ELSEVIER Journal of Pure and Applied Algebra 94 (1994) 127–141

JOURNAL OF PURE AND APPLIED ALGEBRA

Duality, localization and completion*

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Communicated by C.A. Weibel; received 14 June 1991; revised 6 January 1993

Abstract

In this paper we study relative duality theory, with respect to an idempotent kernel functor σ over some commutative ring R and prove that σ-dualizing R-modules are not only locally injective, but (somewhat surprisingly) globally injective. Using a relative version of completion, we show that the endomorphism ring of a σ-dualizing module coincides with the completion of R with respect to σ. In the final part of the paper we consider relative Gorenstein rings, giving an explicit calculation of their generalized local cohomology groups.

Introduction

In this paper, we solve some of the problems that remained open in [4]. Indeed, in the first section, we come back to relative duality theory, and after a careful examination of the functorial aspects involved, we prove that σ-dualizing modules are not only locally injective, as was pointed out in [4], but actually happen to be injective (a fact, which in the relative context is rather unexpected!). Moreover, the local structure of σ-dualizing modules already being determined in [3, 4], we give in Proposition 2.4 a complete global description of these.

In the present text, we also briefly consider a relative version of completion (we hope to come back to this in more detail elsewhere). We show that the endomorphism ring of a σ-dualizing module is the completion of R with respect to σ, cf. Lemma 3.2 and apply this to introduce a relative version of Matlis duality [9].

In the final part of the paper, we restrict to (relative) Gorenstein rings. After an explicit calculation of their generalized local cohomology, cf. Lemma 3.7 and Lemma 3.14, which strengthens similar statements in [5, 6], we calculate the relative Matlis

* It is a pleasure to thank Pepe Mulet and the referee for their useful comments on the subject of this paper.
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SSDI 0022-4049(93)E0117-M

dual of the "highest" generalized local cohomology group of a σ-complete σ-Gorenstein ring and show that it is a well-determined torsion ideal of R, up to isomorphism (generalizing related results in [5, 6]).

Throughout, we will assume the reader to be familiar with the fundamentals of abstract localization theory, such as expounded in [4, 7, 11], for example. However, for the reader's convenience, we will briefly recollect here some of the most relevant definitions and features of this theory.

1. Locally noetherian rings and localization

Recall that an idempotent kernel functor in R-**mod** is a left exact subfunctor σ of the identity, with the property that $\sigma(M/\sigma(M) = 0$, for any R-module M. Any idempotent kernel functor σ defines a torsion class $\mathbf{T}_\sigma$, consisting of all σ-torsion R-modules M, i.e., with $\sigma(M) = M$ and a torsionfree class $\mathbf{F}_\sigma$ consisting of all σ-torsionfree R-modules, i.e., with $\sigma(M) = 0$. It is well known that each of these classes completely determines σ. Whereas the class $\mathbf{F}_\sigma$ is always closed under taking injective hulls, this is not necessarily valid for the class $\mathbf{T}_\sigma$. If it is, then we call σ stable. Note that over a noetherian (commutative) ring, every idempotent kernel functor is stable.

To σ, we may also associate the so-called Gabriel filter $\mathbf{L}(\sigma)$. It consists of all ideals I of R, such that $R/I \in \mathbf{T}_\sigma$ and uniquely determines σ, since for every $M \in R$-**mod** and any $x \in M$, we have $x \in \sigma M$ if and only if there exists some $I \in \mathbf{L}(\sigma)$, with $Ix = 0$. If $\mathfrak{p}$ is a prime ideal of R, then either $R/\mathfrak{p} \in \mathbf{T}_\sigma$ or $R/\mathfrak{p} \in \mathbf{F}_\sigma$, i.e., σ determines a partition $(\mathbf{Z}(\sigma), \mathbf{K}(\sigma))$ of $\mathrm{Spec}(R)$, where $\mathbf{Z}(\sigma) = \mathbf{L}(\sigma) \cap \mathrm{Spec}(R)$ and $\mathbf{K}(\sigma) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R/\mathfrak{p} \in \mathbf{F}_\sigma\}$.

To any idempotent kernel functor σ in R-**mod**, we may associate a localization functor Q_σ. This functor maps σ-isomorphisms, i.e., morphisms in R-**mod** with kernel and cokernel in $\mathbf{T}_\sigma$, to isomorphisms in R-**mod**, and satisfies some obvious universal properties, cf. [7, 11]. For any R-module M there is a canonical σ-isomorphism $j_\sigma : M \to Q_\sigma(M)$, the so-called localization map. We say that M is σ-closed if the associated localization map j_σ is an isomorphism in R-**mod**, and we denote by (R, σ)-**mod** the full subcategory of R-**mod**, consisting of all σ-closed R-modules. As an example, let $\sigma_{R\setminus\mathfrak{p}}$ be defined by letting $\sigma_{R\setminus\mathfrak{p}} M$ consist of all $x \in M$ such that $sx = 0$ for some $s \in R\setminus\mathfrak{p}$, and this for all $M \in R$-**mod**. The associated localization functor is then just the usual localization functor $(-)_\mathfrak{p}$ at the prime ideal $\mathfrak{p}$, while $(R, \sigma_{R\setminus\mathfrak{p}})$-**mod** is just $R_\mathfrak{p}$-**mod**.

We will say that the ring R is σ-noetherian if $Q_\sigma(R)$ is a noetherian object of (R, σ)-**mod**. In this paper, we will always restrict to σ-noetherian rings; we indicate below a short survey of their most significant features, referring to the literature for more details.

An R-module M is said to be σ-finitely generated, if there exists a finitely generated submodule $N \subseteq M$ such that M/N is σ-torsion. We will say that an R-module M is

σ-noetherian if each of its submodules is σ-finitely generated. If R is σ-noetherian as an R-module, then we call R a σ-noetherian ring. If $\sigma = 0$, then we recover usual finiteness notions. Moreover, it is clear that torsion modules or noetherian resp. finitely generated modules are σ-noetherian resp. σ-finitely generated. It is also clear that if $\sigma \leq \tau$ (i.e., $\mathbf{T}_\sigma \subseteq \mathbf{T}_\tau$) and if M is σ-finitely generated resp. σ-noetherian, then it is also τ-finitely generated resp. τ-noetherian.

Let N be a submodule of an R-module M; then M is σ-noetherian (resp. σ-finitely generated) if and only if N and M/N are σ-noetherian (resp. σ-finitely generated). One also easily verifies that an R-module M is σ-noetherian resp. σ-finitely generated if and only if $M/\sigma M$ or $Q_\sigma(M)$ is. On the other hand, it is also clear that M is σ-noetherian if and only if for every ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of R-submodules of M, there exists a positive integer n such that M_m/M_n is σ-torsion (or, equivalently, $Q_\sigma(M_n) = Q_\sigma(M_m)$) for all $m \geq n$ and that this is also equivalent to the fact that $Q_\sigma(M)$ be a noetherian object in (R, σ)-**mod**, which reduces, for $M = R$, to the definition we gave above.

If R is σ-noetherian, then any idempotent kernel functor σ in R-**mod** is completely determined by $\mathbf{K}(\sigma) \subseteq \mathrm{Spec}(R)$. Indeed, in this case an R-module M is σ-torsion if and only if $M_\mathfrak{p} = 0$ for any $\mathfrak{p} \in \mathbf{K}(\sigma)$. Moreover, $\sigma = \bigwedge_{\mathfrak{p} \in \mathbf{C}(\sigma)} \sigma_{R \setminus \mathfrak{p}}$, where the infimum is taken in the obvious sense and where $\mathbf{C}(\sigma)$ consists of all prime ideals $\mathfrak{p}$, which are maximal within $\mathbf{K}(\sigma)$. We can then apply local-global arguments to solve problems in abstract localization theory by classical localization at prime ideals.

Let $\sigma \leq \tau$ be a pair of idempotent kernel functors in R-**mod**, then we denote by $\mathbf{T}_\tau^{\sigma\text{-fg}}$ the full subcategory of R-**mod**, which consists of all σ-finitely generated τ-torsion R-modules.

Since R is σ-noetherian, the idempotent kernel functor σ is completely determined by $\mathbf{C}(\sigma)$. It is easy to see that the set $\mathbf{K}(\sigma) \setminus \mathbf{C}(\sigma)$ is closed under generalization and it determines unambiguously an idempotent kernel functor σ^1 in R-**mod**, with $\mathbf{K}(\sigma^1) = \mathbf{K}(\sigma) \setminus \mathbf{C}(\sigma)$. We call σ^1 the first skeleton of σ. For details and applications concerning this notion, we refer to [4, V, p. 157].

We conclude these preliminaries with some easy technical results, which will be used in the sections below.

Lemma 1.1. *Let σ be an idempotent kernel functor in R-**mod**. If $0 \neq M \in \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$ is σ-torsionfree, then there exists a chain*

$$(*) \quad 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

of R-submodules, such that for each $1 \leq i \leq n$, there is an isomorphism

$$(M_i/M_{i-1})/\sigma(M_i/M_{i-1}) \cong R/\mathfrak{p}_i,$$

for some $\mathfrak{p}_i \in \mathbf{C}(\sigma)$.

Proof. By [4, III-4.4], there exists a chain (*) of R-submodules, such that for each $1 \leq i \leq n$, there is an isomorphism

$$(M_i/M_{i-1})/\sigma(M_i/M_{i-1}) \cong R/\mathfrak{p}_i,$$

for some $\mathfrak{p}_i \in \mathbf{K}(\sigma)$. But $M \in \mathbf{T}_{\sigma^1}$ implies that $\mathfrak{p}_i \in \mathbf{Z}(\sigma^1)$, and so $\mathfrak{p}_i \in \mathbf{C}(\sigma)$. □

Lemma 1.2. *Let σ be an idempotent kernel functor in R-**mod**.*

(1) *If $\mathfrak{p} \in \mathbf{C}(\sigma)$, then $Q_\sigma(R/\mathfrak{p}) = (R/\mathfrak{p})_\mathfrak{p}$.*

(2) *For any $\mathfrak{p} \in \mathbf{C}(\sigma)$ there is a canonical isomorphism*

$$\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p})) \cong Q_\sigma(R/\mathfrak{p}) \cong k(\mathfrak{p}).$$

Proof. (1) by [10, Theorem 6.7] $Q_\sigma(R/\mathfrak{p}) \cong \bigoplus_{\mathfrak{q} \in \mathbf{C}(\sigma)} (R/\mathfrak{p})_\mathfrak{q}$. On the other hand, if $\mathfrak{p} \neq \mathfrak{q} \in \mathbf{C}(\sigma)$, then obviously $\mathfrak{p} \in \mathbf{L}(\sigma_{R \setminus \mathfrak{q}})$ (since $\mathfrak{p}$ and $\mathfrak{q}$ are incomparable), so $(R/\mathfrak{p})_\mathfrak{q} = 0$, since $\mathfrak{p}(R/\mathfrak{p}) = 0$ implies that $R/\mathfrak{p}$ is torsion at $\mathfrak{q}$. It thus follows that $Q_\sigma(R/\mathfrak{p}) = (R/\mathfrak{p})_\mathfrak{p}$, indeed.

(2) Clearly $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p}))$ may be identified with the set Q of all $q \in E(R/\mathfrak{p})$ with the property that $\mathfrak{p}q = 0$. Let $q \in Q$; then, since $E(R/\mathfrak{p})$ is essential over $R/\mathfrak{p}$, there exists some $r \in R$ such that $0 \neq rq \in R/\mathfrak{p}$. In particular, $r \notin \mathfrak{p}$. But then, $L = \mathrm{Ann}(q) \nsubseteq \mathfrak{p}$ and $q \in Q_\sigma(R/\mathfrak{p})$, indeed.

Conversely, if $q \in Q_\sigma(R/\mathfrak{p}) \cong k(\mathfrak{p})$ and if $L = (R/\mathfrak{p} : q)$, then $L \nsubseteq \mathfrak{p}$, so there exists $r \in R \setminus \mathfrak{p}$ which $rq \in R/\mathfrak{p}$. But then $\mathfrak{p}rq = 0$, so $\mathfrak{p}q = 0$, since $k(\mathfrak{p})$ has no $\mathfrak{p}$-torsion as R-module. Hence $q \in Q$, which proves the assertion. □

2. Relative duality

Throughout this text, R will always denote a commutative ring with unit and σ an idempotent kernel functor in R-**mod**, with the property that R is σ-noetherian.

Let $\sigma \leq \tau$ be a pair of idempotent kernel functors in R-**mod**; denote by

$$T : \mathbf{T}_\tau^{\sigma\text{-fg}} \to (R, \sigma)\text{-}\mathbf{mod}$$

a contravariant functor. Thus for any $M \in \mathbf{T}_\tau^{\sigma\text{-fg}}$, the R-module structure on $T(M)$ is induced by the one on M. It has been proved in [4, V-2.2], that there exists a natural transformation

$$\varphi : T \to \mathrm{Hom}_R(-, E),$$

where

$$E = \varinjlim_{I \in \mathbf{L}(\tau)} T(R/I).$$

Moreover, if $T:\mathbf{T}_\tau^{\sigma\text{-fg}} \to (R,\sigma)$-**mod** is left exact and maps σ-isomorphisms in $\mathbf{T}_\tau^{\sigma\text{-fg}}$ to isomorphisms in (R,σ)-**mod**, then the natural transformation

$$\varphi: T \to \mathrm{Hom}_R(-, E)$$

is a natural equivalence.

Denote by

$$\langle T_\tau^{\sigma\text{-fg}}, (R,\sigma)\text{-}\mathbf{mod}\rangle$$

the category of left exact additive contravariant functors

$$T:\mathbf{T}_\tau^{\sigma\text{-fg}} \to (R,\sigma)\text{-}\mathbf{mod},$$

which map σ-isomorphisms to isomorphisms. On the other hand, τ induces, in the obvious way, an idempotent kernel functor $\tilde{\tau}$ in (R,σ)-**mod**. Let $\mathbf{T}_{\tilde{\tau}}$ denote the full subcategory of (R,σ)-**mod** consisting of the τ-torsion R-modules. From [4, V, p. 165], we have the following:

Proposition 2.1. *The categories*

$$\langle \mathbf{T}_\tau^{\sigma-\mathrm{fg}}, (R,\sigma)\text{-}\mathbf{mod}\rangle \quad \textit{and} \quad \mathbf{T}_{\tilde{\tau}}$$

are equivalent by means of the functors

$$\eta: T \mapsto \varinjlim_{I\in\mathbf{L}(\tau)} T(R/I) \quad \textit{and} \quad \nu: N \mapsto \mathrm{Hom}_R(-, N). \qquad \square$$

Let $E \in \mathbf{T}_\tau^{\sigma\text{-fg}}$. It has been pointed out in [4] that

$$T = \mathrm{Hom}_R(-, E): \mathbf{T}_\tau^{\sigma\text{-fg}} \to (R,\sigma)\text{-}\mathbf{mod}$$

is exact if, and only if, $\mathrm{Coker}(\mathrm{Hom}_R(R, E) \to \mathrm{Hom}_R(K, E)) \in \mathbf{T}_\sigma$ for every ideal K of R. Using a "Baer-like" argument, one sees this to be equivalent to the analogous statement for σ-finitely generated R-modules $M' \subseteq M$. An R-module with these properties is said to be σ-locally injective in (R,σ)-**mod**. It is easy to see that if E is σ-locally injective, then $E_\mathfrak{p}$ is an injective $R_\mathfrak{p}$-module for any $\mathfrak{p} \in \mathbf{K}(\sigma)$ (where the terminology). On the other hand, since $Q_\sigma(R)$ is, in general, not a projective generator for (R,σ)-**mod**—unless (R,σ)-**mod** is a full module category—this does not imply such a module to be injective in (R,σ)-**mod** or R-**mod**. We will see below, however, that the module E actually *is* injective, when the functor T is dualizing.

For any $\mathfrak{p} \in \mathbf{C}(\sigma)$, let $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = (R/\mathfrak{p})_\mathfrak{p} = Q_\sigma(R/\mathfrak{p})$.

We then have the following proposition:

Proposition 2.2. *Assume that* $T \in \langle \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}, R, \sigma)$-**mod**$\rangle$, *the category of left exact contravariant functors*

$$T:\mathbf{T}_{\sigma^1}^{\sigma\text{-fg}} \to (R,\sigma)\text{-}\mathbf{mod}$$

which map σ-isomorphisms to isomorphisms. Then the following conditions are equivalent:

(1) *For all* $M \in \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$, *it follows that* $T(M) \in \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$, *and that the natural morphism*

$$Q_\sigma(M) \to TT(M),$$

derived from the natural transformation

$$\mathrm{id}_{R\text{-}\mathbf{mod}} \to TT,$$

is an isomorphism.

(2) *The functor T is exact and for each* $\mathfrak{p} \in \mathbf{C}(\sigma)$, *we have*

$$T(R/\mathfrak{p}) \cong Q_\sigma(R/\mathfrak{p}).$$

Proof. Let us first prove that (1) implies (2). It already follows from [4, V-4.2] that T is exact. On the other hand, if $\mathfrak{p} \in \mathbf{C}(\sigma)$, then $T(R/\mathfrak{p}) \cong T(k(\mathfrak{p}))$. Moreover, since $T(M)$ inherits the R-module structure of M for any $M \in \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$, it is easy to see that $T(R/\mathfrak{p})$ is a $k(\mathfrak{p})$-vectorspace. By hypothesis, $T(R/\mathfrak{p})$ belongs to $\mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$, so, in particular, it is σ-finitely generated. Since T maps σ-isomorhisms to isomorphisms, we may as well assume that $T(R/\mathfrak{p})$ is finitely generated as an R-module, hence also as a $k(\mathfrak{p})$-vectorspace. Assume $T(R/\mathfrak{p}) = k(\mathfrak{p})^n$, for some positive integer n. Then, up to isomorphism,

$$k(\mathfrak{p}) = Q_\sigma(R/\mathfrak{p}) = TT(R/\mathfrak{p}) = T(k(\mathfrak{p})^n) = T(k(\mathfrak{p}))^n = k(\mathfrak{p})^{n^2}.$$

So $n = 1$ and $T(R/\mathfrak{p}) \cong Q_\sigma(R/\mathfrak{p})$.

Conversely, let us prove that (2) implies (1). If $M \in \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$, then $Q_\sigma(M) \in \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$. But the canonical map $M \to Q_\sigma(M)$ is a σ-isomorphism, so $T(M) \cong T(Q_\sigma(M))$, and we may as well assume that M is σ-closed. By Lemma 1.1, we may find a chain

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

of R-modules, such that for each $1 \le i \le n$, there exists an isomorphism

$$(M_i/M_{i-1})/\sigma(M_i/M_{i-1}) \cong R/\mathfrak{p}_i,$$

for some $\mathfrak{p}_i \in \mathbf{C}(\sigma)$. Since T is exact, for each i, we obtain an exact sequence

$$0 \to Q_\sigma(R/\mathfrak{p}_i) \cong T(M_i/M_{i-1}) \to T(M_i) \to T(M_{i-1}) \to 0$$

in (R, σ)-**mod** and, by induction, $T(M)$ is σ-finitely generated and σ^1-torsion.

Next, for $M = R/\mathfrak{p}$, with $\mathfrak{p} \in \mathbf{C}(\sigma)$, clearly $TT(R/\mathfrak{p}) \cong Q_\sigma(R/\mathfrak{p})$. On the other hand, any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in (R, σ)-**mod**, with objects belonging to $\mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$ induces an exact sequence

$$0 \to TT(M') \to TT(M) \to TT(M'') \to 0$$

together with canonical morphisms between the former and the latter exact sequences. If these canonical morphisms are isomorphisms for M' and M'', then so is the canonical morphism $M \to TT(M)$. In particular, using a chain as above, an easy induction argument shows $M \to TT(M)$ to be an isomorphism for any $M \in \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}$. This finishes the proof. □

Definition 2.3. A functor $T \in \langle \mathbf{T}_{\sigma^1}^{\sigma\text{-fg}}, (R, \sigma)\text{-}\mathbf{mod} \rangle$ satisfying the equivalent conditions of Proposition 2.2 is said to be *σ-dualizing*. A σ^1-torsion R-module E such that $\mathrm{Hom}_R(-, E)$ is a σ-dualizing functor is called a *σ-dualizing R-module.*

From Proposition 2.2, it follows immediately that a σ-dualizing R-module E is an internal injective object in (R, σ)-**mod**, hence an injective R-module by [12]. In fact, we may even show

Proposition 2.4. *Let E be a σ-closed σ^1-torsion R-module. Then the following assertions are equivalent*:

(1) *E is σ-dualizing.*

(2) *$E = \bigoplus_{\mathfrak{p} \in \mathbf{C}(\sigma)} E(R/\mathfrak{p})$ up to non-canonical isomorphism.*

Proof. Let us first prove that (1) implies (2). Let E be σ-dualizing R-module and denote by F the injective hull of E. Since E is σ-torsionfree, so is F, hence every indecomposable component of F is of the form $E(R/\mathfrak{p})$ for some $\mathfrak{p} \in \mathbf{K}(\sigma)$. On the other hand, since E is also σ^1-torsion, it follows that these $\mathfrak{p}$ necessarily belong to $\mathbf{C}(\sigma)$. So, $F = \bigoplus_{\mathfrak{p} \in \mathbf{C}(\sigma)} E(R/\mathfrak{p})^{n_\mathfrak{p}}$, for some non-negative integers $n_\mathfrak{p}$.

Pick $\mathfrak{q} \in \mathbf{C}(\sigma)$. From [10] (see also proof of [4, VI-3.5]), it follows that $F_\mathfrak{q} = E(E_\mathfrak{q})$, the injective hull of $E_\mathfrak{q}$ in $R_\mathfrak{q}$-**mod**. Since E is σ-dualizing, $E_\mathfrak{q}$ is a dualizing $R_\mathfrak{q}$-module, hence $F_\mathfrak{q} = E_\mathfrak{q} = E(R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}) = E(k(\mathfrak{q}))$. It follows that

$$F_\mathfrak{q} = \Big(\bigoplus_{\mathfrak{p} \in \mathbf{C}(\sigma)} E(R/\mathfrak{p})^{n_\mathfrak{p}} \Big)_\mathfrak{q} = E(R/\mathfrak{q})_\mathfrak{q}^{n_\mathfrak{q}} = E(k(\mathfrak{q}))^{n_\mathfrak{q}},$$

whence $n_\mathfrak{q} = 1$, and since this holds for all $\mathfrak{q} \in \mathbf{C}(\sigma)$, it follows that $F = \bigoplus_{\mathfrak{p} \in \mathbf{C}(\sigma)} E(R/\mathfrak{p})$

Denote by $i: E \hookrightarrow F = \bigoplus_{\mathfrak{p} \in \mathbf{C}(\sigma)} E(R/\mathfrak{p})$ the canonical inclusion. Then, for any $\mathfrak{q} \in \mathbf{C}(\sigma)$, this induces a monomorphism $i_\mathfrak{q}: E(k(\mathfrak{q})) \subseteq E(k(\mathfrak{q}))$, which then actually is an isomorphism, since $E(k(\mathfrak{q}))$ is an indecomposable injective $R_\mathfrak{q}$-module. Since both E and F are σ-closed, i is an isomorphism. Conversely, let us show that $E = \bigoplus_{\mathfrak{p} \in \mathbf{C}(\sigma)} E(R/\mathfrak{p})$ is σ-dualizing. It is clear that $T = \mathrm{Hom}_R(-, E)$ is exact and maps σ-isomorphisms to isomorphisms in (R, σ)-**mod.** By Proposition 2.2, it thus remains to prove that $T(R/\mathfrak{p}) \cong Q_\sigma(R/\mathfrak{p})$, for all $\mathfrak{p} \in \mathbf{C}(\sigma)$. But, by the previous Lemma, and using the fact that for any $\mathfrak{p} \neq \mathfrak{q} \in \mathbf{C}(\sigma)$ the R-module $E(R/\mathfrak{q})$ is torsion at $\mathfrak{p}$, it follows that

$$T(R/\mathfrak{p}) = \mathrm{Hom}_R(R/\mathfrak{p}, E) = \mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p})) = k(\mathfrak{p}) = Q_\sigma(R/\mathfrak{p}). \qquad \square$$

Example 2.5. Pick a Krull domain R with field of fractions K and let σ be associated to the height-one primes, i.e., $\mathbf{K}(\sigma) = X^{(1)}(R) \cup \{0\}$. Then R is σ-noetherian and σ is stable, cf. [11, XIII-4.6, XIII-5.8]. Now, $\sigma^1 = \sigma_{R\setminus 0}$, whose associated localization functor is just localization at the non-zero elements of R (i.e., $Q_{\sigma^1}(R) = K$). Since, cf. [11, XIII-5.3]

$$E(K/R) = \bigoplus\{K/R_{\mathfrak{p}} \mid \mathfrak{p} \in X^{(1)}(R)\} = \bigoplus\{E(R/\mathfrak{p}) \mid \mathfrak{p} \in X^{(1)}(R)\},$$

we have that $E(K/R)$ is σ-dualizing.

3. Duality and completion

In this section, E denotes a σ-dualizing R-module.

Definition 3.1. Let R be a σ-noetherian ring with σ a stable idempotent kernel functor. We define the *σ-completion* of R as

$$R^\sigma = \varprojlim_{I \in \mathbf{L}(\sigma^1)} Q_\sigma(R/I).$$

In particular, R^σ is σ-closed and the canonical morphism $R \to R^\sigma$ extends to a morphism $Q_\sigma(R) \to R^\sigma$.

Lemma 3.2. *Let R be a σ-noetherian ring with σ a stable idempotent kernel functor. If E is a σ-dualizing R-module, then*

$$R^\sigma \cong \mathrm{End}_R(E).$$

Proof.

$$\begin{aligned} R^\sigma &= \varprojlim_{I \in \mathbf{L}(\sigma^1)} Q_\sigma(R/I) \cong \varprojlim_{I \in \mathbf{L}(\sigma^1)} \mathrm{Hom}_R(\mathrm{Hom}_R(R/I, E), E) \\ &\cong \mathrm{Hom}_R\left(\varinjlim_{I \in \mathbf{L}(\sigma^1)} \mathrm{Hom}_R(R/I, E), E\right) \\ &\cong \mathrm{Hom}_R(\sigma^1(E), E) = \mathrm{End}_R(E). \quad \square \end{aligned}$$

Definition 3.3. Let R be a σ-noetherian ring with σ a stable idempotent kernel functor. We will say that R is *σ-complete* if the canonical morphism $Q_\sigma(R) \to R^\sigma$ is an isomorphism. We denote by D the functor $\mathrm{Hom}_R(-, E)$ in R-**mod**; clearly D coincides with T on $\mathbf{T}^{\sigma\text{-fg}}_{\sigma^1}$.

Definition 3.4. We denote by $\mathbf{T}^{\sigma\text{-fc}}$ the full subcategory of R-**mod**, consisting of all σ-finitely cogenerated, i.e. E-finitely cogenerated R-modules, where $E = \bigoplus_{\mathfrak{p} \in \mathbf{C}(\sigma)} E(R/\mathfrak{p})$. It automatically follows that the modules in $\mathbf{T}^{\sigma\text{-fc}}$ are σ^1-torsion.

Lemma 3.5. *Let R be a σ-noetherian σ-complete ring with σ a stable idempotent kernel functor.*

(1) *If $M \in \mathbf{T}^{\sigma\text{-fg}}$, then $D(M) \in \mathbf{T}^{\sigma\text{-fc}}$.*

(2) *If $M \in \mathbf{T}^{\sigma\text{-fc}}$, then $D(M) \in \mathbf{T}^{\sigma\text{-fg}}$.*

Proof. (1) If M is a σ-finitely generated R-module, then there exists an exact sequence $0 \to N \to M \to M/N \to 0$ with N finitely generated and M/N σ-torsion. Applying the functor $\mathrm{Hom}_R(-, E)$ we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(M/N, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(N, E) \to 0.$$

Since $\mathrm{Hom}_R(M/N, E) = 0$, we obtain $\mathrm{Hom}_R(M, E) \cong \mathrm{Hom}_R(N, E)$, and so we may assume that M is finitely generated. If M is finitely generated, there exists an exact sequence $R^n \to M \to 0$ and so we have the exact sequence

$$0 \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(R^n, E)$$

where

$$\mathrm{Hom}_R(R^n, E) \cong (\mathrm{Hom}_R(R, E))^n \cong E^n,$$

and so $D(M) = \mathrm{Hom}_R(M, E) \in \mathbf{T}^{\sigma\text{-fc}}$.

(2) If M is a σ-finitely cogenerated R-module, then there exists an exact sequence $0 \to M \to E^n$. Applying the functor $\mathrm{Hom}_R(-, E)$, we obtain the exact sequence

$$\mathrm{Hom}_R(E^n, E) \to \mathrm{Hom}_R(M, E) \to 0,$$

where $\mathrm{Hom}_R(E^n, E) \cong (\mathrm{Hom}_R(E, E))^n \cong (R^\sigma)^n \cong Q_\sigma(R)^n$. So, $D(M) = \mathrm{Hom}_R(M, E)$ is σ-finitely generated, indeed. □

Let us denote by $\tilde{\mathbf{T}}^{\sigma\text{-fg}}$, resp. $\tilde{\mathbf{T}}^{\sigma\text{-fc}}$ the full subcategories $\mathbf{T}^{\sigma\text{-fg}} \cap (R, \sigma)$-**mod** resp. $\mathbf{T}^{\sigma\text{-fc}} \cap (R, \sigma)$-**mod** of (R, σ)-**mod**. Then we have the following result:

Theorem 3.6 (Relative Matlis duality). *Assume that R is a σ-noetherian σ-complete ring with σ a stable idempotent kernel functor. Then D defines a natural equivalence between the categories $\tilde{\mathbf{T}}^{\sigma\text{-fg}}$ and $\tilde{\mathbf{T}}^{\sigma\text{-fc}}$.*

Proof. Given $M \in \mathbf{T}^{\sigma\text{-fc}}$, there exists a short exact sequence $0 \to M \to E^n \to C \to 0$. Applying DD to this sequence yields a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & E^n & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_{E^n}} & & \downarrow{\scriptstyle \varphi_C} & & \\
0 & \longrightarrow & DD(M) & \longrightarrow & DD(E^n) & \longrightarrow & DD(C) & \longrightarrow & 0
\end{array}$$

where $\varphi_C : C \to DD(C)$ is defined by $\varphi_C(c)(f) = f(c)$ for all $f \in D(C)$. Since E is an injective cogenerator in (R, σ)-**mod**, it follows that φ_C is a monomorphism, whenever C is not σ-torsion. Since

$$DD(E) = \mathrm{Hom}_R(\mathrm{Hom}_R(E, E), E) \cong \mathrm{Hom}_R(Q_\sigma(R), E) = E,$$

it follows that φ_{E^n} is an isomorphism, hence that φ_M is an isomorphism. If C is σ-torsion, then

$$M \cong Q_\sigma(M) \cong Q_\sigma(E^n) \cong DD(E^n) \cong DD(M)$$

since $D(C) = 0$. Moreover, the above isomorphisms are natural.

Given $M \in \mathbf{T}^{\sigma\text{-fc}}$, since $D(M) \cong D(Q_\sigma(M))$ we may assume that M is finitely generated, so there exists an exact sequence

$$0 \to K \to Q_\sigma(R)^n \to M \to 0$$

Apply D to this exact sequence; its exactness yields an exact sequence

$$0 \to D(M) \to D(Q_\sigma(R)^n) \to D(K) \to 0$$

and subsequently an exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & Q_\sigma(R)^n & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_K} & & \downarrow{\scriptstyle \varphi_{Q_\sigma(R)^n}} & & \downarrow{\scriptstyle \varphi_M} & & \\ 0 & \longrightarrow & DD(K) & \longrightarrow & DD(Q_\sigma(R)^n) & \longrightarrow & DD(M) & \longrightarrow & 0 \end{array}$$

But here

$$\begin{aligned} DD(Q_\sigma(R)^n) &= \mathrm{Hom}_R(\mathrm{Hom}_R(Q_\sigma(R)^n, E), E) \\ &= \mathrm{Hom}_R(E^n, E) = Q_\sigma(R)^n, \end{aligned}$$

so $\varphi_{Q_\sigma(R)^n}$ is an isomorphism and so φ_M is a (natural) isomorphism too. □

An easy (but long) adaptation of the proof of Lemma 7.4 in [6] yields the following stronger version:

Lemma 3.7. *Let $(R, \mathfrak{m})$ be a local Gorenstein ring of dimension n, let $E = E(R/\mathfrak{m})$, and let τ be a non-trivial idempotent kernel functor on R-**mod**. Then*

$$H_\tau^n(R) = E \Big/ \sum_{\substack{\mathfrak{p} \in Z(\tau) \\ \mathrm{ht}(\mathfrak{p}) = n-1}} \bigcup_j \mathrm{Ann}_E(\mathfrak{p}^{(j)}). \qquad \square$$

Definition 3.8. Let τ and τ' be two idempotent kernel functors. The set U of all idempotent kernel functors κ satisfying $\tau \wedge \kappa \leq \tau'$ has a unique maximal element called the *pseudocomplement of τ relative to τ'* and denoted by $(\tau' : \tau)$; see [7, p. 278].

Lemma 3.9. *Let R be a σ-noetherian ring and let τ be an idempotent kernel functor such that $\sigma^1 \leq \tau$. If $\tau_\sigma^* = (\sigma^1 : \tau)$ then*

$$\mathbf{Z}(\tau_\sigma^*) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid V(\mathfrak{p}) \cap \mathbf{Z}(\tau) \subseteq \mathbf{Z}(\sigma^1)\}.$$

Proof. Let $\mathfrak{p} \in \mathbf{Z}(\tau_\sigma^*)$. If $\mathfrak{q} \in V(\mathfrak{p}) \cap \mathbf{Z}(\tau)$, then $\mathfrak{q} \in \mathbf{Z}(\tau_\sigma^*) \cap \mathbf{Z}(\tau)$ and so $\mathfrak{q} \in \mathbf{Z}(\sigma^1)$.

Conversely, let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathbf{Z}(\sigma_\mathfrak{p}) \cap \mathbf{Z}(\tau) = V(\mathfrak{p}) \cap \mathbf{Z}(\tau) \subseteq \mathbf{Z}(\sigma^1)$. By maximality $\sigma_\mathfrak{p} \leq \tau_\sigma^*$ and thus $\mathfrak{p} \in \mathbf{Z}(\tau_\sigma^*)$. □

Corollary 3.10 *Let R be a σ-noetherian ring and let τ be an idempotent kernel functor such that $\sigma^1 \leq \tau$. Then*

$$\mathbf{L}(\tau_\sigma^*) = \bigcap_{\mathfrak{p} \in \mathbf{K}(\sigma^1) \setminus \mathbf{K}(\tau)} \mathbf{L}(\sigma_{R \setminus \mathfrak{p}}).$$

Proof. Let $\mathfrak{q} \in \mathbf{Z}(\tau_\sigma^*)$, then there is $\mathfrak{p} \in \mathbf{K}(\sigma^1) \setminus \mathbf{K}(\tau)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Since $\mathfrak{p} \notin \mathbf{K}(\tau)$, we have $\mathfrak{p} \in \mathbf{Z}(\tau)$, and thus $\mathfrak{p} \in V(\mathfrak{q}) \cap \mathbf{Z}(\tau) \subseteq \mathbf{Z}(\sigma^1)$, so $\mathfrak{p} \notin \mathbf{K}(\sigma^1)$ which is a contradiction. Thus we have shown $\mathbf{Z}(\tau_\sigma^*) \subseteq \bigcap_{\mathfrak{p} \in \mathbf{K}(\sigma^1) \setminus \mathbf{K}(\tau)} \mathbf{Z}(\sigma_{R \setminus \mathfrak{p}})$. Conversely, if $\mathfrak{q} \in \bigcap_{\mathfrak{p} \in \mathbf{K}(\sigma^1) \setminus \mathbf{K}(\tau)} \mathbf{Z}(\sigma_{R \setminus \mathfrak{p}})$ we have $\mathfrak{q} \nsubseteq \mathfrak{p}$ for each $\mathfrak{p} \in \mathbf{K}(\sigma^1) \setminus \mathbf{K}(\tau)$. For any $\mathfrak{p}' \in V(\mathfrak{q}) \cap \mathbf{Z}(\tau)$ we have $\mathfrak{q} \subseteq \mathfrak{p}'$ and $\mathfrak{p} \in \mathbf{Z}(\tau)$, so $\mathfrak{p}' \notin \mathbf{Z}(\tau)$. If $\mathfrak{p}' \notin \mathbf{Z}(\sigma^1)$, then $\mathfrak{p}' \in \mathbf{K}(\sigma^1)$ and $\mathfrak{q} \nsubseteq \mathfrak{p}'$ which is a contradiction. □

Example 3.11. If R is local, which maximal ideal $\mathfrak{m}$, then $\mathbf{Z}(\sigma_{R \setminus \mathfrak{m}}) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \nsubseteq \mathfrak{m}\} = \emptyset$, $\mathbf{K}(\sigma_{R \setminus \mathfrak{m}}) = \operatorname{Spec}(R)$ and $\mathbf{C}(\sigma_{R \setminus \mathfrak{m}}) = \{\mathfrak{m}\}$. Moreover, $\mathbf{Z}(\sigma_\mathfrak{m}) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{m} \subseteq \mathfrak{p}\} = \{\mathfrak{m}\} \cong \mathbf{K}(\sigma_\mathfrak{m}) = \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, so $\sigma_{R \setminus \mathfrak{m}}^1 = \sigma_\mathfrak{m}$, cf. [4, V-1.2].

If τ is an idempotent kernel functor in R-**mod**, then for $\tau_{\sigma_{R \setminus \mathfrak{m}}}^* = (\sigma_\mathfrak{m} : \tau)$ we have

$$\mathbf{Z}(\tau_{\sigma_{R \setminus \mathfrak{m}}}^*) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid V(\mathfrak{p}) \cap \mathbf{Z}(\tau) \subseteq \{\mathfrak{m}\}\},$$

cf. [6].

Definition 3.12. We say that M has *σ-dimension n*, if the Krull dimension of $M_\mathfrak{p}$ is equal to n for all $\mathfrak{p} \in \mathbf{C}(\sigma)$.

If $\sigma\text{-dim}(M) = n$, then $\sigma\text{-dim}(Q_\sigma(M)) = n$, since for all $\mathfrak{p} \in \mathbf{C}(\sigma)$ we have $M_\mathfrak{p} = Q_\sigma(M)_\mathfrak{p}$.

We will call R a *σ-locally Gorenstein ring* if and only if it is σ-noetherian and if $R_\mathfrak{p}$ is a local Gorenstein ring for every $\mathfrak{p} \in \mathbf{K}(\sigma)$.

Example 3.13. Let R be a Krull domain and let σ be the idempotent kernel functor in R-**mod** associated to the height-one prime ideals of R, then R has σ-dimension one and R is certainly σ-locally Gorenstein, since $R_\mathfrak{p}$ is a discrete valuation ring for $\mathfrak{p} \in X^{(1)}(R)$ (and a field, if $\mathfrak{p} = (0)$).

Lemma 3.14. *Let R be a σ-Gorenstein ring with σ-dimension n and assume σ to be stable. Let τ be a non-trivial idempotent kernel functor on R-**mod** such that $\sigma^1 \leq \tau$. Then*

$$H^n_{\tau,\sigma}(R) = Q_\sigma(E/\tau^{n-1}_\sigma(E))$$

where

$$\tau^{n-1}_\sigma(M) = \sum_{\substack{\mathfrak{p}\in \mathbf{K}(\sigma)\setminus \mathbf{K}(\tau)\\ \mathrm{ht}(\mathfrak{p})=n-1}} \tau^s_\mathfrak{p}(M).$$

Proof. We may assume $R = Q_\sigma(R)$. Since R is σ-Gorenstein and the σ-dimension of R is n, by [4, VI-3.34] the Cousin complex is an injective resolution of $Q_\sigma(R)$ in (R, σ)-**mod** and its length is n:

$$0 \to R \to C^0_\sigma(R) \to \cdots \to C^{n-1}_\sigma(R) \xrightarrow{d^{n-1}} C^n_\sigma(R) \to 0.$$

By [4, VI-3.9, 3.8, 3.23], we have that

$$H^n_{\sigma^1/\sigma}(R) \cong H^n_{\sigma^1,\sigma}(R) \cong \bigoplus_{\mathfrak{p}\in \mathbf{C}(\sigma)} H^n_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) \cong C^n_\sigma(R)$$

and by [4, V-4.15] this is a σ-dualizing R-module. Then

$$C^n_\sigma(R) \cong \bigoplus_{\mathfrak{p}\in \mathbf{C}(\sigma)} E(R/\mathfrak{p}) = E.$$

Therefore $C^n_\sigma(R)$ is σ^1-torsion and so τ-torsion. We will use this injective resolution to obtain

$$H^n_{\tau,\sigma}(R) \cong R^n\tilde{\tau}(R) \cong Q_\sigma H^n_\sigma(R),$$

where $\tilde{\tau}$ is the idempotent kernel functor induced by τ in (R, σ)-**mod**. In particular we obtain the following exact sequence:

$$\tau(C^{n-1}_\sigma(R)) \to \tau(C^n_\sigma(R)) \cong E \to H^n_{\tau,\sigma}(R) \to 0.$$

It is possible to interpret $\tau(C^{n-1}_\sigma(R))$ as follows. By [4, VI-3.23]

$$C^{n-1}_\sigma(R) \cong \bigoplus_{\substack{\mathfrak{p}\in \mathbf{K}(\sigma)\\ \mathrm{ht}(\mathfrak{p})=n-1}} H^{n-1}_{\mathfrak{p}R_\mathfrak{p}}(R_{\mathfrak{p}},)$$

but if $\mathfrak{p}\in \mathbf{Z}(\tau)$, then $\tau(H^{n-1}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})) = H^{n-1}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$ and if $\mathfrak{p}\in \mathbf{K}(\tau)$, then $\tau(H^{n-1}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})) = 0$ and then

$$\tau(C^{n-1}_\sigma(R)) \cong \bigoplus_{\substack{\mathfrak{p}\in \mathbf{K}(\sigma)\setminus \mathbf{K}(\tau)\\ \mathrm{ht}(\mathfrak{p})=n-1}} H^{n-1}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) = \bigoplus_{\substack{\mathfrak{p}\in \mathbf{K}(\sigma)\cap \mathbf{Z}(\tau)\\ \mathrm{ht}(\mathfrak{p})=n-1}} H^{n-1}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$$

and there exists an exact sequence

$$\bigoplus_{\substack{\mathfrak{p}\in\mathbf{K}(\sigma)\cap\mathbf{Z}(\tau)\\ \mathrm{ht}(\mathfrak{p})=n-1}} H^{n-1}_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) \xrightarrow{d^{n-1}} E \to H^n_{\tau,\sigma}(R) \to 0$$

It is easy to prove that $\mathrm{Im}(d^{n-1}) \subseteq \tau^{n-1}_\sigma(E)$, then there exists an epimorphism

$$H^n_{\tau,\sigma}(R) = \mathrm{Coker}(d^{n-1}) \to E/\tau^{n-1}_\sigma(E) \to 0$$

and a morphism

$$H^n_{\tau,\sigma}(R) \to Q_\sigma(E/\tau^{n-1}_\sigma(E)).$$

To prove that this morphism is an isomorphism in (R, σ)-**mod** it is sufficient to prove that for every $\mathfrak{p}\in\mathbf{C}(\sigma)$

$$H^n_{\tau,\sigma}(R)_\mathfrak{p} \to Q_\sigma(E/\tau^{n-1}_\sigma(E))_\mathfrak{p}$$

is an isomorphism in R-**mod**. We have

$$H^n_{\tau,\sigma}(R)_\mathfrak{p} \cong H^n_{\tau(\mathfrak{p})}(R_\mathfrak{p})$$

by [4, V-4.12]. If we use Lemma 3.7, we find

$$\begin{aligned} H^n_{\tau(\mathfrak{p})}(R_\mathfrak{p}) &= E(k(\mathfrak{p}))\Bigg/\Bigg(\sum_{\substack{\mathfrak{q}R_\mathfrak{p}\in\mathbf{Z}(\tau(\mathfrak{p}))\\ \mathrm{ht}(\mathfrak{q}R_\mathfrak{p})=n-1}} \bigcup_j \mathrm{Ann}_{E(k(\mathfrak{p}))}(\mathfrak{q}^{(j)}R_\mathfrak{p})\Bigg) \\ &= E(k(\mathfrak{p}))\Bigg/\Bigg(\sum_{\substack{\mathfrak{q}R_\mathfrak{p}\in\mathbf{Z}(\tau(\mathfrak{p}))\\ \mathrm{ht}(\mathfrak{q}R_\mathfrak{p})=n-1}} \tau^s_{\mathfrak{q}R_\mathfrak{p}}(E(k(\mathfrak{p})))\Bigg) \\ &= E(k(\mathfrak{p}))/(\tau(\mathfrak{p})^{n-1}(E(k(\mathfrak{p})))), \end{aligned}$$

where

$$\tau(\mathfrak{p})^{n-1}(E(k(\mathfrak{p}))) = \sum_{\substack{\mathfrak{q}R_\mathfrak{p}\in\mathbf{Z}(\tau(\mathfrak{p}))\\ \mathrm{ht}(\mathfrak{q}R_\mathfrak{p})=n-1}} \tau^s_{\mathfrak{q}R_\mathfrak{p}}(E(k(\mathfrak{p})).$$

On the other hand,

$$Q_\sigma(E/\tau^{n-1}_\sigma(E))_\mathfrak{p} = (E/\tau^{n-1}_\sigma(E))_\mathfrak{p} = E_\mathfrak{p}/\tau^{n-1}_\sigma(E)_\mathfrak{p},$$

but

$$\begin{aligned} \tau^{n-1}_\sigma(E)_\mathfrak{p} &= \sum_{\substack{\mathfrak{q}\in\mathbf{K}(\sigma)\setminus\mathbf{K}(\tau)\\ \mathrm{ht}(\mathfrak{q})=n-1}} \tau^s_\mathfrak{q}(E)_\mathfrak{p} = \sum_{\substack{\mathfrak{q}R_\mathfrak{p}\in\mathbf{Z}(\tau(\mathfrak{p}))\\ \mathrm{ht}(\mathfrak{q}R_\mathfrak{p})=n-1}} \tau^s_{\mathfrak{q}R_\mathfrak{p}}(E_\mathfrak{p}) \\ &= \tau(\mathfrak{p})^{n-1}(E(k(\mathfrak{p}))), \end{aligned}$$

thus

$$Q_\sigma(E/\tau_\sigma^{n-1}(E))_\mathfrak{p} = E(k(\mathfrak{p}))/\tau(\mathfrak{p})^{n-1}(E(k(\mathfrak{p}))),$$

and the result follows. □

Theorem 3.15. *Let R be a σ-Gorenstein σ-complete ring with σ-dimension n and assume σ to be stable. Let τ be a non-trivial idempotent kernel functor on R-**mod** such that* $\sigma^1 \leq \tau$. *Then*

$$D(H^n_{\tau,\sigma}(R)) = \tau^*_\sigma(R).$$

Proof. We may assume $R = Q_\sigma(R)$ and put $K = D(H^n_{\tau,\sigma}(R))$. We have

$$\begin{aligned} K &= \mathrm{Hom}_R(H^n_{\tau,\sigma}(R), E) \\ &= \mathrm{Hom}_R(Q_\sigma(E/\tau_\sigma^{n-1}(E)), E) \\ &= \mathrm{Hom}_R(E/\tau_\sigma^{n-1}(E), E) \\ &\subseteq \mathrm{Hom}_R(E, E) = R^\sigma = Q_\sigma(R) = R. \end{aligned}$$

So K may be viewed as an ideal of R and is τ-closed, since E is τ-closed. Let us consider the exact sequence

$$0 \to I = \tau_\sigma^{n-1}(E) \xrightarrow{i} E \to H^n_{\tau,\sigma}(R) \to 0$$

in (R, σ)-**mod**. We obtain another sequence

$$0 \to K = D(H^n_{\tau,\sigma}(R)) \to R = D(E) \to D(I) \to 0,$$

which is exact in both (R, σ)-**mod**, and R-**mod**. Now, there is an isomorphism $D(I) \cong R/K$, and so

$$K = \mathrm{Ann}(R/K) = \mathrm{Ann}(D(I)).$$

Let us prove $\mathrm{Ann}(D(I)) = \mathrm{Ann}(I)$. Indeed, clearly $\mathrm{Ann}(I) \subseteq \mathrm{Ann}(D(I))$. Conversely, if $r \in \mathrm{Ann}(D(I))$, then for each $f \in D(I)$, we have $rf = 0$. So, in particular $ri = 0$, where $i: I \to E$ is the inclusion, hence $0 = ri(I) = i(rI) = rI$. It follows that $K = \mathrm{Ann}(\tau_\sigma^{n-1}(E))$, and since

$$\tau_\sigma^{n-1}(E) = \sum_{\substack{\mathfrak{p} \in K(\sigma) \setminus K(\tau) \\ \mathrm{ht}(\mathfrak{p}) = n-1}} \tau_\mathfrak{p}^s(E),$$

we get

$$K = \bigcap_{\substack{\mathfrak{p} \in K(\sigma) \setminus K(\tau) \\ \mathrm{ht}(\mathfrak{p}) = n-1}} \mathrm{Ann}(\tau_\mathfrak{p}^s(E)) = \bigcap_{\substack{\mathfrak{p} \in K(\sigma) \setminus K(\tau) \\ \mathrm{ht}(\mathfrak{p}) = n-1}} \bigcap_j \mathfrak{p}^{(j)}$$

as is easily verified by applying Matlis duality, for example. It is easy to see that

$$\{\mathfrak{p}\in\mathbf{K}(\sigma)\backslash\mathbf{K}(\tau)\,|\,\mathrm{ht}(\mathfrak{p})=n-1\}=\{\mathfrak{p}\in\mathbf{K}(\sigma^1)\backslash\mathbf{K}(\tau)\,|\,\mathrm{ht}(\mathfrak{p})=n-1\}$$

since $\mathbf{K}(\sigma^1)=\mathbf{K}(\sigma)\backslash\mathbf{C}(\sigma)=\mathbf{K}(\sigma)\backslash\{\mathfrak{p}\in\mathbf{K}(\sigma)\,|\,\mathrm{ht}(\mathfrak{p})=n\}$.

Finally, we claim that $K=\tau_\sigma^*(R)$, since $a\in K$ if $aR_\mathfrak{p}\subseteq\mathfrak{p}^jR_\mathfrak{p}$ for all j and each $\mathfrak{p}\in\mathbf{K}(\sigma)\backslash\mathbf{K}(\tau)$ with $\mathrm{ht}(\mathfrak{p})=n-1$. By the previous remark, this is equivalent to $aR_\mathfrak{p}\subseteq\cap_j\mathfrak{p}^jR_\mathfrak{p}$ for all $\mathfrak{p}\in\mathbf{K}(\tau)$ with $\mathrm{ht}(\mathfrak{p})=n-1$, i.e., to $aR_\mathfrak{p}=0$ for each of these. Now, concluding, this amounts to

$$\mathrm{Ann}(a)\in\bigcap_{\substack{\mathfrak{p}\in\mathbf{K}(\sigma^1)\backslash\mathbf{K}(\tau)\\ \mathrm{ht}(\mathfrak{p})=n-1}}\mathbf{L}(\sigma_{R\backslash\mathfrak{p}}),$$

i.e., to $a\in\tau_\sigma^*(R)$, indeed. □

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