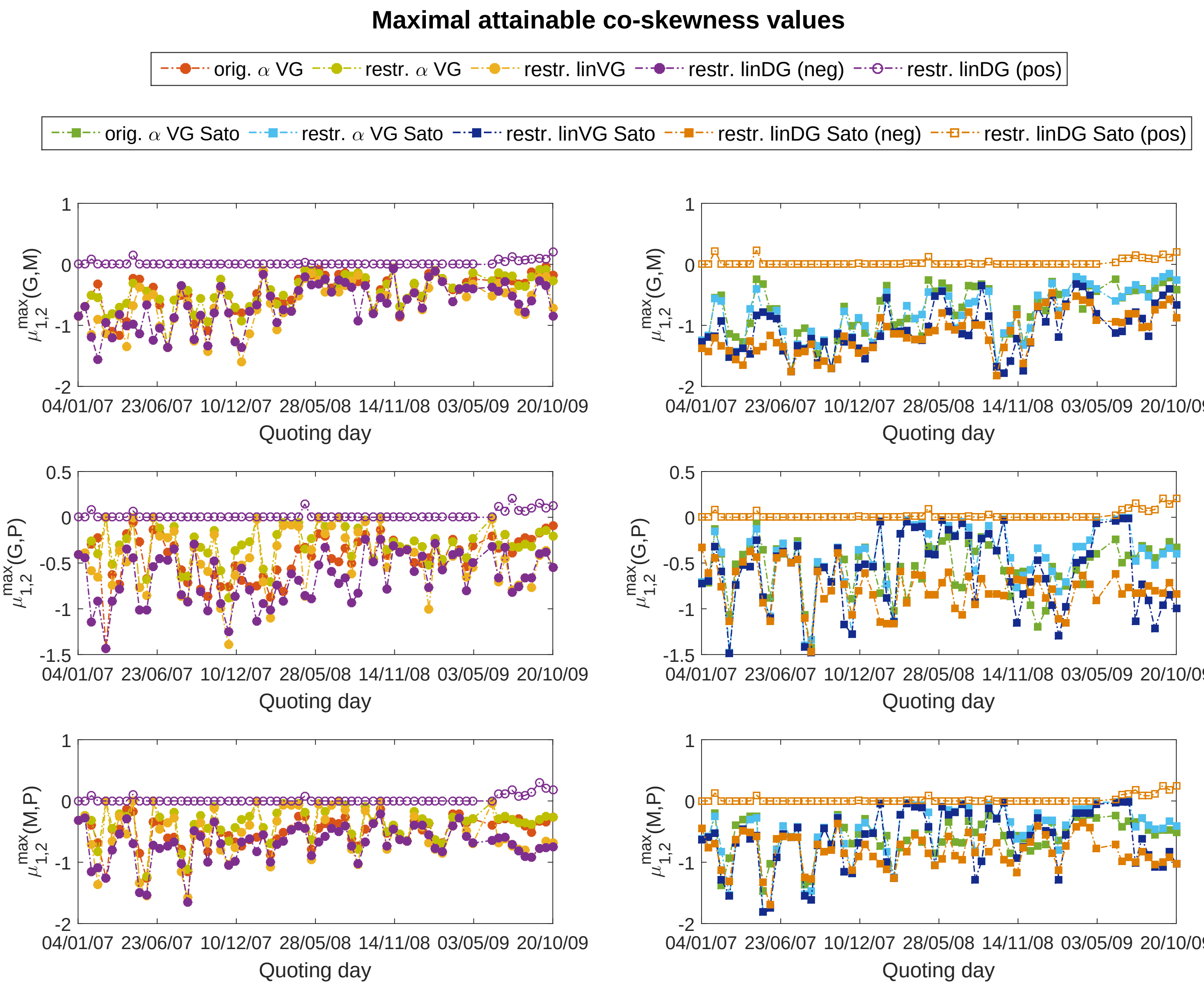


Figure 9: Comparison of the maximal attainable co-skewness $\mu_{1,2}^{\max}$ values under the Lévy (left) and Sato (right) models. Note that the co-skewness under the restricted linear DG models can reach both negative and (small) positive values.

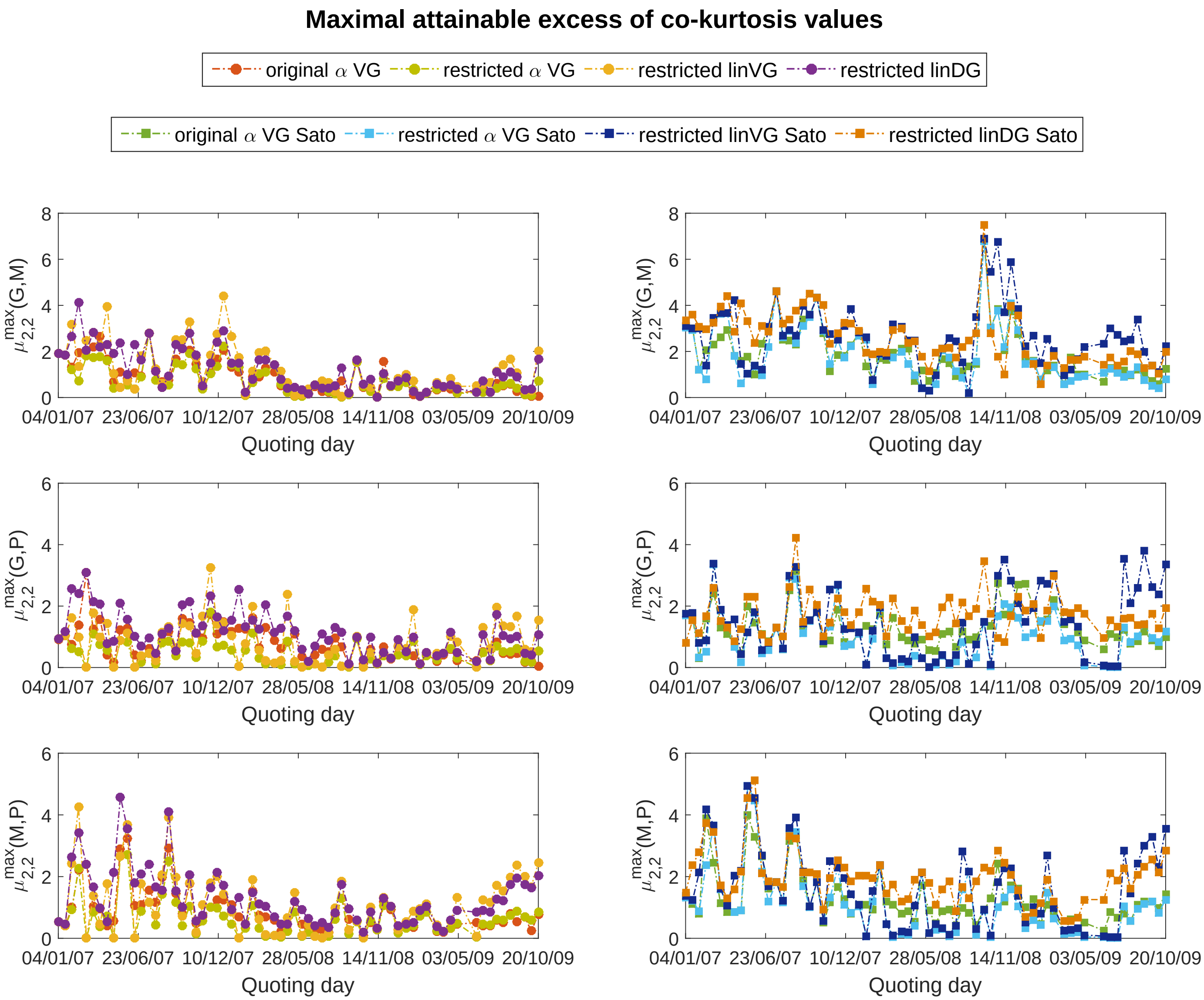


Figure 10: Comparison of the maximal attainable excess of co-kurtosis $\mu_{2,2}^{\max}$ values under the Lévy (left) and Sato (right) models.

Appendix

A Proof of Theorem 1.

Due to the independence of X and Z, we have:

$$\begin{aligned}
\Psi_Y(u) &= \Psi_X(u) + \Psi_Z(au) \\
&= \mathrm{i}\gamma_X u - \frac{\sigma_X^2}{2}u^2 + \int_{\mathbb{R}} \left(\exp(\mathrm{i}ux) - 1 - \mathrm{i}ux\mathbf{1}_{\{|x|<1\}}\right) \frac{h_1(x)}{|x|}\mathrm{d}x \\
&\quad + \mathrm{i}\gamma_Z au - \frac{\sigma_Z^2}{2}a^2u^2 + \int_{\mathbb{R}} \left(\exp(\mathrm{i}aux) - 1 - \mathrm{i}aux\mathbf{1}_{\{|x|<1\}}\right) \frac{h_2(x)}{|x|}\mathrm{d}x,
\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}
\Psi_Y(u) = \mathrm{i}&\left(\gamma_X + \gamma_Z a + \int_{\mathbb{R}} \left(x\left(\mathbf{1}_{\{|x|<1\}} - \mathbf{1}_{\{|x|<|a|\}}\right)\right) \frac{h_2(\frac{x}{a})}{|x|}\mathrm{d}x\right) u \\
&- \frac{\left(\sigma_X^2 + a^2\sigma_Z^2\right)}{2}u^2 + \int_{\mathbb{R}} \left(\exp(\mathrm{i}ux) - 1 - \mathrm{i}ux\mathbf{1}_{\{|x|<1\}}\right) \left(\frac{h_1(x) + h_2(\frac{x}{a})}{|x|}\right) \mathrm{d}x.
\end{aligned}$$

Set $h(x) := h_1(x) + h_2(\frac{x}{a})$. In order for $\nu_Y = \frac{h(x)}{|x|}\mathrm{d}x$ to be a Lévy measure, it must hold that:

P1 ν_Y is defined on $\mathbb{R} \setminus \{0\}$,

P2 $\int_{\mathbb{R}} \min\left(1, |x|^2\right) \frac{h(x)}{|x|}\mathrm{d}x < \infty$.

Moreover, Y is self-decomposable if

P3 $h(x) \geq 0$,

P4 $h(x)$ increasing for negative x and decreasing for positive x.

It is easy to see that the properties P1, P3 and P4 are fullfilled. For the property P2 to hold, it must hold that

$$\int_{\mathbb{R}} \min\left(1, |x|^2\right) \frac{h_2\left(\frac{x}{a}\right)}{|x|}\mathrm{d}x < \infty,$$

which is true since:

$$\int_{\mathbb{R}} \min\left(1, |x|^2\right) \frac{h_2\left(\frac{x}{a}\right)}{|x|}\mathrm{d}x \leq \max\left(1, |a|^2\right) \int_{\mathbb{R}} \min\left(1, |y|^2\right) \frac{h_2\left(y\right)}{|y|}\mathrm{d}x < \infty.$$

B Term structure of Lévy standardized co-moments

Let $\mathbf{X} = \{\mathbf{X}_t, t \geq 0\}$ be a multivariate Lévy process with dependent marginals $X_t^{(j)}, j = 1, \ldots, n$ and joint charactertistic function $\phi_{\mathbf{X}}(\mathbf{u}; t)$. The (m, n)-th standardized co-moment of $X_t^{(j)}$ and $X_t^{(k)}$ is given by:

$$\mu_{m,n}\left(X_t^{(j)}, X_t^{(k)}\right) = \frac{\mathbb{E}\left[\left(X_t^{(j)} - \mathbb{E}\left[X_t^{(j)}\right]\right)^m \left(X_t^{(k)} - \mathbb{E}\left[X_t^{(k)}\right]\right)^n\right]}{\left(\sqrt{\mathrm{Var}\left[X_t^{(j)}\right]}\right)^m \left(\sqrt{\mathrm{Var}\left[X_t^{(k)}\right]}\right)^n}, \quad m, n \geq 0.$$

It is easily shown that $\mathrm{Var}[X_t^{(j)}] = t\mathrm{Var}[X^{(j)}], \forall j = 1, \dots, n$ by using the infinitely divisibility property of Lévy processes (see f.i. [21]):

$$\phi_Y(u;t) = (\phi_Y(u;1))^t .$$

The (m,n)-th central co-moment is given in terms of the joint charactistic function of $X_t^{(j)}$ and $X_t^{(k)}$ as follows:

$$\mathbb{E}\left[\left(X_t^{(j)} - \mathbb{E}\left[X_t^{(j)}\right]\right)^m \left(X_t^{(k)} - \mathbb{E}\left[X_t^{(k)}\right]\right)^n\right] = \frac{\partial^{m+n}}{\partial(\mathrm{i}u)^m \partial(\mathrm{i}v)^n} \left(e^{-\mathrm{i}\left(u\mathbb{E}\left[X_t^{(j)}\right] + v\mathbb{E}\left[X_t^{(k)}\right]\right)} \phi_{X^{(j)}X^{(k)}}(u,v;t)\right)\Bigg|_{u=0,v=0},$$

which can be calculated using the multivariate product rule for partial derivatives in combination with the multivariate version of Faà di Bruno's formula.

Proposition B.1 (The multivariate product rule [18])**.** *Let a and b be continuous functions of $(x_1, \dots, x_n)$. Then:*

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n}(a \cdot b) = \sum_S \frac{\partial^{(|S|)} a}{\prod_{j \in S} \partial x_j} \cdot \frac{\partial^{(n-|s|)} b}{\prod_{j \notin S} \partial x_j},$$

where the index S runs through the set of all subsets of $\{1, \dots, n\}$.

Proposition B.2 (The multivariate version of Faà di Bruno's formula [18])**.**

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} f(y) = \sum_{\pi \in \Pi} f^{(|\pi|)}(y) \cdot \prod_{B \in \pi} \frac{\partial^{|B|} y}{\prod_{j \in B} \partial x_j},$$

where $y = g(x_1, \dots, x_n), n \in \mathbb{N}$ and π runs through the set Π of all partitions of the set $\{1, \dots, n\}$.

Using these two formulae with $a = e^{-\mathrm{i}\left(u\mathbb{E}[X_t^{(j)}] + v\mathbb{E}[X_t^{(k)}]\right)}$ and $b = f(g(u,v)) = g(u,v)^t$, with $g(u,v) = \phi_{X^{(j)}X^{(k)}}(u,v;1)$, together with the fact that $\mathbb{E}[Y_t] = t\mathbb{E}[Y]$ for any Lévy process $Y = \{Y_t, t \geq 0\}$, we can prove that:

$$\mathbb{E}\left[\left(X_t^{(j)} - \mathbb{E}\left[X_t^{(j)}\right]\right)\left(X_t^{(k)} - \mathbb{E}\left[X_t^{(k)}\right]\right)^2\right] = t\left(\mathrm{Cov}\left(X_1^{(j)}, \left(X_1^{(k)}\right)^2\right) - 2\mathbb{E}\left[X_1^{(k)}\right]\mathrm{Cov}\left(X_1^{(j)}, X_1^{(k)}\right)\right),$$

$$\mathbb{E}\left[\left(X_t^{(j)} - \mathbb{E}\left[X_t^{(j)}\right]\right)^2\left(X_t^{(k)} - \mathbb{E}\left[X_t^{(k)}\right]\right)\right] = t\left(\mathrm{Cov}\left(\left(X_1^{(j)}\right)^2, X_1^{(k)}\right) - 2\mathbb{E}\left[X_1^{(j)}\right]\mathrm{Cov}\left(X_1^{(j)}, X_1^{(k)}\right)\right).$$

Hence, the numerator in the co-skewnesses $\mu_{1,2}$ and $\mu_{2,1}$ scales as t, while the denominator scales as $t^{3/2}$ and thus, the co-skewness scales like $1/\sqrt{t}$ over the term.

In a similar way, we obtain:

$$\begin{aligned}\mathbb{E}\left[\left(X_t^{(j)} - \mathbb{E}\left[X_t^{(j)}\right]\right)\left(X_t^{(k)} - \mathbb{E}\left[X_t^{(k)}\right]\right)^3\right] &= 3t^2\mathrm{Var}\left[X_1^{(k)}\right]\mathrm{Cov}\left(X_1^{(j)}, X_1^{(k)}\right) \\ &+ t\left(6\mathbb{E}\left[X_1^{(k)}\right]^2\mathrm{Cov}\left(X_1^{(j)}, X_1^{(k)}\right) - 3\mathbb{E}\left[X_1^{(k)}\right]\mathrm{Cov}\left(X_1^{(j)}, \left(X_1^{(k)}\right)^2\right)\right. \\ &\left. -3\mathbb{E}\left[\left(X_1^{(k)}\right)^2\right]\mathrm{Cov}\left(X_1^{(j)}, X_1^{(k)}\right) + \mathrm{Cov}\left(X_1^{(j)}, \left(X_1^{(k)}\right)^3\right)\right),\end{aligned}$$

$$\mathbb{E}\left[\left(X_t^{(j)}-\mathbb{E}\left[X_t^{(j)}\right]\right)^3\left(X_t^{(k)}-\mathbb{E}\left[X_t^{(k)}\right]\right)\right] = 3t^2\mathrm{Var}\left[X_1^{(j)}\right]\mathrm{Cov}\left(X_1^{(j)},X_1^{(k)}\right)$$
$$+\,t\left(6\mathbb{E}\left[X_1^{(j)}\right]^2\mathrm{Cov}\left(X_1^{(j)},X_1^{(k)}\right)-3\mathbb{E}\left[X_1^{(j)}\right]\mathrm{Cov}\left(\left(X_1^{(j)}\right)^2,X_1^{(k)}\right)\right.$$
$$\left.-3\mathbb{E}\left[\left(X_1^{(j)}\right)^2\right]\mathrm{Cov}\left(X_1^{(j)},X_1^{(k)}\right)+\mathrm{Cov}\left(\left(X_1^{(j)}\right)^3,X_1^{(k)}\right)\right),$$
$$\mathbb{E}\left[\left(X_t^{(j)}-\mathbb{E}\left[X_t^{(j)}\right]\right)^2\left(X_t^{(k)}-\mathbb{E}\left[X_t^{(k)}\right]\right)^2\right] = t^2\left(\mathrm{Var}\left[X_1^{(j)}\right]\mathrm{Var}\left[X_1^{(k)}\right]+2\mathrm{Cov}\left(X_1^{(j)},X_1^{(k)}\right)^2\right)$$
$$+\,t\left(6\mathbb{E}\left[X_1^{(j)}\right]\mathbb{E}\left[X_1^{(k)}\right]\mathrm{Cov}\left(X_1^{(j)},X_1^{(k)}\right)+\mathrm{Cov}\left(\left(X_1^{(j)}\right)^2,\left(X_1^{(k)}\right)^2\right)\right.$$
$$-2\mathbb{E}\left[X_1^{(j)}X_1^{(k)}\right]\mathrm{Cov}\left(X_1^{(j)},X_1^{(k)}\right)-2\mathbb{E}\left[X_1^{(k)}\right]\mathrm{Cov}\left(\left(X_1^{(j)}\right)^2,X_1^{(k)}\right)$$
$$\left.-2\mathbb{E}\left[X_1^{(j)}\right]\mathrm{Cov}\left(X_1^{(j)},\left(X_1^{(k)}\right)^2\right)\right).$$

Hence, the numerator of the co-kurtoses scales as t^2+t, while the denominator scales as t^2. The co-kurtosis is thus the sum of a constant term independent of t and the excess of co-kurtosis, which scales like $1/t$.

C Co-moments linear versus subordinated model setting

Imposing the following condition on both the αVG and the linear VG models, we can compare the range of attainable values for linear and non-linear dependence measures between the two models:

$$\frac{\tilde{\sigma}_j^2}{\tilde{k}_j\tilde{\theta}_j^2}=\frac{\sigma_Z^2}{k_Z\theta_Z^2}=c,\quad \forall j=1,\dots,n, \tag{C.1}$$

for some $c>0$. Indeed, it then turns out that the two models have exactly the same marginal structure. Under the decoupled calibration procedure, where the idiosyncratic parameters are calibrated seperately from the systemic parameters, the upper bound on the common parameter is the same under the restricted αVG models as under the restricted linear VG models:

$$c_1<\frac{1}{k^*}\quad\text{and}\quad\frac{1}{k_Z}<\frac{1}{k^*},$$

where $\frac{1}{k^*}=\min_j\left(\frac{1}{\tilde{k}_j}\right)$.

C.1 Correlation

The correlation between $Y_1^{(i)}$ and $Y_1^{(j)}$, $i\neq j$, under the restricted αVG model is given by:

$$\mu_{1,1}^{\alpha VG}\left(Y_1^{(i)},Y_1^{(j)}\right)=\mathrm{sign}(\tilde{\theta}_i\tilde{\theta}_j)\frac{\sqrt{\tilde{k}_i\tilde{k}_j}c_1}{c+1},$$

and under the restricted linear VG model we have:

$$\mu_{1,1}^{linVG}\left(Y_1^{(i)},Y_1^{(j)}\right)=\mathrm{sign}(\tilde{\theta}_i\tilde{\theta}_j)\frac{\sqrt{\tilde{k}_i\tilde{k}_j}}{k_Z},$$

where $\text{sign}(\tilde{\theta}_i\tilde{\theta}_j) = +1$ under both models due to the imposed constraints (C.1). Hence, the ratio of the maximal attainable correlation coefficients is given by:

$$\frac{\mu_{1,1}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{\mu_{1,1}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} = \frac{1}{c+1},$$

which is smaller than 1 since $c > 0$.

C.2 Co-skewness

We will focus on $\mu_{1,2}$, the result for $\mu_{2,1}$ then follows by symmetry.

The co-skewness between $Y_1^{(i)}$ and $Y_1^{(j)}$, $i \neq j$, under the αVG model is given by:

$$\mu_{1,2}^{\alpha VG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{\tilde{\theta}_i\tilde{k}_i\tilde{k}_j\left(2\tilde{k}_j\tilde{\theta}_j^2 + \tilde{\sigma}_j^2\right)c_1}{\sqrt{\tilde{\sigma}_i^2 + \tilde{k}_i\tilde{\theta}_i^2}\left(\tilde{\sigma}_j^2 + \tilde{k}_j\tilde{\theta}_j^2\right)},$$

which reduces to

$$\mu_{1,2}^{\alpha VG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{\sqrt{\tilde{k}_i}\tilde{k}_j(2+c)c_1}{\text{sign}(\tilde{\theta}_i)(c+1)^{3/2}},$$

under the restricted αVG model. Under the restricted linear VG model it is given by:

$$\mu_{1,2}^{linVG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{\tilde{\theta}_i\tilde{\theta}_j^2\tilde{k}_i\tilde{k}_j^2\left(2 + 3\frac{\sigma_Z^2}{k_Z\theta_Z^2}\right)}{k_Z\sqrt{\tilde{\sigma}_i^2 + \tilde{k}_i\tilde{\theta}_i^2}\left(\tilde{\sigma}_j^2 + \tilde{k}_j\tilde{\theta}_j^2\right)} = \frac{\sqrt{\tilde{k}_i}\tilde{k}_j(2+3c)}{\text{sign}(\tilde{\theta}_i)k_Z(c+1)^{3/2}}.$$

The ratio of the maximal attainable co-skewness can thus be written as:

$$\frac{\mu_{1,2}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{\mu_{1,2}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} = \frac{2+c}{2+3c} < 1.$$

Note that we assume $\tilde{\theta}_i$ to be the same under both models, hence the sign of the co-skewness will be equal under both model settings.

C.3 Co-kurtosis

We will focus on the excesses of co-kurtosis $C_{1,3}$ and $C_{2,2}$. The result for $C_{3,1}$ then follows by symmetry. From the formulas in Appendix B, we have that the excesses of co-kurtosis can be written in terms of the standardized co-moments as follows:

$$\begin{aligned}
C_{1,3}(Y_t^{(j)}, Y_t^{(k)}) &= \mu_{1,3}(Y_t^{(j)}, Y_t^{(k)}) - 3\mu_{1,1}(Y_1^{(j)}, Y_1^{(k)}), \\
C_{3,1}(Y_t^{(j)}, Y_t^{(k)}) &= \mu_{3,1}(Y_t^{(j)}, Y_t^{(k)}) - 3\mu_{1,1}(Y_1^{(j)}, Y_1^{(k)}), \\
C_{2,2}(Y_t^{(j)}, Y_t^{(k)}) &= \mu_{2,2}(Y_t^{(j)}, Y_t^{(k)}) - 1 - 2(\mu_{1,1}(Y_1^{(j)}, Y_1^{(k)}))^2.
\end{aligned}$$

The symmetric excess of co-kurtosis under the αVG model is given by:

$$C_{2,2}^{\alpha VG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{c_1 \tilde{k}_i \tilde{k}_j \left(\tilde{\sigma}_i^2 \left(2\tilde{k}_j \tilde{\theta}_j^2 + \tilde{\sigma}_j^2\right) + 2\tilde{k}_i \tilde{\theta}_i^2 \left(3\tilde{k}_j \tilde{\theta}_j^2 + \tilde{\sigma}_j^2\right)\right)}{\left(\tilde{\theta}_i^2 \tilde{k}_i + \tilde{\sigma}_i^2\right)\left(\tilde{\theta}_j^2 \tilde{k}_j + \tilde{\sigma}_j^2\right)}$$

and reduces to

$$C_{2,2}^{\alpha VG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{\tilde{k}_i \tilde{k}_j (c^2 + 4c + 6) c_1}{(1+c)^2} > 0$$

in the restricted model setting, while under the restricted linear VG model it is given by:

$$C_{2,2}^{linVG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{\tilde{k}_i^2 \tilde{k}_j^2 \tilde{\theta}_i^2 \tilde{\theta}_j^2 \left(6k_Z^3 \theta_Z^4 + 12k_Z^2 \theta_Z^2 \sigma_Z^2 + 3k_Z \sigma_Z^4\right)}{k_Z^4 \theta_Z^4 \left(\tilde{\theta}_i^2 \tilde{k}_i + \tilde{\sigma}_i^2\right)\left(\tilde{\theta}_j^2 \tilde{k}_j + \tilde{\sigma}_j^2\right)} = \frac{3\tilde{k}_i \tilde{k}_j (c^2 + 4c + 2)}{k_Z (1+c)^2} > 0.$$

The ratio of the maximal attainable excess of co-kurtosis is therefore given by:

$$\frac{C_{2,2}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{C_{2,2}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} = \frac{c^2 + 4c + 6}{3(c^2 + 4c + 2)},$$

which is smaller than 1 if and only if

$$c^2 + 4c > 0,$$

which is always true, since $c > 0$.

For the asymmetric excess of co-kurtosis $C_{1,3}$, we have:

$$C_{1,3}^{\alpha VG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{6c_1 \tilde{k}_i \tilde{k}_j^2 \tilde{\theta}_i \tilde{\theta}_j \left(\tilde{k}_j \tilde{\theta}_j^2 + \tilde{\sigma}_j^2\right)}{\sqrt{\tilde{k}_i \tilde{\theta}_i^2 + \tilde{\sigma}_i^2}\left(\tilde{k}_j \tilde{\theta}_j^2 + \tilde{\sigma}_j^2\right)^{3/2}},$$

under the αVG model. Imposing condition (6.4), this reduces to:

$$C_{1,3}^{\alpha VG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{6\sqrt{\tilde{k}_i}\tilde{k}_j^{3/2}(1+c)c_1}{\operatorname{sign}(\tilde{\theta}_i \tilde{\theta}_j)(1+c)^2}.$$

Under the restricted linear VG model we have:

$$C_{1,3}^{linVG}\left(Y_1^{(i)}, Y_1^{(j)}\right) = \frac{3\tilde{k}_i \tilde{k}_j^3 \tilde{\theta}_i \tilde{\theta}_j^3 \left(2 + 4\frac{\sigma_Z^2}{k_Z \theta_Z^2} + \frac{\sigma_Z^4}{k_Z^2 \theta_Z^4}\right)}{k_Z \sqrt{\tilde{k}_i \tilde{\theta}_i^2 + \tilde{\sigma}_i^2}\left(\tilde{k}_j \tilde{\theta}_j^2 + \tilde{\sigma}_j^2\right)^{3/2}} = \operatorname{sign}(\tilde{\theta}_i \tilde{\theta}_j)\frac{3\sqrt{\tilde{k}_i}\tilde{k}_j^{3/2}(2 + 4c + c^2)}{k_Z (1+c)^2},$$

where $\operatorname{sign}(\tilde{\theta}_i \tilde{\theta}_j) = +1$ under both models due to the imposed constraints, leading to

$$\frac{C_{1,3}^{\alpha VG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)}{C_{1,3}^{linVG,\max}\left(Y_1^{(i)}, Y_1^{(j)}\right)} = \frac{2(1+c)}{2 + 4c + c^2} < 1.$$

References

[1] Ané, T. and Geman, H. (1996). Stochastic subordination. *RISK*, **9**(9), 145-149.

[2] Ballotta, L. and Bonfiglioli, E. (2014). Multivariate asset models using Lévy processes and applications. *The European Journal of Finance*, 1-31.

[3] Barndorff-Nielsen, O., Kent, J. and Sorensen, M. (1982). Normal variance-mean mixtures and z distributions. *International Statistical Review*, **50**(2), 145-159.

[4] Barndorff-Nielsen, O. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics*, textbf2(1), 41-68.

[5] Barndorff-Nielsen, O., Pedersen, J. and Sato, K. (2001). Multivariate subordination, self-decomposability and stability. *Advances in Applied Probability*, **33**(1), 160-187.

[6] Carr, P. and Madan, D.B. (1998). Option valuation using the fast Fourier transform. *Journal of Computational Finance*, **2**(4), 61-73.

[7] Boen, L. and Guillaume, F. (2016). Towards a Δ-Gamma Sato model. SSRN report (https://ssrn.com/abstract=2900900). Submitted.

[8] Breeden, D. and Litzenberger, R. (1978). Prices of state contingent claims implicit in option prices. *Journal of Business*, **51**, 621-651.

[9] Buchmann, B., Lu, K., and Madan, D.B. (2016). Weak subordination of multivariate Lévy processes. *Working paper.*

[10] Carr, P., Geman, H., Madan, D.B. and Yor, M. (2007). Self-decomposability and Option Pricing. *Mathematical Finance*, **17**(1), 31-57.

[11] Cont, R. and Tankov, P. (2004). Financial Modelling with Lévy processes. *Chapman and Hall/CRC Financial Mathematics Series.* CRC Press.

[12] Eberlein, E and Madan, D.B. (2010). On correlating Lévy processes. *Journal of Risk*, **13**(1), 3-16.

[13] Finlay, R. and Seneta, E. (2008). Option pricing with VG-like models. *International Journal of Theoretical and Applied Finance*, **11**(8), 943-955.

[14] Gradshteyn, I.S. and Ryzhik, I.M. (2014).Table of integrals, series, and products. *Academic press.*

[15] Guillaume, F. (2012). Sato two-factor models for multivariate option pricing. *Journal of Computational Finance*, **15**(4), 159-192.

[16] Guillaume, F. (2013). The αVG model for multivariate asset pricing: Calibration and extension. *Review of Derivatives Research*, **16**(1), 25-52.

[17] Guillaume, F. (2017). Multivariate option pricing models with Lévy and Sato VG marginal processes. Working paper.

[18] Hardy, M. (2006).Combinatorics of partial derivatives. *The Electronic Journal of Combinatorics*, **13**(1), 13p.

[19] Harris, L. (1986). Cross-security tests of the mixture of distributions hypothesis. *Journal of Financial and Quantitative Analysis*, **21**(1), 39-46.

[20] Kawai, R. (2009). A multivariate Lévy process model with linear correlation. *Quantitative Finance*, **9**(5), 597-606.

[21] Konikov, M. and Madan, D.B. (2001). Option pricing using Variance Gamma Markov chains. *Review of Derivatives Research*, **5**, 81-115.

[22] Küchler, U. and Tappe, S. (2008). Bilateral gamma distributions and processes in financial mathematics. *Stochastic Processes and their Applications.*, **118**(2), 261-283.

[23] Küchler, U. and Tappe, S. (2013). Tempered stable distributions and applications to financial mathematics. *Stochastic processes and their applications.*

[24] Kyprianou, A. (2006). Fluctuations of Lévy Processes with Applications. Introductory Lectures. *Springer Science & Business Media*. Springer Berlin Heidelberg.

[25] Leoni, P. and Schoutens W. (2008). Multivariate smiling. *Wilmott magazine*

[26] Lo, A. W., Wang J., (2000). Trading volume: Definitions, data analysis, and implications of portfolio theory. *The Review of Financial Studies*, **31**(2), 257-300.

[27] Luciano, E., Schoutens, W. (2006). A multivariate jump-driven financial asset model. *Quantitative Finance*, **6**(5), 385-402.

[28] Luciano E., Semeraro P. (2010). Multivariate time changes for Lévy asset models: Characterization and calibration. *Journal of Computational and Applied Mathematics*, **233**, 1937-1953.

[29] Luciano, E., Marena, M. and Semeraro, P. (2016). Dependence calibration and portfolio fit with factor-based subordinators. *Quantitative Finance*, **16**(7), 1037-1052.

[30] Madan, D.B., Seneta, E. (1990). The variance gamma (V.G.) model for share market returns. *Journal of Business*, **63**, 511-524.

[31] Madan, D.B. (2011). *Joint risk-neutral laws and hedging.* **IIE transactions**, **43**(12), 840-850.

[32] Marfé, R. (2012). A multivariate pure-jump model with multi-factorial dependence structure. *International Journal of Theoretical and Applied Finance*, **15**(4), 1250028-(1-30).

[33] Sato K. (1999). Lévy Processes and Infinitely Divisble Distributions. *Cambridge University Press.*

[34] Sato K. (2000). Subordination and self-decomposability. *Statistics and Probability Letters*, **54**(3), 317-324.

[35] Schoutens, W. (2003). *Lévy processes in finance: Pricing financial derivatives.* New York: Wiley.

[36] Semeraro, P. (2008). A multivariate variance gamma model for financial applications. *International Journal of Theoretical and Applied Finance*, **11**(1), 1-18.

[37] Tian, Y. (2011). Extracting risk-neutral density and its moments from American option prices. *The Journal of Derivatives*, **18**, 17-34.

[38] Vasicek, O. (1987). Probability of loss on loan portfolio. Memo. San Francisco: KMV Corporation.