The n-fold compound option

Liesbeth Thomassen\textsuperscript{1} and Martine Van Wouwe\textsuperscript{1}

Abstract

This paper revisits the compound options as introduced by R. Geske \cite{2}. Geske presented a theory for pricing an option on an option which he defined as a compound option. He developed a closed form expression for this kind of options. In this paper we will extend the notion of compound option to the n-fold compound option or compound option of order n. Moreover an interesting relationship between a $k$-variate normal distribution function and a $(k+1)$-variate normal distribution function is proved for this intention.

keywords: financial, n-fold compound options, multivariate normal CDF.

1 Introduction

As was mentioned by Geske \cite{2}, any opportunity with a choice whose value depends on an underlying asset can be viewed as an option. The specific opportunity for an option are its boundary conditions. Many opportunities have a sequential nature, where latter opportunities are available only if earlier opportunities are undertaken. Such is the nature of the compound option (by Geske \cite{2}) or option on an option.

We reintroduce the concept of a compound (call) option.

A 2-fold compound call option (or compound option of order 2) is a call option on a call option i.e. a call option with the underlying being a call option itself.

So such a contract entitles one to the following payoff at $t_1$

$$\max \{C(t_1, S(t_1), t_2, K_2), K_1\},$$

\textsuperscript{1}University of Antwerp, Belgium
that is, at \( t_1 \), the investor (holder) receives the maximum of the amount \( K_1 \) and the value of a European call on the asset \( S \) with exercise date and price given by \( t_2 \) and \( K_2 \) respectively. In other words the investor holds a call, exercisable at \( t_1 \), on the underlying call which is exercisable at \( t_2 \).

In this paper this idea is generalized to a compound call of order \( n \) (with exercise date and price given by \( t_1 \) and \( K_1 \)) with as underlying asset a compound call of order \( n-1 \) (with exercise date and price given by \( t_2 \) and \( K_2 \)) which itself is a call on a call of order \( n-2 \) ... until the final underlying asset, a European call, to be a call of order 1 (with exercise date and price given by \( t_n \) and \( K_n \)).

The price at time \( t_0 \) of such a call of order \( n \) is denoted by

\[
C^{(n)}(t_0, S(t_0; (t_j, K_j)_{j=1}^n)),
\]

with \((t_j, K_j)\) the exercise date and price of the call of order \((n-j+1)\), underlying the call of order \(n\).

In section 2 the valuation equation for this \(n\)-fold compound option (compound of order \(n\)) is presented and proved by induction. The proof is based on the PDE representation form and involves a result on a relationship between the \(k\)-variate and \((k+1)\)-variate normal distribution function which is treated in appendix A.

The reason for the consideration of these \(n\)-fold compound options is the possible application of such derived financial products in the field of

- growing business
- insurance business where the relevance of such products was shown by Simon and Van Wouwe [6] to leave the insured the opportunity to get out of a life insurance contract on certain surrender dates and to be in the possibility to put a price on such an opportunity.

Notations:

\( V \) : current market value of the firm,
\( S \) : current market value of the stock, viewed as a call option on the value of the firm \( V \),
\( C \) : current value of the compound call-option,
\( t \) : current time,
\( t^* \) : maturity date of investment for the compound call option \( C \),
\( T \) : maturity date of investment for the call option \( S \),
\( r \) : risk-free rate of interest,
\( \sigma \) : instantaneous variance of the return on the assets of the firm,
\( K \) : exercise price for the compound call option \( C \),

\( M \) : exercise price for the call option \( S \),

\( N_1(\cdot) \) : univariate cumulative normal distribution function,

\( N_2(h, k : \rho) \) : bivariate cumulative normal distribution function

with \( h \) and \( k \) as upper limits and \( \rho \) as the correlation coefficient,

\( \nabla \) : solution of the equation \( S(V, t^*) = K \).

The economic assumptions, which are to be considered for the valuation equation of a compound option in a continuous time and using a hedging argument, are:

- there is no credit risk, only market risk,

- the market is maximally efficient, i.e. it is infinitely liquid and does not exhibit any friction,

- continuous trading is possible,

- the time evolution of the asset price is stochastic and exhibits geometric Brownian motion,

- the risk-free interest rate \( r \) and the volatility \( \sigma \) are constant,

- the underlying pays no dividends,

- the underlying is arbitrarily divisible,

- the market is arbitrage-free.

According to the article of Geske [2] we have the following partial differential equations:

\[
\frac{\partial C}{\partial t} = rC - rV \frac{\partial C}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 C}{\partial V^2} \tag{1}
\]

\[
\frac{\partial S}{\partial t} = rS - rV \frac{\partial S}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 S}{\partial V^2}, \tag{2}
\]

with boundary conditions:

\[ C_{t^*} = \max (0, S_{t^*} - K) \tag{3} \]

\[ S_T = \max (0, V_T - M). \tag{4} \]

Solving equation (2) leads to the well-known Black-Scholes-Merton equation:

\[ S(V, t) = V N_1(d_1) - M e^{-r(T-t)} N_1(d_2), \]

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with
\[ d_2 = \frac{\ln \frac{V}{M} + (r - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \]

\[ d_1 = d_2 + \sigma \sqrt{T - t}. \]

However, there is a slight difference for the value of the compounded call option, which was showed by Geske [2]:

\[ C(V, t) = V N_2(h_1, d_1; \rho) - M e^{-r(T-t)} N_2(h_2, d_2; \rho) - K e^{-r(t^*-t)} N_1(h_2), \]

with
\[ h_2 = \frac{\ln \frac{V}{P} + \left( r - \frac{1}{2} \sigma^2 \right)(t^*-t)}{\sigma \sqrt{t^*-t}} \]

\[ h_1 = h_2 + \sigma \sqrt{t^*-t}. \]
2 Valuation of the n-fold compound option

Definition 2.1 By induction:
A compound call option (of order 2) is a call option on a call option. This can be generalized to a compound call of order $k+1$ (with exercise date and price given by $t_1$ and $K_1$) with an underlying call of order $k$.

Now consider an expansion of the symbols from paragraph 1:
- $t_i$ : maturity date of investment for the compound call option $C_i$
- $K_i$ : exercise price for the compound call option $C_i$
- $C_i$ : current value of the compound call option on the option $C_{i+1}$
- $N_k(a_1, a_2, \ldots, a_k; A)$ : $k$-variate cumulative normal distribution function with $a_i$ as upper limits and $A$ as the correlation matrix.

Because each $C_i$ is function of the value of the firm $V$ and the time $t$, these calls all have the same PDE:
\[
\frac{\partial C_i}{\partial t} = r_i C_i - r_i V \cdot \frac{\partial C_i}{\partial V} - \frac{1}{2} \sigma^2 V^2 \cdot \frac{\partial^2 C_i}{\partial V^2},
\]
but with a different boundary condition:
\[
C_i(V, t_i) = \max(0, C_{i+1}(V, t_i) - K_i).
\]

The most outer call is simply derived according to Black, Scholes and Merton, while the next one can be defined following the method of Geske [2].

Further we repeatedly add a time step and solve the corresponding PDE.

This results in the following theorem:

Theorem 2.1 Suppose for $s = k+1, k, \ldots, 2$ the calls $C_s$ are known and given by:
\[
C_s = V \cdot N_{k+2-s} \left( a_s, a_{s+1}, \ldots, a_{k+1}; A_s^{k+2-s} \right) - \sum_{m=s}^{k+1} K_m \cdot e^{-r(m-t)} \cdot N_{m+1-s} \left( b_s, b_{s+1}, \ldots, b_m; A_s^{m+1-s} \right),
\]

where we use the notations
\[
a_{\ell} = b_{\ell} + \sigma \sqrt{t_\ell - t} \quad \ell = 2, 3, \ldots, k+1
\]
\[
b_{\ell} = \frac{\ln V + (r - \frac{\sigma^2}{2})(t_\ell - t)}{\sigma \sqrt{t_\ell - t}} \quad \ell = 2, 3, \ldots, k+1
\]
\[
V_\ell \quad \text{solution of the equation} \ C_{\ell+1}(V, t_{\ell}) = K_\ell \quad \ell = 2, 3, \ldots, k
\]
\[ V_{k+1} = M \]
\[ \rho_{ij} = \sqrt{\frac{t_i - t}{t_j - t}} \quad i \neq j \]
\[ A_s^\ell = (a_{ij}^\ell)_{i,j=1,2,\ldots,\ell} \quad \text{where} \quad \begin{cases} a_{ii} = 1 \\ a_{ij} = \rho_{i+s-1,j+s-1} \quad i \neq j. \end{cases} \]

Then the \((k+1)\) fold compound option can be found to be:

\[ C_1 = V \cdot N_{k+1}(a_1, a_2, \ldots, a_{k+1}; A_1^{k+1}) - \sum_{m=1}^{k+1} K_m e^{-r(t_m - t)} \cdot N_m(b_1, b_2, \ldots, b_m; A_1^m). \]

Proof
Since \(C_1\) is a call option, the following PDE holds for \(C_1\):

\[ \frac{\partial C_1}{\partial t} = r \cdot C_1 - r \cdot V \cdot \frac{\partial C_1}{\partial V} - \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 C_1}{\partial V^2}, \]

with \(C_1(V, t_1) = \max(0, C_2(V, t_1) - K_1)\) as boundary condition.
Making use of the results in appendix B, the PDE for \(C_1\) can be transformed into a diffusion equation:

\[ \frac{\partial \tilde{x}}{\partial s} = \frac{\partial^2 \tilde{x}}{\partial p^2}, \]

with adjusted boundary conditions for the variables \(p\) and \(s\)

\[ \tilde{x}(p, 0) = \begin{cases} C_2(V, t_1) - K_1 & \text{if } V \geq \tilde{V}_1 \\ 0 & \text{if } V < \tilde{V}_1 \end{cases} \]

(\(\tilde{V}_1\) being the solution of \(C_2 - K_1 = 0\).)
(see [7] for a proof of the monotonicity of \(C\) with respect to \(V\))

or

\[ \tilde{x}(p, 0) = \begin{cases} V \cdot N_k(d_2, d_3, \ldots, d_{k+1}; A_2^k) - \sum_{m=2}^{k+1} K_m e^{-r(t_m - t_1)} \cdot N_{m-1}(f_2, f_3, \ldots, f_m; A_2^{m-1}) - K_1 & \text{if } p \geq 0 \\ 0 & \text{if } p < 0. \end{cases} \]
By introducing the notations
\[ d_\ell = f_\ell + \sigma \sqrt{t_\ell - t_1} \quad \ell = 2, 3, ..., k + 1 \]
\[ f_\ell = \frac{\ln \frac{V}{\sqrt[4]{V_{\ell}}}}{\sqrt[4]{V_{\ell}}} \frac{(r - \frac{\sigma^2}{2})(t_\ell - t_1)}{\sigma \sqrt{t_\ell - t_1}} \quad \ell = 2, 3, ..., k + 1 \]
and for \( \ell = 2, ..., k \) the matrices
\[ A_2^\ell = (a_{ij}^\ell)_{i,j=1,2,...,\ell} \]
where
\[ a_{ii} = 1 \]
\[ a_{ij} = a_{ji} = \rho_{i+1,j+1}|_{t_1} \quad i < j \]
\[ = \frac{t_{i+1} - t_1}{t_{j+1} - t_1} \]
we find the following expression for \( V \) at \( t = t_1 \):
\[ V = \sqrt[4]{V_1} \exp \left( \frac{\frac{1}{2}\sigma^2}{r - \frac{1}{2}\sigma^2} p \right) \],
which will be used later on.
Using a Green’s function as delta-function, the expression for \( \ddot{x}(p, s) \) can be written as:
\[ \ddot{x}(p, s) = \int_{-\infty}^{+\infty} \ddot{x}(p', 0).G(p - p', s) \, dp' \]
with \( G(p - p', s) = \frac{1}{\sqrt{4\pi s}} e^{-\frac{(p - p')^2}{4s}} \),
or
\[ \ddot{x}(p, s) = \int_0^{+\infty} \sqrt[4]{V_1} \exp \left( \frac{\frac{1}{2}\sigma^2}{r - \frac{1}{2}\sigma^2} p' \right).N_k(d_2, ..., d_{k+1}; A_2^k).\frac{1}{\sqrt{4\pi s}} \exp \left( -\frac{(p - p')^2}{4s} \right) \, dp' \]
\[ - \int_0^{+\infty} \sum_{m=2}^{k+1} K_m e^{-r(t_m - t_1)} . N_{m-1}(f_2, ..., f_m; A_2^{m-1}).\frac{1}{\sqrt{4\pi s}} \exp \left( -\frac{(p - p')^2}{4s} \right) \, dp' \]
\[ - \int_0^{+\infty} K_1 \frac{1}{\sqrt{4\pi s}} \exp \left( -\frac{(p - p')^2}{4s} \right) \, dp'. \]
A substitution of \( b = \frac{p' - p}{\sqrt{2s}} \) in all of the three integrals and a second substitution
\( b' = b - \sigma \sqrt{t_1 - t} \) in the first integral, lead to the following expression:
\[ \ddot{x}(p, s) = \int_{-d_1}^{+\infty} V \exp(t_1 - t).N_k(d_2^*, ..., d_{k+1}^*; A_2^k).\frac{1}{\sqrt{2\pi}} \exp \left( -\frac{b'^2}{2} \right) \, db' \]
\[- \int_{-f_1}^{+\infty} K_m e^{-r(t_m-t_1)} N_{m-1}(f_1, \ldots, f_m, A_2^{m-1}) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{b^2}{2} \right) \, db \]
\[- \int_{-f_1}^{+\infty} K_1 \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{b^2}{2} \right) \, db. \]

In these calculations the new integration boundaries can be found as:
\[- f_1 = - \frac{\ln \frac{V}{V_1} + \left( r - \frac{1}{2} \sigma^2 \right)(t_1 - t)}{\sigma \sqrt{t_1 - t}} \]
\[- d_1 = - (f_1 + \sigma \sqrt{t_1 - t}), \]
while for \( \ell = 2, 3, \ldots, k + 1 \) we get:
\[ f_\ell = \frac{\ln \frac{V_1}{V_\ell} + \frac{1}{2} \frac{\sigma^2}{r - \frac{1}{2} \sigma^2} p' + \left( r - \frac{1}{2} \sigma^2 \right)(t_\ell - t_1)}{\sigma \sqrt{t_\ell - t_1}} \]
\[ f'_\ell = \frac{b_\ell + \rho_\ell b}{\sqrt{1 - \rho_\ell^2}} \quad \text{if} \quad \rho_\ell = \frac{t_1 - t}{t_\ell - t} \]
and
\[ d_\ell = \frac{\ln \frac{V_1}{V_\ell} + \frac{1}{2} \frac{\sigma^2}{r - \frac{1}{2} \sigma^2} p' + \left( r + \frac{1}{2} \sigma^2 \right)(t_\ell - t_1)}{\sigma \sqrt{t_\ell - t_1}} \]
\[ d'_\ell = \frac{a_\ell + \rho_\ell b'}{\sqrt{1 - \rho_\ell^2}}. \]

An application of theorem A.1 in the first integral leads to the final expression for \( \tilde{x}(p, s) \):
\[ \tilde{x}(p, s) = V e^{r(t_1 - t)} N_{k+1}(d_1, a_2, \ldots, a_{k+1}; A_1^{k+1}) \]
\[ - \sum_{m=2}^{k+1} K_m e^{-r(t_m-t_1)} N_m(f_1, b_2, \ldots, b_m; A_1^m) - K_1 N_1(f_1), \]

where the correlation matrices can be written as:
\[ A_1^{k+1} = (a_{ij})_{i,j=1,2,\ldots,k+1} \quad \text{with} \quad \begin{cases} a_{ii} = 1 \\ a_{ij} = a_{ji} = \rho_{ij} \quad \text{if} \ i < j \end{cases} \]
and for $m = 2, 3, \cdots, k + 1$:

$$A_1^m = (a_{ij})_{i,j=1,2,\cdots,m} \quad \text{with} \quad \begin{cases} a_{ii} = 1 \\ a_{ij} = a_{ji} = \rho_{ij} & \text{if } i < j \end{cases}$$

The final form for the $(k+1)$ fold compound call option therefore equals

$$C_1(V, t) = V.N_{k+1}(a_1, a_2, \ldots, a_{k+1}; A_1^{k+1})$$

$$- \sum_{m=2}^{k+1} K_m e^{-r(t_m - t)}.N_m(b_1, b_2, \ldots, b_m; A_1^m) - K_1 e^{-r(t_1 - t)}.N_1(b_1),$$

with

$$a_{\ell} = b_{\ell} + \sigma \sqrt{t_\ell - t} \quad \ell = 1, 2, \ldots, k + 1$$

$$b_{\ell} = \frac{\ln \frac{V}{V_{\ell}} + (r - \sigma^2/2)(t_\ell - t)}{\sigma \sqrt{t_\ell - t}} \quad \ell = 1, 2, \ldots, k + 1$$

$$\nabla_{\ell} \quad \text{determined by} C_{\ell+1}(V, t_{\ell}) = K_{\ell} \quad \ell = 1, 2, \ldots, k$$

$$\nabla_{k+1} = M$$

$$\rho_{ij} = \sqrt{\frac{t_i - t}{t_j - t}} \quad i < j$$

$$A_1^\ell = (a_{ij}^\ell)_{i,j=1,2,\ldots,\ell} \quad \text{where} \quad \begin{cases} a_{ii} = 1 \\ a_{ij} = a_{ji} = \rho_{ij} & \text{if } i < j \end{cases}$$

3 Conclusions

The notion of a n-fold compound option is introduced as a generalization of the compound option by Geske [2]. The closed-form analytic expression for this n-fold compound option is proved by induction and by using some interesting results on the relationship between $(k+1)$-variate normal distributions and k-variate normal distributions.
A useful relationship between the $k+1$ th multivariate distribution function and the $k$ th multivariate distribution function

In theorem A.1 we will need the following two lemmas about a matrix and its inverse. Both of the lemmas can be proved in a straightforward way.

Lemma A.1 If

\[
A = \begin{bmatrix}
1 & \frac{-a_{12} \sqrt{1 - a_{12}^2}}{1} & \cdots & \frac{-a_{1k} \sqrt{1 - a_{1k}^2}}{1} \\
0 & \frac{1}{\sqrt{1 - a_{12}^2}} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{\sqrt{1 - a_{1k}^2}}
\end{bmatrix}
\]

then

\[
A^{-1} = \begin{bmatrix}
1 & a_{12} & \cdots & a_{1k} \\
0 & \frac{1}{\sqrt{1 - a_{12}^2}} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{\sqrt{1 - a_{1k}^2}}
\end{bmatrix}
\]

by the use of the principle of partitioning.

Lemma A.2

If \(A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & B^{-1} \end{bmatrix}\), then \(A^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & B^{-1} \end{bmatrix}\).

Let \(N_k\) be the k-variate normal distribution function and \(N_{k-1}\) the (k-1)-variate normal distribution function, the following expression can be determined between \(N_k\) and \(N_{k-1}\).
Theorem A.1

\[
\int_{-m_1}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} N_{k-1} \left( \frac{m_2 + \rho_{12} x_1}{\sqrt{1 - \rho_{12}^2}}, \ldots, \frac{m_k + \rho_{1k} x_1}{\sqrt{1 - \rho_{1k}^2}} ; B \right) \, dx_1 = N_k(m_1, \ldots, m_k; C),
\]

with \( C = (c_{ij})_{i,j=1,\ldots,k} \) a symmetric matrix with

\[
\begin{align*}
  c_{11} &= 1 \\
  c_{1j} &= \rho_{1j} \\
  c_{ij} &= \rho_{1i} \rho_{1j} + \sqrt{(1 - \rho_{1i}^2)(1 - \rho_{1j}^2)} b_{i-1,j-1}
\end{align*}
\]

and where for convenience we put

\[ N_0 = 1. \]

Proof by induction
For \( k = 1 \) the result is straightforward.

For the second part of the proof we first rewrite the integral as

\[
\int_{-m_1}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2} N_{k-1} \left( \frac{m_2 + \rho_{12} x_1}{\sqrt{1 - \rho_{12}^2}}, \ldots, \frac{m_k + \rho_{1k} x_1}{\sqrt{1 - \rho_{1k}^2}} ; B \right) \, dx_1 = \int_{-m_1}^{+\infty} \cdots \int_{-m_k}^{+\infty} \frac{1}{\sqrt{(2\pi)^k \det B}} e^{-\frac{1}{2} P(x_1, \ldots, x_k)} \, dx_1 \, dx_2 \cdots \, dx_k,
\]

with \( P(x_1, \ldots, x_k) = [x_1 \ x_2 \ \cdots \ \ x_k] \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & \ddots & \\ \vdots & & & B^{-1} \\ 0 & & & \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \).

Making use of the substitution

\[ y_j = x_j \sqrt{1 - \rho_{1j}^2} + \rho_{1j} x_1 \quad \text{for } j = 2, 3, \ldots, k \]

we can rewrite this expression as

\[
\int_{-m_1}^{+\infty} \cdots \int_{-m_k}^{+\infty} \frac{1}{\sqrt{(2\pi)^k \det B} \prod_{j=2}^{k} (1 - \rho_{1j}^2)} e^{-\frac{1}{2} P^*(x_1, y_2, \ldots, y_k)} \, dx_1 \, dy_2 \cdots \, dy_k.
\]
In this formula, we introduced the matrix:

\[ P^*(x_1, y_2, \ldots, y_k) = [x_1 \ y_2 \ \cdots \ y_k].D.\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \sqrt{1 - \rho_{1k}} \\ 0 & \cdots & 0 & \sqrt{1 - \rho_{1k}} \end{bmatrix}.D^t.\begin{bmatrix} x_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}, \]

where

\[
D = \begin{bmatrix}
1 & \frac{-\rho_{12}}{\sqrt{1 - \rho_{12}^2}} & \cdots & \frac{-\rho_{1k}}{\sqrt{1 - \rho_{1k}^2}} \\
0 & \frac{1}{\sqrt{1 - \rho_{12}^2}} & & \nu \\
\vdots & & \ddots & \sqrt{1 - \rho_{1k}^2} \\
0 & \nu & \cdots & \frac{1}{\sqrt{1 - \rho_{1k}^2}}
\end{bmatrix}.
\]

Since we want to express the k-variate integration by means of a k-variate normal CDF, we now have to determine the correlation matrix \( C \) with:

\[
C^{-1} = D.\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \sqrt{1 - \rho_{1k}} \\ 0 & \cdots & 0 & \sqrt{1 - \rho_{1k}} \end{bmatrix}.D^t.
\]

An application of lemma A.1 and A.2 leads to

\[
C = \begin{bmatrix}
1 & \frac{0}{\rho_{12}} & \cdots & \frac{0}{\rho_{1k}} \\
\frac{0}{\rho_{12}} & \frac{1}{\sqrt{1 - \rho_{12}^2}} & & \nu \\
\frac{0}{\rho_{1k}} & \nu & \cdots & \frac{1}{\sqrt{1 - \rho_{1k}^2}} \\
0 & \nu & \cdots & \frac{1}{\sqrt{1 - \rho_{1k}^2}}
\end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{1 - \rho_{1k}} \end{bmatrix}.
\]
or

\[
C = \begin{bmatrix}
1 & \rho_{12} & \cdots & \rho_{1k} \\
\rho_{12} & 1 & \cdots & \rho_{1k} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{1k} & \rho_{1k} & \cdots & 1
\end{bmatrix} \begin{bmatrix}
F \\
O \\
O \\
O
\end{bmatrix},
\]

with \( F \) obtained by partitioning as

\[
\begin{bmatrix}
\rho_{12} \\
\vdots \\
\rho_{1k}
\end{bmatrix} \begin{bmatrix}
\rho_{12} & \cdots & \rho_{1k}
\end{bmatrix} + \begin{bmatrix}
\sqrt{1 - \rho_{12}^2} & \cdots & O \\
O & \ddots & \sqrt{1 - \rho_{1k}^2} \\
O & \cdots & \sqrt{1 - \rho_{1k}^2}
\end{bmatrix} \cdot \begin{bmatrix}
\sqrt{1 - \rho_{12}^2} \\
O \\
O
\end{bmatrix}.
\]

Hence this results in:

\[
C = (c_{ij})_{i,j=1,\ldots,k} \quad \text{with} \quad \begin{cases} 
  c_{11} = 1 \\
  c_{ij} = \rho_{ij} \\
  c_{ij} = \rho_{1i} \rho_{1j} + \sqrt{(1 - \rho_{1i}^2)(1 - \rho_{1j}^2)} b_{i-1,j-1}
\end{cases}
\]

Now, since \( P^*(x_1, y_2, \ldots, y_k) \) can be written as

\[
P^*(x_1, y_2, \ldots, y_k) = \begin{bmatrix} x_1 & y_2 & \cdots & y_k \end{bmatrix} C^{-1} \begin{bmatrix} x_1 \\
y_2 \\
\vdots \\
y_k
\end{bmatrix}
\]

and

\[
\det C = \prod_{j=2}^{k} (1 - \rho_{1j}^2) \cdot \det B
\]

for the integral we find

\[
\int_{-m_1}^{+\infty} \cdots \int_{-m_k}^{+\infty} \frac{1}{\sqrt{(2\pi)^k \det B \prod_{j=2}^{k} (1 - \rho_{1j}^2)}} e^{-\frac{1}{2} P^*(x_1, y_2, \ldots, y_k)} dx_1 dy_2 \cdots dy_k
\]

\[
= \int_{-m_1}^{+\infty} \cdots \int_{-m_k}^{+\infty} \frac{1}{\sqrt{(2\pi)^k \det C}} e^{-\frac{1}{2} P^*(x_1, y_2, \ldots, y_k)} dx_1 dy_2 \cdots dy_k
\]

or

\[
= N_k(m_1, \ldots, m_k; C),
\]

which completes the proof.
B From PDE to diffusion equation

Consider the PDE:
\[
\frac{\partial C_i}{\partial t} = r_i C_i - r_i V_i \frac{\partial C_i}{\partial V_i} - \frac{1}{2} \sigma^2 V_i^2 \frac{\partial^2 C_i}{\partial V_i^2},
\]
with boundary condition at time \( t_i \) given by
\[
C_i(V, t_i) = \max(0, C_{i+1}(V, t_i) - K_i).
\]
Let \( V_i \) be defined as the value for which \( C_{i+1}(V_i, t_i) = K_i \).
Making use of some substitutions, we can rewrite this PDE as a diffusion equation.
Indeed, first we choose \( w \) as
\[
w = \ln \frac{V}{V_i},
\]
and we define the function \( x(w, t) \) as
\[
x(w, t) = e^{r(t_i - t)} C_i(V, t) = e^{r(t_i - t)} C_i(V_i e^w, t).
\]

Secondly, we rescale the independent variables as
\[
\begin{align*}
w' &= \frac{r - \frac{1}{2} \sigma^2}{\frac{1}{2} \sigma^2} w \\
s &= \left(\frac{r - \frac{1}{2} \sigma^2}{\frac{1}{2} \sigma^2}\right)^2 (t_i - t),
\end{align*}
\]
we define the function \( \hat{x}(w', s) \)
\[
\hat{x}(w', s) = x(w, t).
\]
With
\[
p = w' + s,
\]
finally we rewrite the dependent variable as
\[
\hat{x}(p, s) = \hat{x}(w', s).
\]
Then it follows in a straightforward way that this last function satisfies the diffusion equation
\[
\frac{\partial \hat{x}}{\partial s} = \frac{\partial^2 \hat{x}}{\partial p^2}.
\]
C  Green’s function

Consider Green’s function

\[ G(z - z', t') = \frac{1}{\sqrt{4\pi t'}} e^{-\frac{(z - z')^2}{4t'}} , \]

which clearly satisfies the diffusion equation

\[ \frac{\partial G}{\partial t'} = \frac{\partial^2 G}{\partial z^2} . \]

Note that \( G \) behaves like a delta-function in \( t' = 0 \):

* If \( z \neq z' \) and \( t' \to 0 \) then \( G(z - z', t') \to 0 \)

* If \( z = z' \) and \( t' \to 0 \) then \( G(z - z', t') \to \infty \)

* 

\[
\int_{-\infty}^{+\infty} G(z - z', t') dz' = -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{q^2}{2}\right) dq = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{q^2}{2}\right) dq = 1
\]
Corresponding author

Thomassen Liesbeth
University of Antwerp
Faculty of Applied Economics UFSIA-RUCA
Middelheimlaan 1, G224
B2020 Antwerp
tel. (+32) 3.218.07.83
mail: liesbeth.thomassen@ua.ac.be

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References


